ESSENTIAL NORMALITY OF POLYNOMIAL-GENERATED SUBMODULES: HARDY SPACE AND BEYOND

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Abstract. Recently, Douglas and Wang proved that for each polynomial q, the submodule [q] of the Bergman module generated by q is essentially normal [9]. Using improved techniques, we show that the Hardy-space analogue of this result holds, and more.

1. Introduction

Let **B** be the unit ball in \mathbb{C}^n . Throughout the paper, the complex dimension n is always assumed to be greater than or equal to 2. Recall that the Drury-Arveson space H_n^2 is the Hilbert space of analytic functions on **B** with $(1 - \langle \zeta, z \rangle)^{-1}$ as its reproducing kernel. The space H_n^2 is naturally considered as a Hilbert module over the polynomial ring $\mathbb{C}[z_1, \ldots, z_n]$. In [3-6], Arveson raised the question of whether graded submodules \mathcal{M} of H_n^2 are essentially normal. That is, for the restricted operators

$$Z_{\mathcal{M},j} = M_{z_j} | \mathcal{M}, \quad 1 \le j \le n,$$

on \mathcal{M} , do commutators $[Z^*_{\mathcal{M},j}, Z_{\mathcal{M},i}]$ belong to the Schatten class \mathcal{C}_p for p > n? This problem is commonly referred to as the Arveson conjecture.

Numerous papers have been written on this problem [4,6,7,10,13,14]. In particular, Guo and Wang showed that the answer to the above question is affirmative if \mathcal{M} is generated by a homogeneous polynomial [14]. In [8], Douglas proposed analogous essential normality problems for submodules of the Bergman module $L^2_a(\mathbf{B}, dv)$.

As it turns out, the Bergman space case is more tractable. In fact, the Bergman space version of the problem was recently solved by Douglas and Wang in [9] for *arbitrary* polynomials. In that paper, Douglas and Wang showed that for any polynomial $q \in \mathbf{C}[z_1, \ldots, z_n]$, the submodule [q] of the Bergman module generated by q is p-essentially normal for p > n. What is especially remarkable is that [9] contains many novel ideas.

The present paper grew out of a remark in [9]. Toward the end of [9], Douglas and Wang commented

"It seems likely that the argument in this paper can be generalized to obtain the same result for the Hardy and the Drury-Arveson spaces. However, while we believe that both results hold, perhaps techniques from [9,8] may be needed to complete the proofs."

While the Drury-Arveson space case is out of reach at the moment, in this paper we will settle the Hardy space case mentioned above, and we will go a little farther than that.

Keywords: Essential normality, submodule, polynomial.

The key realization is that Bergman space, Hardy space and Drury-Arveson space are all members of a family of reproducing-kernel Hilbert spaces of analytic functions on **B** parametrized by a real-valued parameter $-n \leq t < \infty$. In fact, the spaces corresponding to the values $t \in \mathbf{Z}_+$ were used in an essential way in the proofs in [9]. Our main observation is that if one considers other values of t, then one will see how to extend the techniques in [9] beyond the Bergman space case. In short, in this paper we establish the analogue of the main result in [9] for spaces with parameter $-2 < t < \infty$. Before stating the result, let us first introduce these spaces.

For each real number $-n \leq t < \infty$, let $\mathcal{H}^{(t)}$ be the Hilbert space of analytic functions on **B** with the reproducing kernel

$$\frac{1}{(1-\langle \zeta, z \rangle)^{n+1+t}}.$$

Alternately, one can describe $\mathcal{H}^{(t)}$ as the completion of $\mathbf{C}[z_1, \ldots, z_n]$ with respect to the norm $\|\cdot\|_t$ arising from the inner product $\langle \cdot, \cdot \rangle_t$ defined according to the following rules: $\langle z^{\alpha}, z^{\beta} \rangle_t = 0$ whenever $\alpha \neq \beta$,

$$\langle z^{\alpha}, z^{\alpha} \rangle_t = \frac{\alpha!}{\prod_{j=1}^{|\alpha|} (n+t+j)}$$

if $\alpha \in \mathbf{Z}_{+}^{n} \setminus \{0\}$, and $\langle 1, 1 \rangle_{t} = 1$. Here and throughout the paper, we use the conventional multi-index notation [15,page 3].

Obviously, $\mathcal{H}^{(0)}$ is the Bergman space $L^2_a(\mathbf{B}, dv)$. One can view the Bergman space $\mathcal{H}^{(0)} = L^2_a(\mathbf{B}, dv)$ as a benchmark, against which the other spaces in the family should be compared. Note that for each $-1 < t < \infty$, $\mathcal{H}^{(t)}$ is a weighted Bergman space.

