# ESSENTIAL NORMALITY OF POLYNOMIAL-GENERATED SUBMODULES: HARDY SPACE AND BEYOND 

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Abstract. Recently, Douglas and Wang proved that for each polynomial $q$, the submodule $[q]$ of the Bergman module generated by $q$ is essentially normal [9]. Using improved techniques, we show that the Hardy-space analogue of this result holds, and more.

## 1. Introduction

Let $\mathbf{B}$ be the unit ball in $\mathbf{C}^{n}$. Throughout the paper, the complex dimension $n$ is always assumed to be greater than or equal to 2. Recall that the Drury-Arveson space $H_{n}^{2}$ is the Hilbert space of analytic functions on $\mathbf{B}$ with $(1-\langle\zeta, z\rangle)^{-1}$ as its reproducing kernel. The space $H_{n}^{2}$ is naturally considered as a Hilbert module over the polynomial ring $\mathbf{C}\left[z_{1}, \ldots, z_{n}\right]$. In [3-6], Arveson raised the question of whether graded submodules $\mathcal{M}$ of $H_{n}^{2}$ are essentially normal. That is, for the restricted operators

$$
Z_{\mathcal{M}, j}=M_{z_{j}} \mid \mathcal{M}, \quad 1 \leq j \leq n
$$

on $\mathcal{M}$, do commutators $\left[Z_{\mathcal{M}, j}^{*}, Z_{\mathcal{M}, i}\right.$ ] belong to the Schatten class $\mathcal{C}_{p}$ for $p>n$ ? This problem is commonly referred to as the Arveson conjecture.

Numerous papers have been written on this problem [4,6,7,10,13,14]. In particular, Guo and Wang showed that the answer to the above question is affirmative if $\mathcal{M}$ is generated by a homogeneous polynomial [14]. In [8], Douglas proposed analogous essential normality problems for submodules of the Bergman module $L_{a}^{2}(\mathbf{B}, d v)$.

As it turns out, the Bergman space case is more tractable. In fact, the Bergman space version of the problem was recently solved by Douglas and Wang in [9] for arbitrary polynomials. In that paper, Douglas and Wang showed that for any polynomial $q \in$ $\mathbf{C}\left[z_{1}, \ldots, z_{n}\right]$, the submodule $[q]$ of the Bergman module generated by $q$ is $p$-essentially normal for $p>n$. What is especially remarkable is that [9] contains many novel ideas.

The present paper grew out of a remark in [9]. Toward the end of [9], Douglas and Wang commented
"It seems likely that the argument in this paper can be generalized to obtain the same result for the Hardy and the Drury-Arveson spaces. However, while we believe that both results hold, perhaps techniques from [9,8] may be needed to complete the proofs."

While the Drury-Arveson space case is out of reach at the moment, in this paper we will settle the Hardy space case mentioned above, and we will go a little farther than that.

[^0]The key realization is that Bergman space, Hardy space and Drury-Arveson space are all members of a family of reproducing-kernel Hilbert spaces of analytic functions on $\mathbf{B}$ parametrized by a real-valued parameter $-n \leq t<\infty$. In fact, the spaces corresponding to the values $t \in \mathbf{Z}_{+}$were used in an essential way in the proofs in [9]. Our main observation is that if one considers other values of $t$, then one will see how to extend the techniques in [9] beyond the Bergman space case. In short, in this paper we establish the analogue of the main result in [9] for spaces with parameter $-2<t<\infty$. Before stating the result, let us first introduce these spaces.

For each real number $-n \leq t<\infty$, let $\mathcal{H}^{(t)}$ be the Hilbert space of analytic functions on $\mathbf{B}$ with the reproducing kernel

$$
\frac{1}{(1-\langle\zeta, z\rangle)^{n+1+t}} .
$$

Alternately, one can describe $\mathcal{H}^{(t)}$ as the completion of $\mathbf{C}\left[z_{1}, \ldots, z_{n}\right]$ with respect to the norm $\|\cdot\|_{t}$ arising from the inner product $\langle\cdot, \cdot\rangle_{t}$ defined according to the following rules: $\left\langle z^{\alpha}, z^{\beta}\right\rangle_{t}=0$ whenever $\alpha \neq \beta$,

$$
\left\langle z^{\alpha}, z^{\alpha}\right\rangle_{t}=\frac{\alpha!}{\prod_{j=1}^{|\alpha|}(n+t+j)}
$$

if $\alpha \in \mathbf{Z}_{+}^{n} \backslash\{0\}$, and $\langle 1,1\rangle_{t}=1$. Here and throughout the paper, we use the conventional multi-index notation [15, page 3].

Obviously, $\mathcal{H}^{(0)}$ is the Bergman space $L_{a}^{2}(\mathbf{B}, d v)$. One can view the Bergman space $\mathcal{H}^{(0)}=L_{a}^{2}(\mathbf{B}, d v)$ as a benchmark, against which the other spaces in the family should be compared. Note that for each $-1<t<\infty, \mathcal{H}^{(t)}$ is a weighted Bergman space.

Let $S$ denote the unit sphere $\left\{z \in \mathbf{C}^{n}:|z|=1\right\}$ in $\mathbf{C}^{n}$. Let $\sigma$ be the positive, regular Borel measure on $S$ that is invariant under the orthogonal group $O(2 n)$, i.e., the group of isometries on $\mathbf{C}^{n} \cong \mathbf{R}^{2 n}$ that fix 0 . We take the usual normalization $\sigma(S)=1$. Recall that the Hardy space $H^{2}(S)$ is the closure of $\mathbf{C}\left[z_{1}, \ldots, z_{n}\right]$ in $L^{2}(S, d \sigma)$.

Obviously, $\mathcal{H}^{(-1)}$ is just the Hardy space $H^{2}(S)$. Moreover, $\mathcal{H}^{(-n)}$ is none other than the Drury-Arveson space $H_{n}^{2}$.

It is well known that for each $-n \leq t<-1$, the tuple of multiplication operators $\left(M_{z_{1}}, \ldots, M_{z_{n}}\right)$ is not jointly subnormal on $\mathcal{H}^{(t)}$ [1, Theorem 3.9]. In other words, if $-n \leq$ $t<-1$, then $\mathcal{H}^{(t)}$ is more like the Drury-Arveson space than the Hardy space. The practical consequence of this is that it is difficult to do estimates on $\mathcal{H}^{(t)}$ if $-n \leq t<-1$.

Let $q \in \mathbf{C}\left[z_{1}, \ldots, z_{n}\right]$. For each $-n \leq t<\infty$, let $[q]^{(t)}$ denote the closure of

$$
\left\{q f: f \in \mathbf{C}\left[z_{1}, \ldots, z_{n}\right]\right\}
$$

in $\mathcal{H}^{(t)}$. Since $\mathcal{H}^{(t)}$ is a Hilbert module over $\mathbf{C}\left[z_{1}, \ldots, z_{n}\right],[q]^{(t)}$ is a submodule. For each $j \in\{1, \ldots, n\}$, define submodule operator

$$
Z_{q, j}^{(t)}=M_{z_{j}} \mid[q]^{(t)} .
$$

Recall that the submodule $[q]^{(t)}$ is said to be $p$-essentially normal if the commutators $\left[Z_{q, j}^{(t) *}, Z_{q, i}^{(t)}\right], i, j \in\{1, \ldots, n\}$, all belong to the Schatten class $\mathcal{C}_{p}$. With the foregoing preparation, we are now ready to state our result.

Theorem 1.1. Let $q$ be an arbitrary polynomial in $\mathbf{C}\left[z_{1}, \ldots, z_{n}\right]$. Then for each real number $-2<t<\infty$, the submodule $[q]^{(t)}$ of $\mathcal{H}^{(t)}$ is $p$-essentially normal for every $p>n$.

Clearly, the Hardy-space case mentioned in [9] is settled by applying Theorem 1.1 to the special case $t=-1$.

On the other hand, it is a real pity that the requirement $t>-2$ in Theorem 1.1 does not allow us to capture any Drury-Arveson space in dimensions $n \geq 2$. But as a consolation, Theorem 1.1 does cover spaces $\mathcal{H}^{(t)}$ for $-2<t<-1$, which, as we mentioned, are more Drury-Arveson-like than Hardy-like.

On the technical side, this paper does offer some improvement over [9]. As the authors of [9] stated, the key step in the proof of their result rests on weighted norm estimates given in Section 3 in that paper. At the core of their weighted estimates is an argument using a covering lemma. This is where we offer the most significant improvement. In this paper, the covering-lemma argument of [9] is done away with entirely. In its place, we use a much simpler argument based on Fubini's theorem.