Let S denote the unit sphere $\{z \in \mathbf{C}^n : |z| = 1\}$ in \mathbf{C}^n . Let σ be the positive, regular Borel measure on S that is invariant under the orthogonal group O(2n), i.e., the group of isometries on $\mathbf{C}^n \cong \mathbf{R}^{2n}$ that fix 0. We take the usual normalization $\sigma(S) = 1$. Recall that the Hardy space $H^2(S)$ is the closure of $\mathbf{C}[z_1, \ldots, z_n]$ in $L^2(S, d\sigma)$.

Obviously, $\mathcal{H}^{(-1)}$ is just the Hardy space $H^2(S)$. Moreover, $\mathcal{H}^{(-n)}$ is none other than the Drury-Arveson space H^2_n .

It is well known that for each $-n \leq t < -1$, the tuple of multiplication operators $(M_{z_1}, \ldots, M_{z_n})$ is not jointly subnormal on $\mathcal{H}^{(t)}$ [1,Theorem 3.9]. In other words, if $-n \leq t < -1$, then $\mathcal{H}^{(t)}$ is more like the Drury-Arveson space than the Hardy space. The practical consequence of this is that it is difficult to do estimates on $\mathcal{H}^{(t)}$ if $-n \leq t < -1$.

Let $q \in \mathbf{C}[z_1, \ldots, z_n]$. For each $-n \leq t < \infty$, let $[q]^{(t)}$ denote the closure of

$$\{qf: f \in \mathbf{C}[z_1, \dots, z_n]\}$$

in $\mathcal{H}^{(t)}$. Since $\mathcal{H}^{(t)}$ is a Hilbert module over $\mathbf{C}[z_1, \ldots, z_n]$, $[q]^{(t)}$ is a submodule. For each $j \in \{1, \ldots, n\}$, define submodule operator

$$Z_{q,j}^{(t)} = M_{z_j} | [q]^{(t)}.$$

Recall that the submodule $[q]^{(t)}$ is said to be *p*-essentially normal if the commutators $[Z_{q,j}^{(t)*}, Z_{q,i}^{(t)}], i, j \in \{1, \ldots, n\}$, all belong to the Schatten class C_p . With the foregoing preparation, we are now ready to state our result.

Theorem 1.1. Let q be an arbitrary polynomial in $\mathbf{C}[z_1, \ldots, z_n]$. Then for each real number $-2 < t < \infty$, the submodule $[q]^{(t)}$ of $\mathcal{H}^{(t)}$ is p-essentially normal for every p > n.

Clearly, the Hardy-space case mentioned in [9] is settled by applying Theorem 1.1 to the special case t = -1.

On the other hand, it is a real pity that the requirement t > -2 in Theorem 1.1 does not allow us to capture any Drury-Arveson space in dimensions $n \ge 2$. But as a consolation, Theorem 1.1 does cover spaces $\mathcal{H}^{(t)}$ for -2 < t < -1, which, as we mentioned, are more Drury-Arveson-like than Hardy-like.

On the technical side, this paper does offer some improvement over [9]. As the authors of [9] stated, the key step in the proof of their result rests on weighted norm estimates given in Section 3 in that paper. At the core of their weighted estimates is an argument using a covering lemma. This is where we offer the most significant improvement. In this paper, the covering-lemma argument of [9] is done away with entirely. In its place, we use a much simpler argument based on Fubini's theorem.

In fact, using Fubini's theorem-based argument in place of covering-lemma argument is a situation with which we are quite familiar. See, for example, the proofs of Proposition 2.6 and Lemma 5.2 in [11].

There are many technical contributions made in [9]. Perhaps, the most important among these is Lemma 3.2 in that paper. This lemma will again be the basis for analysis here. The reader will see that with the combination of [9,Lemma 3.2] and our Fubini's theorem-based argument, the analysis part of the proof is actually easy.

As it was the case in [9], an essential role in the proof is played by the number operator N introduced by Arveson in [2]. Recall that, for a polynomial $f(z) = \sum_{\alpha} c_{\alpha} z^{\alpha}$,

$$(Nf)(z) = \sum_{\alpha} c_{\alpha} |\alpha| z^{\alpha}$$

Here as well as in [9], the proof boils down to the estimate of an operator series where the k-th term has the operator

$$(N+1+n+t)^{-k-1}$$

as a factor, $k \ge 0$. Douglas and Wang's idea is to factor the above in the form

$$(N+1+n+t)^{-k-1} = (N+1+n+t)^{-1/2} \cdot (N+1+n+t)^{-k-(1/2)},$$

"reserve" the factor $(N + 1 + n + t)^{-1/2}$ for establishing the requisite Schatten-class membership, and use the other factor, $(N + 1 + n + t)^{-k-(1/2)}$, to boost the weight of the space. This is another place where [9] and the present paper differ. Instead of factoring, we will apply the whole of $(N + 1 + n + t)^{-k-1}$ to boost weight. Proposition 4.2 below allows us to recover an equivalent of $(N + 1 + n + t)^{-1/2}$ at the end of the estimate. This is why we are able to push t below -1.