In fact, using Fubini's theorem-based argument in place of covering-lemma argument is a situation with which we are quite familiar. See, for example, the proofs of Proposition 2.6 and Lemma 5.2 in [11].

There are many technical contributions made in [9]. Perhaps, the most important among these is Lemma 3.2 in that paper. This lemma will again be the basis for analysis here. The reader will see that with the combination of [9,Lemma 3.2] and our Fubini's theorem-based argument, the analysis part of the proof is actually easy.

As it was the case in [9], an essential role in the proof is played by the number operator $N$ introduced by Arveson in [2]. Recall that, for a polynomial $f(z)=\sum_{\alpha} c_{\alpha} z^{\alpha}$,

$$
(N f)(z)=\sum_{\alpha} c_{\alpha}|\alpha| z^{\alpha}
$$

Here as well as in [9], the proof boils down to the estimate of an operator series where the $k$-th term has the operator

$$
(N+1+n+t)^{-k-1}
$$

as a factor, $k \geq 0$. Douglas and Wang's idea is to factor the above in the form

$$
(N+1+n+t)^{-k-1}=(N+1+n+t)^{-1 / 2} \cdot(N+1+n+t)^{-k-(1 / 2)},
$$

"reserve" the factor $(N+1+n+t)^{-1 / 2}$ for establishing the requisite Schatten-class membership, and use the other factor, $(N+1+n+t)^{-k-(1 / 2)}$, to boost the weight of the space. This is another place where [9] and the present paper differ. Instead of factoring, we will apply the whole of $(N+1+n+t)^{-k-1}$ to boost weight. Proposition 4.2 below allows us
to recover an equivalent of $(N+1+n+t)^{-1 / 2}$ at the end of the estimate. This is why we are able to push $t$ below -1 .

The rest of the paper is organized as follows. Since the analysis part of the proof is now easy, we will take care of that first, in Sections 2 and 3 . Section 4 contains a brief discussion of the relation between the natural embedding $\mathcal{H}^{(t)} \rightarrow \mathcal{H}^{(t+1)}$ and norm ideals. Section 5, which mirrors Section 2 in [9], contains the proof of our result.

## 2. Derivative on the Disc

Write $D$ for the open unit disc $\{z \in \mathbf{C}:|z|<1\}$ in the complex plane. Let $d A$ be the area measure on $D$ with the normalization $A(D)=1$. The unit circle $\{\tau \in \mathbf{C}:|\tau|=1\}$ will be denote by $\mathbf{T}$. Furthermore, let $d m$ be the Lebesgue measure on $\mathbf{T}$ with the normalization $m(\mathbf{T})=1$. For convenience, we write $\partial$ for the one-variable differentiation $d / d z$ on $\mathbf{C}$.

Our first lemma is basically a restatement of Lemma 3.2 in [9].
Lemma 2.1. Suppose that $g$ is a one-variable polynomial of degree $K \geq 1$, and that $f$ is analytic on $D$. Then for each $k \in \mathbf{N}$ we have

$$
\left|\left(\partial^{k} g\right)(0) f(0)\right|^{2} \leq 2^{2 k+2}(K!)^{2} \int|g f|^{2} d A
$$

Proof. For each $0 \leq r<1$, let $g_{r}(z)=g(r z)$ and $f_{r}(z)=f(r z)$. We only need to consider the case $1 \leq k \leq K$. For such a $k$, Lemma 3.2 in [9] tells us that $\left|\left(\partial^{k} g_{r}\right)(0) f_{r}(0)\right| \leq$ $K!\int_{\mathbf{T}}\left|g_{r} f_{r}\right| d m$. Since $\left(\partial^{k} g_{r}\right)(0)=r^{k}\left(\partial^{k} g\right)(0)$ and $f_{r}(0)=f(0)$, we have

$$
\begin{aligned}
\left|\left(\partial^{k} g\right)(0) f(0)\right| & =2 \int_{1 / 2}^{1} r^{-k}\left|\left(\partial^{k} g_{r}\right)(0) f_{r}(0)\right| d r \leq 2 K!\int_{1 / 2}^{1} r^{-k} \int_{\mathbf{T}}\left|g_{r}(\tau) f_{r}(\tau)\right| d m(\tau) d r \\
& \leq 2^{k+1} K!\int_{1 / 2}^{1} 2 r \int_{\mathbf{T}}|g(r \tau) f(r \tau)| d m(\tau) d r \leq 2^{k+1} K!\int|g f| d A
\end{aligned}
$$

Squaring both sides and applying the Cauchy-Schwarz inequality, the lemma follows.
For each $z \in D$, define the disc $D(z)=\{w \in D:|w-z|<(1 / 2)(1-|z|)\}$.
Lemma 2.2. For all $w \in D$ and $x \in(-1, \infty)$, we have

$$
\int \frac{\left(1-|z|^{2}\right)^{x}}{A(D(z))} \chi_{D(z)}(w) d A(z) \leq 2^{2 \max \{x, 0\}+5}\left(1-|w|^{2}\right)^{x}
$$

Proof. Let $w \in D$, and let $z \in D$ be such that $w \in D(z)$. Then we have $1-|w| \leq$ $1-|z|+|z-w|<(3 / 2)(1-|z|)$. Also, $1-|z| \leq 1-|w|+|w-z| \leq 1-|w|+(1 / 2)(1-|z|)$. After cancellation, we find $(1 / 2)(1-|z|) \leq 1-|w|$. Thus

$$
\begin{equation*}
(2 / 3)(1-|w|) \leq 1-|z| \leq 2(1-|w|) \quad \text { whenever } \quad w \in D(z) \tag{2.1}
\end{equation*}
$$

From this we obtain that for $w \in D(z)$ and $x \in(-1, \infty)$,

$$
\left(1-|z|^{2}\right)^{x} \leq\left\{\begin{array}{lll}
2^{2 x}\left(1-|w|^{2}\right)^{x} & \text { if } & 0 \leq x<\infty \\
3\left(1-|w|^{2}\right)^{x} & \text { if } & -1<x<0
\end{array}\right.
$$

Thus, to complete the proof, it suffices to show that

$$
\begin{equation*}
\int \frac{\chi_{D(z)}(w)}{A(D(z))} d A(z) \leq 9 \tag{2.2}
\end{equation*}
$$

for every $w \in D$. For each $w \in D$, let $G(w)=\{z \in D: w \in D(z)\}$. If $z \in G(w)$, then $|z-w| \leq(1 / 2)(1-|z|) \leq 1-|w|$ by $(2.1)$. Hence $A(G(w)) \leq(1-|w|)^{2}$. On the other hand, if $z \in G(w)$, then $A(D(z))=(1 / 4)(1-|z|)^{2} \geq(1 / 3)^{2}(1-|w|)^{2}$, also by (2.1). Clearly, (2.2) follows from these two inequalities.

Proposition 2.3. Suppose that $g$ is a one-variable polynomial of degree $K \geq 1$, and that $f$ is analytic on $D$. Then for all $k \in \mathbf{N}$ and $t \in(0, \infty)$ satisfying the condition $t-2 k>-1$,

$$
\begin{aligned}
\int\left|\left(\partial^{k} g\right)(z) f(z)\right|^{2} & \left(1-|z|^{2}\right)^{t} d A(z) \\
& \leq 2^{6 k+2 \max \{t-2 k, 0\}+7}(K!)^{2} \int|g(w) f(w)|^{2}\left(1-|w|^{2}\right)^{t-2 k} d A(w)
\end{aligned}
$$

Proof. Define $g_{z}(u)=g(z+(1 / 2)(1-|z|) u)$ and $f_{z}(u)=f(z+(1 / 2)(1-|z|) u)$ for each $z \in D$. Then $2^{-k}(1-|z|)^{k}\left(\partial^{k} g\right)(z)=\left(\partial^{k} g_{z}\right)(0)$ and $f(z)=f_{z}(0)$. By Lemma 2.1,

$$
\begin{aligned}
\left|\left(\partial^{k} g\right)(z) f(z)\right|^{2} & =\frac{2^{2 k}\left|\left(\partial^{k} g_{z}\right)(0) f_{z}(0)\right|^{2}}{(1-|z|)^{2 k}} \leq \frac{2^{4 k+2}(K!)^{2}}{(1-|z|)^{2 k}} \int\left|g_{z}(u) f_{z}(u)\right|^{2} d A(u) \\
& =\frac{2^{4 k+2}(K!)^{2}}{(1-|z|)^{2 k}} \cdot \frac{1}{A(D(z))} \int_{D(z)}|g(w) f(w)|^{2} d A(w)
\end{aligned}
$$

Therefore, if $t-2 k>-1$, then

$$
\begin{aligned}
\int \mid\left(\partial^{k} g\right)(z) & \left.f(z)\right|^{2}\left(1-|z|^{2}\right)^{t} d A(z) \\
& \leq 2^{6 k+2}(K!)^{2} \int \frac{\left(1-|z|^{2}\right)^{t-2 k}}{A(D(z))}\left(\int_{D(z)}|g(w) f(w)|^{2} d A(w)\right) d A(z) \\
& =2^{6 k+2}(K!)^{2} \int\left\{\int \frac{\left(1-|z|^{2}\right)^{t-2 k}}{A(D(z))} \chi_{D(z)}(w) d A(z)\right\}|g(w) f(w)|^{2} d A(w) \\
& \leq 2^{6 k+2 \max \{t-2 k, 0\}+7}(K!)^{2} \int\left(1-|w|^{2}\right)^{t-2 k}|g(w) f(w)|^{2} d A(w),
\end{aligned}
$$

where the last step is an application of Lemma 2.2. This completes the proof.

## 3. Derivatives on the Ball

Recall that there is a constant $A_{0} \in\left(2^{-n}, \infty\right)$ such that

$$
\begin{equation*}
2^{-n} r^{n} \leq \sigma(\{\xi \in S:|1-\langle u, \xi\rangle|<r\}) \leq A_{0} r^{n} \tag{3.1}
\end{equation*}
$$

for all $u \in S$ and $0<r \leq 2$ [15,Proposition 5.1.4]. For each $z \in \mathbf{B}$, define the subset

$$
T(z)=\left\{w \in \mathbf{B}:|1-\langle w, z\rangle|<2\left(1-|z|^{2}\right), 1-|w|^{2}>(1 / 2)\left(1-|z|^{2}\right)\right\}
$$

of the unit ball. We begin our estimates with the properties of the set $T(z)$.
Let $d v$ be the volume measure on $\mathbf{B}$ with the normalization $v(\mathbf{B})=1$.
Lemma 3.1. There is a constant $0<C_{3.1}<\infty$ such that for all $\zeta \in \mathbf{B}$ and $x \in(-1, \infty)$,

$$
\int\left(1-|z|^{2}\right)^{x-n-1} \chi_{T(z)}(\zeta) d v(z) \leq C_{3.1} 2^{\max \{x, 0\}}\left(1-|\zeta|^{2}\right)^{x}
$$

Proof. Let $\zeta, z \in \mathbf{B}$ be such that $\zeta \in T(z)$. Then we have $1-|\zeta|^{2} \leq 2(1-|\zeta|) \leq 2|1-\langle\zeta, z\rangle|$ $<4\left(1-|z|^{2}\right)$. Combining this with the condition $1-|\zeta|^{2}>(1 / 2)\left(1-|z|^{2}\right)$, we have

$$
\begin{equation*}
(1 / 4)\left(1-|\zeta|^{2}\right) \leq 1-|z|^{2} \leq 2\left(1-|\zeta|^{2}\right) \tag{3.2}
\end{equation*}
$$

Therefore, for $x \in(-1, \infty)$ we have

$$
\left(1-|z|^{2}\right)^{x} \leq\left\{\begin{array}{llc}
2^{x}\left(1-|\zeta|^{2}\right)^{x} & \text { if } & 0 \leq x<\infty \\
4\left(1-|\zeta|^{2}\right)^{x} & \text { if } & -1<x<0
\end{array}\right.
$$

Thus, to complete the proof, it suffices to show that there is a $0<C<\infty$ such that

$$
\begin{equation*}
\int \frac{\chi_{T(z)}(\zeta)}{\left(1-|z|^{2}\right)^{n+1}} d v(z) \leq C \tag{3.3}
\end{equation*}
$$

for every $\zeta \in \mathbf{B}$. Given a $\zeta \in \mathbf{B}$, consider the set $\Omega(\zeta)=\{z \in \mathbf{B}: \zeta \in T(z)\}$. Write $\zeta=|\zeta| \eta$ with $\eta \in S$. If $z=|z| \xi \in \Omega(\zeta)$, where $\xi \in S$, then $|1-\langle\eta, \xi\rangle| \leq 2|1-\langle\zeta, z\rangle|<4\left(1-|z|^{2}\right)$ $<8\left(1-|\zeta|^{2}\right)$. Also, $1-|z| \leq|1-\langle\zeta, z\rangle|<4\left(1-|\zeta|^{2}\right)$ if $z \in \Omega(\zeta)$. Hence

$$
\Omega(\zeta) \subset\left\{r \xi: 0<1-r<4\left(1-|\zeta|^{2}\right) ; \xi \in S,|1-\langle\eta, \xi\rangle|<8\left(1-|\zeta|^{2}\right)\right\}
$$

By (3.1) and the decomposition $d v=2 n r^{2 n-1} d r d \sigma$, there is a $0<C_{1}<\infty$ such that $v(\Omega(\zeta)) \leq C_{1}\left(1-|\zeta|^{2}\right)^{n+1}$ for every $\zeta \in \mathbf{B}$. By $(3.2),\left(1-|z|^{2}\right)^{-n-1} \leq 4^{n+1}\left(1-|\zeta|^{2}\right)^{-n-1}$ when $z \in \Omega(\zeta)$. Clearly, (3.3) follows from these two inequalities.
Lemma 3.2. There is a constant $0<\epsilon<1$ such that for each $0 \leq a<1$, the set $T((a, 0, \ldots, 0))$ contains the polydisc

$$
\begin{equation*}
P_{a}=\left\{\left(a+u, \zeta_{2}, \ldots, \zeta_{n}\right):|u|<\epsilon\left(1-a^{2}\right),\left|\zeta_{j}\right|<\epsilon \sqrt{1-a^{2}}, 2 \leq j \leq n\right\} \tag{3.4}
\end{equation*}
$$

Proof. Given an $a \in[0,1)$, write $\alpha=(a, 0, \ldots, 0)$. Let $0<\epsilon<1$, and suppose that $u$ and $\zeta_{2}, \ldots, \zeta_{n}$ satisfy the conditions $|u|<\epsilon\left(1-a^{2}\right)$ and $\left|\zeta_{j}\right|<\epsilon \sqrt{1-a^{2}}, 2 \leq j \leq n$. Then consider the vector $w=\left(a+u, \zeta_{2}, \ldots, \zeta_{n}\right)$. We have $|1-\langle w, \alpha\rangle|=\left|1-a^{2}-a u\right|<(1+\epsilon)(1-$ $a^{2}$ ). Moreover, $1-|w|^{2}=1-|a+u|^{2}-\left(\left|\zeta_{2}\right|^{2}+\cdots+\left|\zeta_{n}\right|^{2}\right) \geq 1-|a+u|^{2}-(n-1) \epsilon^{2}\left(1-a^{2}\right)$. On the other hand, $1-|a+u|^{2}=1-\left(a^{2}+2 \operatorname{Re}(a u)+|u|^{2}\right) \geq 1-a^{2}-3|u| \geq(1-3 \epsilon)\left(1-a^{2}\right)$. Hence $1-|w|^{2} \geq(1-(n+2) \epsilon)\left(1-a^{2}\right)$. Thus $\epsilon=\{3(n+2)\}^{-1}$ suffices for our purpose.

As usual, write $\partial_{1}, \ldots, \partial_{n}$ for the differentiations with respect to the complex variables $z_{1}, \ldots, z_{n}$. For each vector $b=\left(b_{1}, \ldots, b_{n}\right) \in \mathbf{C}^{n}$, define the directional derivative

$$
\partial_{b}=b_{1} \partial_{1}+\cdots+b_{n} \partial_{n}
$$

Lemma 3.3. There is a constant $0<C_{3.3}<\infty$ such that the following estimate holds: Suppose that $q \in \mathbf{C}\left[z_{1}, \ldots, z_{n}\right]$ and that $\operatorname{deg}(q)=K \geq 1$. Let $f \in \mathbf{C}\left[z_{1}, \ldots, z_{n}\right]$. If $z$ and $b$ are vectors in $\mathbf{B} \backslash\{0\}$ satisfying the relation $\langle b, z\rangle=0$, then

$$
\left|\left(\partial_{b} q\right)(z) f(z)\right|^{2} \leq \frac{C_{3.3}(K!)^{2}}{\left(1-|z|^{2}\right)^{n+2}} \int_{T(z)}|q f|^{2} d v
$$

Proof. Consider the special case where $z=\alpha=(a, 0, \ldots, 0)$ for some $0<a<1$. Let $\epsilon$ be the constant provided by Lemma 3.2. Define the polydisc

$$
Y=\left\{\left(a+u, 0, \zeta_{3}, \ldots, \zeta_{n}\right):|u|<\epsilon\left(1-a^{2}\right),\left|\zeta_{j}\right|<\epsilon \sqrt{1-a^{2}}, 3 \leq j \leq n\right\}
$$

For each $y \in Y$, we define the one-varible polynomial $q_{y}(w)=q\left(y+\epsilon \sqrt{1-a^{2}} w e_{2}\right)$, where $e_{2}=(0,1,0, \ldots, 0)$. Similarly, define $f_{y}(w)=f\left(y+\epsilon \sqrt{1-a^{2}} w e_{2}\right)$ on $D$. Since $\left(\partial q_{y}\right)(0)=$ $\epsilon \sqrt{1-a^{2}}\left(\partial_{2} q\right)(y)$ and $f_{y}(0)=f(y)$, we apply Lemma 2.1 to obtain

$$
\left|\left(\partial_{2} q\right)(y) f(y)\right|^{2}=\frac{\left|\left(\partial q_{y}\right)(0) f_{y}(0)\right|^{2}}{\epsilon^{2}\left(1-a^{2}\right)} \leq \frac{16(K!)^{2}}{\epsilon^{2}\left(1-a^{2}\right)} \int\left|q_{y}(w) f_{y}(w)\right|^{2} d A(w)
$$

Making the substitution $\zeta_{2}=\epsilon \sqrt{1-a^{2}} w$, we find that

$$
\left|\left(\partial_{2} q\right)(y) f(y)\right|^{2} \leq \frac{16(K!)^{2}}{\epsilon^{4}\left(1-a^{2}\right)^{2}} \int_{\left|\zeta_{2}\right|<\epsilon \sqrt{1-a^{2}}}\left|q\left(y+\zeta_{2} e_{2}\right) f\left(y+\zeta_{2} e_{2}\right)\right|^{2} d A\left(\zeta_{2}\right)
$$

Now, integrating both sides over $Y$, we see that

$$
\epsilon^{2 n-2}\left(1-a^{2}\right)^{n}\left|\left(\partial_{2} q\right)(\alpha) f(\alpha)\right|^{2} \leq \int_{Y}\left|\left(\partial_{2} q\right)(y) f(y)\right|^{2} d y \leq \frac{16 C(K!)^{2}}{\epsilon^{4}\left(1-a^{2}\right)^{2}} \int_{P_{a}}|q f|^{2} d v
$$

where $P_{a}$ is given by (3.4) and $C$ accounts for the normalization constants for the measures involved. Since Lemma 3.2 tells us that $P_{a} \subset T(\alpha)$, we have

$$
\left|\left(\partial_{2} q\right)(\alpha) f(\alpha)\right|^{2} \leq \frac{16 \epsilon^{-(2 n+2)} C(K!)^{2}}{\left(1-a^{2}\right)^{n+2}} \int_{T(\alpha)}|q f|^{2} d v
$$

Obviously, the above inequality also holds if we replace $\partial_{2}$ by $\partial_{j}$ for any $2 \leq j \leq n$. Applying these and the Cauchy-Schwarz inequality, we see that

$$
\left|\left(\partial_{b} q\right)(\alpha) f(\alpha)\right|^{2} \leq(n-1) \frac{16 \epsilon^{-(2 n+2)} C(K!)^{2}}{\left(1-a^{2}\right)^{n+2}} \int_{T(\alpha)}|q f|^{2} d v \quad \text { if }\langle b, \alpha\rangle=0, b \in \mathbf{B}
$$

This proves the lemma in the special case where $z=\alpha=(a, 0, \ldots, 0), 0<a<1$. The general case follows from this special case and the following easily-verified relations: If $U$ is any unitary transformation on $\mathbf{C}^{n}$ and $w, b \in \mathbf{B}$, then $U T(w)=T(U w)$ and $\left(\partial_{b}(q \circ U)\right)(w)=\left(\partial_{U b} q\right)(U w)$.

Following [9], for each pair of $i \neq j$ in $\{1, \ldots, n\}$ we define $L_{i, j}=\bar{z}_{j} \partial_{i}-\bar{z}_{i} \partial_{j}$.
Proposition 3.4. There is a constant $1 \leq C_{3.4}<\infty$ such that the following estimate holds: Suppose that $q \in \mathbf{C}\left[z_{1}, \ldots, z_{n}\right]$ and that $\operatorname{deg}(q)=K \geq 1$. Let $f \in \mathbf{C}\left[z_{1}, \ldots, z_{n}\right]$. Then for every positive number $t>0$ and all integers $i \neq j$ in $\{1, \ldots, n\}$, we have

$$
\int\left|\left(L_{i, j} q\right)(z) f(z)\right|^{2}\left(1-|z|^{2}\right)^{t} d v(z) \leq C_{3.4} 2^{t}(K!)^{2} \int|q(\zeta) f(\zeta)|^{2}\left(1-|\zeta|^{2}\right)^{t-1} d v(\zeta)
$$

Proof. It follows from Lemma 3.3 that

$$
\left|\left(L_{i, j} q\right)(z) f(z)\right|^{2} \leq \frac{C_{3.3}(K!)^{2}}{\left(1-|z|^{2}\right)^{n+2}} \int_{T(z)}|q(\zeta) f(\zeta)|^{2} d v(\zeta)
$$

$z \in \mathbf{B}$. Multiplying both sides by $\left(1-|z|^{2}\right)^{t}$ and integrating, we find that

$$
\begin{aligned}
\int\left|\left(L_{i, j} q\right)(z) f(z)\right|^{2} & \left(1-|z|^{2}\right)^{t} d v(z) \\
& \leq C_{3.3}(K!)^{2} \int\left(\left(1-|z|^{2}\right)^{t-n-2} \int_{T(z)}|q(\zeta) f(\zeta)|^{2} d v(\zeta)\right) d v(z) \\
& =C_{3.3}(K!)^{2} \int\left\{\int\left(1-|z|^{2}\right)^{t-n-2} \chi_{T(z)}(\zeta) d v(z)\right\}|q(\zeta) f(\zeta)|^{2} d v(\zeta)
\end{aligned}
$$

Applying Lemma 3.1 with $x=t-1$ to the $\{\cdots\}$ above, the proposition follows.
Write $R=z_{1} \partial_{1}+\cdots+z_{n} \partial_{n}$, the radial derivative in $n$ variables. We will denote the one-variable radial derivative by $\mathcal{R}$. For each polynomial $h$ and each $\xi \in S$, define the "slice" function $h_{\xi}(z)=h(z \xi), z \in D$. If $q$ is a polynomial in $n$ variables, then for every $\xi \in S$ we have the relation $\left(\mathcal{R} q_{\xi}\right)(z)=(R q)_{\xi}(z)$.

Proposition 3.5. There is a constant $1 \leq C_{3.5}<\infty$ such that the following estimate holds: Suppose that $q \in \mathbf{C}\left[z_{1}, \ldots, z_{n}\right]$ and that $\operatorname{deg}(q)=K \geq 1$. Let $f \in \mathbf{C}\left[z_{1}, \ldots, z_{n}\right]$. Then for each pair of $k \in \mathbf{N}$ and $t \in(0, \infty)$ satisfying the condition $t-2 k>-1$,

$$
\int\left|\left(R^{k} q\right)(\zeta) f(\zeta)\right|^{2}\left(1-|\zeta|^{2}\right)^{t} d v(\zeta) \leq C_{3.5}^{K(k+t)}(K!)^{2} \int|q(\zeta) f(\zeta)|^{2}\left(1-|\zeta|^{2}\right)^{t-2 k} d v(\zeta)
$$

Proof. As in [9], we need the following relation between $d v, d \sigma$ and $d A$ : Since $d v=$ $2 n r^{2 n-1} d r d \sigma, d A=2 r d r d m$, and $d \sigma$ is invariant under rotation, we have

$$
\begin{equation*}
\int g d v=n \int\left(\int g(z \xi)|z|^{2 n-2} d A(z)\right) d \sigma(\xi) \tag{3.5}
\end{equation*}
$$

By Lemma 3.6 in [9], for each $k \in \mathbf{N}$,

$$
\begin{equation*}
\mathcal{R}^{k}=\sum_{j=1}^{k} a_{j}^{(k)} z^{j} \partial^{j} \quad \text { with } \quad\left|a_{j}^{(k)}\right|<(j+1)^{k} \tag{3.6}
\end{equation*}
$$

Since the degree of $q$ equals $K$, for each $\xi \in S$ we have

$$
\left(R^{k} q\right)_{\xi}(z)=\left(\mathcal{R}^{k} q_{\xi}\right)(z)=\sum_{j=1}^{\min \{k, K\}} a_{j}^{(k)} z^{j}\left(\partial^{j} q_{\xi}\right)(z)
$$

Given $f$, for each $\xi \in S$ we define the "rigged" slice function $f^{(\xi)}(z)=z^{n-1} f(z \xi), z \in D$. Applying first (3.6) and then Proposition 2.3, when $t-2 k>-1$, we have

$$
\begin{aligned}
& \int\left|\left(\mathcal{R}^{k} q_{\xi}\right)(z) f^{(\xi)}(z)\right|^{2}\left(1-|z|^{2}\right)^{t} d A(z) \\
& \quad \leq K(K+1)^{2 k} \sum_{j=1}^{\min \{k, K\}} \int\left|\left(\partial^{j} q_{\xi}\right)(z) f^{(\xi)}(z)\right|^{2}\left(1-|z|^{2}\right)^{t} d A(z) \\
& \quad \leq K(K+1)^{2 k} \sum_{j=1}^{\min \{k, K\}} 2^{6 j+2 \max \{t-2 j, 0\}+7}(K!)^{2} \int\left|q_{\xi}(z) f^{(\xi)}(z)\right|^{2}\left(1-|z|^{2}\right)^{t-2 j} d A(z) \\
& \quad \leq K^{2}(K+1)^{2 k} 2^{6 k+2 t+7}(K!)^{2} \int\left|q_{\xi}(z) f^{(\xi)}(z)\right|^{2}\left(1-|z|^{2}\right)^{t-2 k} d A(z) \\
& \quad \leq C_{3.5}^{K(k+t)}(K!)^{2} \int\left|q_{\xi}(z) f^{(\xi)}(z)\right|^{2}\left(1-|z|^{2}\right)^{t-2 k} d A(z)
\end{aligned}
$$

By the relations $\left(\mathcal{R}^{k} q_{\xi}\right)(z)=\left(R^{k} q\right)_{\xi}(z), f^{(\xi)}(z)=z^{n-1} f(z \xi)$ and $|z|=|z \xi|$, we now have

$$
\begin{aligned}
& \int\left|\left(R^{k} q\right)(z \xi) f(z \xi)\right|^{2}\left(1-|z \xi|^{2}\right)^{t}|z|^{2 n-2} d A(z) \\
& \quad \leq C_{3.5}^{K(k+t)}(K!)^{2} \int|q(z \xi) f(z \xi)|^{2}\left(1-|z \xi|^{2}\right)^{t-2 k}|z|^{2 n-2} d A(z)
\end{aligned}
$$

Integrating both sides with respect to the measure $d \sigma$ on $S$ and applying (3.5), the proposition follows.

Proposition 3.6. There is a constant $1 \leq C_{3.6}<\infty$ such that the following estimate holds: Suppose that $q \in \mathbf{C}\left[z_{1}, \ldots, z_{n}\right]$ and that $\operatorname{deg}(q)=K \geq 1$. Let $f \in \mathbf{C}\left[z_{1}, \ldots, z_{n}\right]$. Then for each $t \in(1, \infty)$ and each $j \in\{1, \ldots, n\}$, we have

$$
\begin{equation*}
\int\left|\left(\partial_{j} q\right)(\zeta) f(\zeta)\right|^{2}\left(1-|\zeta|^{2}\right)^{t} d v(\zeta) \leq C_{3.6}^{K t}(K!)^{2} \int|q(\zeta) f(\zeta)|^{2}\left(1-|\zeta|^{2}\right)^{t-2} d v(\zeta) \tag{3.7}
\end{equation*}
$$

Proof. There is a $C$ such that for every analytic function $h$ on $\mathbf{B}$ and every $t>0$, we have

$$
\begin{equation*}
\int_{|\zeta|<1 / 2}|h(\zeta)|^{2}\left(1-|\zeta|^{2}\right)^{t} d v(\zeta) \leq C\left(\frac{16}{7}\right)^{t} \int_{1 / 2 \leq|\zeta|<3 / 4}|h(\zeta)|^{2}\left(1-|\zeta|^{2}\right)^{t} d v(\zeta) \tag{3.8}
\end{equation*}
$$

Now apply Proposition 3.4 and the case $k=1$ in Proposition 3.5: by the identity $|z|^{2} \partial_{j}=$ $\bar{z}_{j} R+\sum_{i \neq j} z_{i} L_{j, i},(3.7)$ obviously holds if $\left(\partial_{j} q\right)(\zeta)$ is replaced by $|\zeta|^{2}\left(\partial_{j} q\right)(\zeta)$ on the righthand side. The extra factor $|\zeta|^{2}$ is then removed by using (3.8).

## 4. Embedding and Norm Ideals

For a bounded operator $A$, we write its $s$-numbers as $s_{1}(A), \ldots, s_{k}(A), \ldots$ as usual. Recall that, for each $1 \leq p<\infty$, the formula

$$
\begin{equation*}
\|A\|_{p}^{+}=\sup _{k \geq 1} \frac{s_{1}(A)+s_{2}(A)+\cdots+s_{k}(A)}{1^{-1 / p}+2^{-1 / p}+\cdots+k^{-1 / p}} \tag{4.1}
\end{equation*}
$$

defines a symmetric norm for operators [12,Section III.14]. On any Hilbert space $\mathcal{H}$, the set $\mathcal{C}_{p}^{+}=\left\{A \in \mathcal{B}(\mathcal{H}):\|A\|_{p}^{+}<\infty\right\}$ is a norm ideal [12,Section III.2]. It is well known that if $p<p^{\prime}$, then $\mathcal{C}_{p}^{+}$is contained in the Schatten class $\mathcal{C}_{p^{\prime}}$.

For a non-increasing sequence of non-negative numbers $\left\{a_{1}, \ldots, a_{k}, \ldots\right\}$, if $a_{1}+\cdots+$ $a_{k} \leq C\left(1^{-1 / p}+\cdots+k^{-1 / p}\right)$, then $k a_{k} \leq C\left(1^{-1 / p}+\cdots+k^{-1 / p}\right)$. It follows that if $p>1$ and if $T \in \mathcal{C}_{p}^{+}$, then there is a $0<C(T)<\infty$ such that $s_{k}(T) \leq C(T) k^{-1 / p}$ for every $k \in$ $\mathbf{N}$. Thus if $p>1$ and if $B$ is a bounded operator such that $B^{*} B \in \mathcal{C}_{p}^{+}$, then $B \in \mathcal{C}_{2 p}^{+}$.

Proposition 4.1. For each $t \geq-n$, let $I^{(t)}: \mathcal{H}^{(t)} \rightarrow \mathcal{H}^{(t+1)}$ be the natural embedding. Then $I^{(t) *} I^{(t)} \in \mathcal{C}_{n}^{+}$.

Proof. Expanding the reproducing kernel $(1-\langle\zeta, z\rangle)^{-(n+1+t)}$, we see that the standard orthonormal basis for $\mathcal{H}^{(t)}$ is $\left\{e_{\alpha}^{(t)}: \alpha \in \mathbf{Z}_{+}^{n}\right\}$, where

$$
\begin{equation*}
e_{\alpha}^{(t)}(\zeta)=\left(\frac{1}{\alpha!} \prod_{j=1}^{|\alpha|}(n+t+j)\right)^{1 / 2} \zeta^{\alpha}, \quad \alpha \neq 0 \tag{4.2}
\end{equation*}
$$

and $e_{0}^{(t)}(\zeta)=1$. Given these orthonormal bases, it is straightforward to verify that

$$
I^{(t) *} I^{(t)} e_{\alpha}^{(t)}=\frac{n+1+t}{n+1+|\alpha|+t} e_{\alpha}^{(t)}, \quad \alpha \in \mathbf{Z}_{+}^{n}
$$

This formula gives us all the $s$-numbers for $I^{(t) *} I^{(t)}$. By (4.1), $I^{(t) *} I^{(t)} \in \mathcal{C}_{n}^{+}$.
Proposition 4.2. Suppose that $E$ is a linear subspace of $\mathbf{C}\left[z_{1}, \ldots, z_{n}\right]$ and that $t \geq-n$. Let $E^{(t)}$ be the closure of $E$ in $\mathcal{H}^{(t)}$, and let $\mathcal{E}^{(t)}$ be the orthogonal projection from $\mathcal{H}^{(t)}$ to $E^{(t)}$. Suppose that $A \in \mathcal{B}\left(\mathcal{H}^{(t)}\right)$, and suppose that there is a $C$ such that

$$
\begin{equation*}
\|A g\|_{t} \leq C\|g\|_{t+1} \tag{4.3}
\end{equation*}
$$

for every $g \in E$. Then $A \mathcal{E}^{(t)} \in \mathcal{C}_{2 n}^{+}$.
Proof. By (4.3), for each $g \in E$ we have

$$
\left\langle A^{*} A g, g\right\rangle_{t}=\|A g\|_{t}^{2} \leq C\|g\|_{t+1}^{2}=C^{2}\left\|I^{(t)} g\right\|_{t+1}^{2}=C^{2}\left\langle I^{(t)} g, I^{(t)} g\right\rangle_{t+1}=C^{2}\left\langle I^{(t) *} I^{(t)} g, g\right\rangle_{t} .
$$

That is, the operator inequality $\left(A \mathcal{E}^{(t)}\right)^{*} A \mathcal{E}^{(t)} \leq C^{2} \mathcal{E}^{(t)} I^{(t) *} I^{(t)} \mathcal{E}^{(t)}$ holds on $\mathcal{H}^{(t)}$. Thus $s_{j}\left(\left(A \mathcal{E}^{(t)}\right)^{*} A \mathcal{E}^{(t)}\right) \leq s_{j}\left(C^{2} \mathcal{E}^{(t)} I^{(t) *} I^{(t)} \mathcal{E}^{(t)}\right)$ for each $j \in \mathbf{N}$ [12,Lemma II.1.1]. By Proposition 4.1, $\left(A \mathcal{E}^{(t)}\right)^{*} A \mathcal{E}^{(t)} \in \mathcal{C}_{n}^{+}$. Since $n \geq 2$, this implies $A \mathcal{E}^{(t)} \in \mathcal{C}_{2 n}^{+}$.

## 5. Proof of Theorem 1.1

For each $t \geq-n$ and each polynomial $q$, we write $M_{q}^{(t)}$ for the operator of multiplication by $q$ on the space $\mathcal{H}^{(t)}$. Keep in mind that the notation "*" is $t$-specific: $M_{q}^{(t) *}$ means the adjoint of $M_{q}^{(t)}$ with respect to the inner product $\langle\cdot, \cdot\rangle_{t}$.

Proposition 5.1. Let $q \in \mathbf{C}\left[z_{1}, \ldots, z_{n}\right], 1 \leq j \leq n$ and $t \geq-n$. For $f \in \mathbf{C}\left[z_{1}, \ldots, z_{n}\right]$ satisfying the condition $f(0)=0$, we have

$$
M_{z_{j}}^{(t) *} M_{q}^{(t)} f-M_{q}^{(t)} M_{z_{j}}^{(t) *} f=\sum_{k=0}^{\infty}(N+1+n+t)^{-k-1}\left(M_{\partial_{j} R^{k} q}^{(t)}-M_{z_{j}}^{(t) *} M_{R^{k+1} q}^{(t)}\right) f .
$$

Proof. The main idea is that both sides are linear with respect to both $q$ and $f$. Therefore the proof is a matter of straightforward verification in the special case of $q=z^{\alpha}$ and $f=z^{\beta}, \beta \neq 0$, using (4.2). The details of the verification are similar to the Bergman space case (see the proof of Proposition 2.1 in [9]).
Proposition 5.2. Let $t \geq-n$ and $\ell \in \mathbf{N}$. (1) For each $f \in \mathbf{C}\left[z_{1}, \ldots, z_{n}\right]$ satisfying the condition $\left(\partial^{\alpha} f\right)(0)=0$ for $|\alpha|<\ell$ and each non-negative integer $k$, we have

$$
\left\|(N+1+n+t)^{-k-1} f\right\|_{t}^{2} \leq \frac{(n+2 k+2+t+\ell)^{\ell}}{(\ell+1+n+t)^{2 k+2}}\|f\|_{2 k+2+t}^{2}
$$

(2) For each $f \in \mathbf{C}\left[z_{1}, \ldots, z_{n}\right]$ satisfying the condition $\left(\partial^{\alpha} f\right)(0)=0$ for $|\alpha|<\ell+1$, each non-negative integer $k$ and each $1 \leq j \leq n$, we have

$$
\left\|(N+1+n+t)^{-k-1}\left(M_{z_{j}}^{(t) *}-M_{z_{j}}^{(t+2 k+2) *}\right) f\right\|_{t}^{2} \leq(2 k+4)^{4} \frac{(n+2 k+4+t+\ell)^{2 \ell}}{(\ell+1+n+t)^{2 k+2}}\|f\|_{2 k+4+t}^{2}
$$

Proof. For (1), it suffices to consider the case where $f$ is a homogeneous polynomial of degree $m \geq \ell$, as it was the case for the corresponding part in [9]. For such an $f$,

$$
\begin{align*}
\left\|(N+1+n+t)^{-k-1} f\right\|_{t}^{2} & =\frac{\|f\|_{t}^{2}}{(m+1+n+t)^{2 k+2}} \\
& =\frac{\|f\|_{2 k+2+t}^{2}}{(m+1+n+t)^{2 k+2}} \prod_{j=1}^{m} \frac{n+2 k+2+t+j}{n+t+j}  \tag{4.2}\\
& =\frac{\|f\|_{2 k+2+t}^{2}}{(m+1+n+t)^{2 k+2}} \prod_{j=1}^{2 k+2} \frac{n+m+t+j}{n+t+j}
\end{align*}
$$

where the last $=$ is obtained by considering $\prod_{j=1}^{2 k+2+m}(n+t+j)$. Since $m \geq \ell$, for each $j \geq 1$ we have $(n+m+t+j) /(m+1+n+t) \leq(n+\ell+t+j) /(\ell+1+n+t)$. Hence

$$
\begin{aligned}
\|(N+1 & +n+t)^{-k-1} f\left\|_{t}^{2} \leq\right\| f \|_{2 k+2+t}^{2} \prod_{j=1}^{2 k+2} \frac{n+\ell+t+j}{(\ell+1+n+t)(n+t+j)} \\
& =\frac{\|f\|_{2 k+2+t}^{2}}{(\ell+1+n+t)^{2 k+2}} \prod_{j=1}^{\ell} \frac{n+2 k+2+t+j}{n+t+j} \leq \frac{(n+2 k+2+t+\ell)^{\ell}}{(\ell+1+n+t)^{2 k+2}}\|f\|_{2 k+2+t}^{2} .
\end{aligned}
$$

This proves (1).
Let $e_{j}$ be the element in $\mathbf{Z}_{+}^{n}$ whose $j$-th component is 1 and whose other components are 0 . To prove (2), first note that (4.2) gives us

$$
M_{z_{j}}^{(t) *} z^{\alpha}=\frac{\alpha_{j}}{n+t+|\alpha|} z^{\alpha-e_{j}}
$$

whenever the $j$-th component $\alpha_{j}$ of $\alpha$ is greater than 0 . Hence

$$
\begin{align*}
\left(M_{z_{j}}^{(t) *}-M_{z_{j}}^{(2 k+2+t) *}\right) z^{\alpha} & =\frac{\alpha_{j}(2 k+2)}{(n+t+|\alpha|)(n+2 k+2+t+|\alpha|)} z^{\alpha-e_{j}} \\
& =\frac{2 k+2}{n+t+|\alpha|} M_{z_{j}}^{(2 k+2+t) *} z^{\alpha} . \tag{5.1}
\end{align*}
$$

For $f \in \mathbf{C}\left[z_{1}, \ldots, z_{n}\right]$ with $f(0)=0,(N+n+t)^{-1} f$ is well defined. Thus we can define

$$
f_{t, k}=(N+1+n+2 k+2+t)(N+n+t)^{-1} f .
$$

We have $\left\|f_{t, k}\right\|_{\tau} \leq(2 k+4)\|f\|_{\tau}$ for every $\tau \geq-n$. Obviously, (5.1) implies

$$
\left(M_{z_{j}}^{(t) *}-M_{z_{j}}^{(2 k+2+t) *}\right) f=(2 k+2) M_{z_{j}}^{(2 k+2+t) *}(N+1+n+2 k+2+t)^{-1} f_{t, k}
$$

Now suppose that $\left(\partial^{\alpha} f\right)(0)=0$ for $|\alpha|<\ell+1$. Applying (1) twice, we have

$$
\begin{aligned}
\|(N & +1+n+t)^{-k-1}\left(M_{z_{j}}^{(t) *}-M_{z_{j}}^{(2 k+2+t) *}\right) f \|_{t}^{2} \\
& =(2 k+2)^{2}\left\|(N+1+n+t)^{-k-1} M_{z_{j}}^{(2 k+2+t) *}(N+1+n+2 k+2+t)^{-1} f_{t, k}\right\|_{t}^{2} \\
& \leq(2 k+2)^{2} \frac{(n+2 k+2+t+\ell)^{\ell}}{(\ell+1+n+t)^{2 k+2}}\left\|M_{z_{j}}^{(2 k+2+t) *}(N+1+n+2 k+2+t)^{-1} f_{t, k}\right\|_{2 k+2+t}^{2} \\
& \leq(2 k+2)^{2} \frac{(n+2 k+2+t+\ell)^{\ell}}{(\ell+1+n+t)^{2 k+2}}\left\|(N+1+n+2 k+2+t)^{-1} f_{t, k}\right\|_{2 k+2+t}^{2} \\
& \leq(2 k+2)^{2} \frac{(n+2 k+2+t+\ell)^{\ell}}{(\ell+1+n+t)^{2 k+2}} \cdot \frac{(n+2 k+4+t+\ell)^{\ell}}{(\ell+1+n+2 k+2+t)^{2}}\left\|f_{t, k}\right\|_{2 k+4+t}^{2} \\
& \leq(2 k+2)^{2} \frac{(n+2 k+4+t+\ell)^{2 \ell}}{(\ell+1+n+t)^{2 k+2}}(2 k+4)^{2}\|f\|_{2 k+4+t}^{2} .
\end{aligned}
$$

This completes the proof of (2).
For each real number $t>-1$, define

$$
a_{n, t}=\frac{1}{n!} \prod_{j=1}^{n}(t+j)
$$

Using (4.2) and [15,Proposition 1.4.9], it is straightforward to verify that

$$
\begin{equation*}
\langle f, g\rangle_{t}=a_{n, t} \int f(\zeta) \overline{g(\zeta)}\left(1-|\zeta|^{2}\right)^{t} d v(\zeta) \tag{5.2}
\end{equation*}
$$

for $f, g \in \mathcal{H}^{(t)}$, $t>-1$. In other words, if $t>-1$, then $\mathcal{H}^{(t)}$ is the weighted Bergman space $L_{a}^{2}\left(\mathbf{B}, a_{n, t}\left(1-|\zeta|^{2}\right)^{t} d v(\zeta)\right)$.

Our next step requires the assumption that $t>-2$.
Proposition 5.3. Let real number $t>-2$ and integer $K \geq 1$ be given. Then there is a constant $C_{5.3}=C_{5.3}(n, K, t)$ such that the following estimate holds: Let $q \in \mathbf{C}\left[z_{1}, \ldots, z_{n}\right]$ be such that $\operatorname{deg}(q)=K$. Suppose that $f \in \mathbf{C}\left[z_{1}, \ldots, z_{n}\right]$ satisfies the condition $\left(\partial^{\alpha} f\right)(0)=0$ for $|\alpha| \leq \ell+1$, where $\ell \in \mathbf{N}$. Then for every integer $k \geq 0$ and every $j \in\{1, \ldots, n\}$,

$$
\begin{aligned}
\|(N+1+n+t)^{-k-1}\left(M_{\partial_{j} R^{k} q}^{(t)}\right. & \left.-M_{z_{j}}^{(t) *} M_{R^{k+1} q}^{(t)}\right) f \|_{t} \\
& \leq \frac{(n+2 k+4+t+\ell)^{\ell+2}}{(\ell+1+n+t)^{k+1}} C_{5.3}^{k+1}\|q f\|_{t+1}
\end{aligned}
$$

Proof. Since
$M_{\partial_{j} R^{k} q}^{(t)}-M_{z_{j}}^{(t) *} M_{R^{k+1} q}^{(t)}=\left(M_{\partial_{j} R^{k} q}^{(t)}-M_{z_{j}}^{(2 k+2+t) *} M_{R^{k+1} q}^{(t)}\right)-\left(M_{z_{j}}^{(t) *}-M_{z_{j}}^{(2 k+2+t) *}\right) M_{R^{k+1} q}^{(t)}$,
we have

$$
\begin{equation*}
\left\|(N+1+n+t)^{-k-1}\left(M_{\partial_{j} R^{k} q}^{(t)}-M_{z_{j}}^{(t) *} M_{R^{k+1} q}^{(t)}\right) f\right\|_{t} \leq A+B \tag{5.3}
\end{equation*}
$$

where

$$
\begin{aligned}
& A=\left\|(N+1+n+t)^{-k-1}\left(M_{\partial_{j} R^{k} q}^{(t)}-M_{z_{j}}^{(2 k+2+t) *} M_{R^{k+1} q}^{(t)}\right) f\right\|_{t} \quad \text { and } \\
& B=\left\|(N+1+n+t)^{-k-1}\left(M_{z_{j}}^{(t) *}-M_{z_{j}}^{(2 k+2+t) *}\right) M_{R^{k+1} q}^{(t)} f\right\|_{t} .
\end{aligned}
$$

We estimate $A$ and $B$ separately. For $A$, we apply Proposition 5.2(1), which gives us

$$
\begin{align*}
A & \leq \frac{(n+2 k+2+t+\ell)^{\ell}}{(\ell+1+n+t)^{k+1}}\left\|\left(M_{\partial_{j} R^{k} q}^{(t)}-M_{z_{j}}^{(2 k+2+t) *} M_{R^{k+1} q}^{(t)}\right) f\right\|_{2 k+2+t} \\
& =\frac{(n+2 k+2+t+\ell)^{\ell}}{(\ell+1+n+t)^{k+1}}\left\|\left(M_{\partial_{j} R^{k} q}^{(2 k+2+t)}-M_{z_{j}}^{(2 k+2+t) *} M_{R^{k+1} q}^{(2 k+2+t)}\right) f\right\|_{2 k+2+t} \tag{5.4}
\end{align*}
$$

Since $t>-2$, we have $2 k+2+t>0$ for each $k \geq 0$. Hence $\mathcal{H}^{(2 k+2+t)}$ is a weighted Bergman space. By (5.2), we have

$$
\begin{aligned}
& \left\|\left(M_{\partial_{j} R^{k} q}^{(2 k+2+t)}-M_{z_{j}}^{(2 k+2+t) *} M_{R^{k+1} q}^{(2 k+2+t)}\right) f\right\|_{2 k+2+t} \\
& \quad \leq a_{n, 2 k+2+t}^{1 / 2}\left(\int\left|\left\{\left(\partial_{j} R^{k} q\right)(z)-\bar{z}_{j}\left(R^{k+1} q\right)(z)\right\} f(z)\right|^{2}\left(1-|z|^{2}\right)^{2 k+2+t} d v(z)\right)^{1 / 2}
\end{aligned}
$$

The identity $\partial_{j}-\bar{z}_{j} R=\left(1-|z|^{2}\right) \partial_{j}+\sum_{i \neq j} z_{i} L_{j, i}$ then leads to

$$
\begin{align*}
\|\left(M_{\partial_{j} R^{k} q}^{(2 k+2+t)}-\right. & \left.M_{z_{j}}^{(2 k+2+t) *} M_{R^{k+1} q}^{(2 k+2+t)}\right) f \|_{2 k+2+t} \\
& \leq a_{n, 2 k+2+t}^{1 / 2}\left(\int\left|\left(\partial_{j} R^{k} q\right)(z) f(z)\right|^{2}\left(1-|z|^{2}\right)^{2 k+4+t} d v(z)\right)^{1 / 2} \\
& +a_{n, 2 k+2+t}^{1 / 2} \sum_{i \neq j}\left(\int\left|\left(L_{j, i} R^{k} q\right)(z) f(z)\right|^{2}\left(1-|z|^{2}\right)^{2 k+2+t} d v(z)\right)^{1 / 2} \tag{5.5}
\end{align*}
$$

Applying Propositions 3.6 and 3.5 , we have

$$
\begin{align*}
\int \mid\left(\partial_{j} R^{k} q\right)(z) & \left.f(z)\right|^{2}\left(1-|z|^{2}\right)^{2 k+4+t} d v(z) \\
& \leq C_{3.6}^{K(2 k+4+t)}(K!)^{2} \int\left|\left(R^{k} q\right)(z) f(z)\right|^{2}\left(1-|z|^{2}\right)^{2 k+2+t} d v(z) \\
& \leq\left(C_{3.6} C_{3.5}\right)^{K(3 k+4+t)}(K!)^{4} \int|q(z) f(z)|^{2}\left(1-|z|^{2}\right)^{2+t} d v(z) \tag{5.6}
\end{align*}
$$

Since $1+t>-1$, we can apply Propositions 3.4 and 3.5 to obtain

$$
\begin{align*}
\int \mid\left(L_{j, i} R^{k} q\right)(z) & \left.f(z)\right|^{2}\left(1-|z|^{2}\right)^{2 k+2+t} d v(z) \\
& \leq C_{3.4} 2^{2 k+2+t}(K!)^{2} \int\left|\left(R^{k} q\right)(z) f(z)\right|^{2}\left(1-|z|^{2}\right)^{2 k+1+t} d v(z) \\
& \leq C_{3.4}\left(2 C_{3.5}\right)^{K(3 k+2+t)}(K!)^{4} \int|q(z) f(z)|^{2}\left(1-|z|^{2}\right)^{1+t} d v(z) \tag{5.7}
\end{align*}
$$

By the assumption $t>-2$, we have $a_{n, 1+t} \geq(n!)^{-1}(2+t)^{n}$. Also note that $a_{n, 2 k+2+t} \leq$ $(n!)^{-1}(n+2 k+2+t)^{n}$. Combining (5.5), (5.6), (5.7) and (5.2), we see that there is a $C_{1}$ that depends only on $n, K$ and $t(>-2)$ such that

$$
\left\|\left(M_{\partial_{j} R^{k} q}^{(2 k+2+t)}-M_{z_{j}}^{(2 k+2+t) *} M_{R^{k+1} q}^{(2 k+2+t)}\right) f\right\|_{2 k+2+t} \leq C_{1}^{k+1}\|q f\|_{t+1}
$$

Recalling (5.4), this gives us

$$
\begin{equation*}
A \leq \frac{(n+2 k+2+t+\ell)^{\ell}}{(\ell+1+n+t)^{k+1}} C_{1}^{k+1}\|q f\|_{t+1} \tag{5.8}
\end{equation*}
$$

It follows from Proposition 5.2(2) that

$$
B \leq \frac{(n+2 k+4+t+\ell)^{\ell+2}}{(\ell+1+n+t)^{k+1}}\left\|M_{R^{k+1} q}^{(t)} f\right\|_{2 k+4+t}
$$

Applying (5.2) and Proposition 3.5, we obtain

$$
\begin{aligned}
\left\|M_{R^{k+1} q}^{(t)} f\right\|_{2 k+4+t}^{2} & =a_{n, 2 k+4+t} \int\left|\left(R^{k+1} q\right)(z) f(z)\right|^{2}\left(1-|z|^{2}\right)^{2 k+4+t} d v(z) \\
& \leq a_{n, 2 k+4+t} C_{3.5}^{K(3 k+5+t)}(K!)^{2} \int|q(z) f(z)|^{2}\left(1-|z|^{2}\right)^{2+t} d v(z)
\end{aligned}
$$

Thus there is a $C_{2}$ that depends only on $n, K$ and $t(>-2)$ such that $\left\|M_{R^{k+1} q}^{(t)} f\right\|_{2 k+4+t} \leq$ $C_{2}^{k+1}\|q f\|_{t+1}$. Consequently,

$$
B \leq \frac{(n+2 k+4+t+\ell)^{\ell+2}}{(\ell+1+n+t)^{k+1}} C_{2}^{k+1}\|q f\|_{t+1}
$$

Combining this with (5.8) and (5.3), the proof of the proposition is complete.
Proof of Theorem 1.1. Let $q \in \mathbf{C}\left[z_{1}, \ldots, z_{n}\right]$ be such that $\operatorname{deg}(q)=K, K \geq 1$. Let $t>-2$ also be given. For this pair of $K$ and $t$, let $C_{5.3}=C_{5.3}(n, K, t)$ be the constant provided by Proposition 5.3. Let $\ell \in \mathbf{N}$ satisfy the condition

$$
\begin{equation*}
\ell+1+n+t>2 C_{5.3} . \tag{5.9}
\end{equation*}
$$

With this $\ell$, we now define

$$
E=\left\{q f: f \in \mathbf{C}\left[z_{1}, \ldots, z_{n}\right],\left(\partial^{\alpha} f\right)(0)=0 \text { for }|\alpha| \leq \ell+1\right\}
$$

For the given $q$, let $Q^{(t)}$ denote the orthogonal projection from $\mathcal{H}^{(t)}$ onto $\mathcal{H}^{(t)} \ominus[q]^{(t)}$. Let $j \in\{1, \ldots, n\}$, and let $f \in \mathbf{C}\left[z_{1}, \ldots, z_{n}\right]$ be such that $\left(\partial^{\alpha} f\right)(0)=0$ for $|\alpha| \leq \ell+1$. Then

$$
Q^{(t)} M_{z_{j}}^{(t) *} q f=Q^{(t)} M_{z_{j}}^{(t) *} M_{q}^{(t)} f=Q^{(t)}\left(M_{z_{j}}^{(t) *} M_{q}^{(t)}-M_{q}^{(t)} M_{z_{j}}^{(t) *}\right) f
$$

Applying Propositions 5.1 and 5.3, we have

$$
\begin{align*}
\left\|Q^{(t)} M_{z_{j}}^{(t) *} q f\right\|_{t} & \leq \sum_{k=0}^{\infty}\left\|(N+1+n+t)^{-k-1}\left(M_{\partial_{j} R^{k} q}^{(t)}-M_{z_{j}}^{(t) *} M_{R^{k+1} q}^{(t)}\right) f\right\|_{t} \\
& \leq \sum_{k=0}^{\infty} \frac{(n+2 k+4+t+\ell)^{\ell+2}}{(\ell+1+n+t)^{k+1}} C_{5.3}^{k+1}\|q f\|_{t+1} . \tag{5.10}
\end{align*}
$$

Set

$$
C=\sum_{k=0}^{\infty} \frac{(n+2 k+4+t+\ell)^{\ell+2}}{(\ell+1+n+t)^{k+1}} C_{5.3}^{k+1}
$$

Then (5.9) ensures that $C<\infty$. Thus (5.10) can be restated as

$$
\left\|Q^{(t)} M_{z_{j}}^{(t) *} g\right\|_{t} \leq C\|g\|_{t+1} \quad \text { for every } g \in E
$$

Let $E^{(t)}$ be the closure of $E$ in $\mathcal{H}^{(t)}$, and let $\mathcal{E}^{(t)}: \mathcal{H}^{(t)} \rightarrow E^{(t)}$ be the orthogonal projection. By Proposition 4.2, the above implies that

$$
Q^{(t)} M_{z_{j}}^{(t) *} \mathcal{E}^{(t)} \in \mathcal{C}_{2 n}^{+}
$$

Obviously, $E^{(t)}$ is a subspace of $[q]^{(t)}$ of finite codimension. That is, if $P^{(t)}$ denotes the orthogonal projection from $\mathcal{H}^{(t)}$ onto $[q]^{(t)}$, then $\operatorname{rank}\left(P^{(t)}-\mathcal{E}^{(t)}\right)<\infty$. Therefore

$$
Q^{(t)} M_{z_{j}}^{(t) *} P^{(t)} \in \mathcal{C}_{2 n}^{+}
$$

Combining this with the well-known fact that $\left[M_{z_{j}}^{(t) *}, M_{z_{i}}^{(t)}\right] \in \mathcal{C}_{n}^{+}$, it follows from a routine argument that $\left[Z_{q, j}^{(t) *}, Z_{q, i}^{(t)}\right] \in \mathcal{C}_{n}^{+}, i, j \in\{1, \ldots, n\}$. This completes the proof.

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