The rest of the paper is organized as follows. Since the analysis part of the proof is now easy, we will take care of that first, in Sections 2 and 3. Section 4 contains a brief discussion of the relation between the natural embedding $\mathcal{H}^{(t)} \to \mathcal{H}^{(t+1)}$ and norm ideals. Section 5, which mirrors Section 2 in [9], contains the proof of our result.

2. Derivative on the Disc

Write D for the open unit disc $\{z \in \mathbf{C} : |z| < 1\}$ in the complex plane. Let dA be the area measure on D with the normalization A(D) = 1. The unit circle $\{\tau \in \mathbf{C} : |\tau| = 1\}$ will be denote by **T**. Furthermore, let dm be the Lebesgue measure on **T** with the normalization $m(\mathbf{T}) = 1$. For convenience, we write ∂ for the one-variable differentiation d/dz on **C**.

Our first lemma is basically a restatement of Lemma 3.2 in [9].

Lemma 2.1. Suppose that g is a one-variable polynomial of degree $K \ge 1$, and that f is analytic on D. Then for each $k \in \mathbf{N}$ we have

$$|(\partial^k g)(0)f(0)|^2 \le 2^{2k+2}(K!)^2 \int |gf|^2 dA.$$

Proof. For each $0 \leq r < 1$, let $g_r(z) = g(rz)$ and $f_r(z) = f(rz)$. We only need to consider the case $1 \leq k \leq K$. For such a k, Lemma 3.2 in [9] tells us that $|(\partial^k g_r)(0)f_r(0)| \leq K! \int_{\mathbf{T}} |g_r f_r| dm$. Since $(\partial^k g_r)(0) = r^k (\partial^k g)(0)$ and $f_r(0) = f(0)$, we have

$$\begin{aligned} |(\partial^k g)(0)f(0)| &= 2\int_{1/2}^1 r^{-k} |(\partial^k g_r)(0)f_r(0)| dr \le 2K! \int_{1/2}^1 r^{-k} \int_{\mathbf{T}} |g_r(\tau)f_r(\tau)| dm(\tau) dr \\ &\le 2^{k+1}K! \int_{1/2}^1 2r \int_{\mathbf{T}} |g(r\tau)f(r\tau)| dm(\tau) dr \le 2^{k+1}K! \int |gf| dA. \end{aligned}$$

Squaring both sides and applying the Cauchy-Schwarz inequality, the lemma follows. \Box

For each $z \in D$, define the disc $D(z) = \{w \in D : |w - z| < (1/2)(1 - |z|)\}$. Lemma 2.2. For all $w \in D$ and $x \in (-1, \infty)$, we have

$$\int \frac{(1-|z|^2)^x}{A(D(z))} \chi_{D(z)}(w) dA(z) \le 2^{2\max\{x,0\}+5} (1-|w|^2)^x.$$

Proof. Let $w \in D$, and let $z \in D$ be such that $w \in D(z)$. Then we have $1 - |w| \le 1 - |z| + |z - w| < (3/2)(1 - |z|)$. Also, $1 - |z| \le 1 - |w| + |w - z| \le 1 - |w| + (1/2)(1 - |z|)$. After cancellation, we find $(1/2)(1 - |z|) \le 1 - |w|$. Thus

(2.1)
$$(2/3)(1-|w|) \le 1-|z| \le 2(1-|w|)$$
 whenever $w \in D(z)$.

From this we obtain that for $w \in D(z)$ and $x \in (-1, \infty)$,

$$(1 - |z|^2)^x \le \begin{cases} 2^{2x}(1 - |w|^2)^x & \text{if } 0 \le x < \infty \\ \\ 3(1 - |w|^2)^x & \text{if } -1 < x < 0 \end{cases}$$

Thus, to complete the proof, it suffices to show that

(2.2)
$$\int \frac{\chi_{D(z)}(w)}{A(D(z))} dA(z) \le 9$$

for every $w \in D$. For each $w \in D$, let $G(w) = \{z \in D : w \in D(z)\}$. If $z \in G(w)$, then $|z-w| \leq (1/2)(1-|z|) \leq 1-|w|$ by (2.1). Hence $A(G(w)) \leq (1-|w|)^2$. On the other hand, if $z \in G(w)$, then $A(D(z)) = (1/4)(1-|z|)^2 \geq (1/3)^2(1-|w|)^2$, also by (2.1). Clearly, (2.2) follows from these two inequalities. \Box

Proposition 2.3. Suppose that g is a one-variable polynomial of degree $K \ge 1$, and that f is analytic on D. Then for all $k \in \mathbb{N}$ and $t \in (0, \infty)$ satisfying the condition t - 2k > -1,

$$\int |(\partial^k g)(z)f(z)|^2 (1-|z|^2)^t dA(z)$$

$$\leq 2^{6k+2\max\{t-2k,0\}+7} (K!)^2 \int |g(w)f(w)|^2 (1-|w|^2)^{t-2k} dA(w).$$

Proof. Define $g_z(u) = g(z + (1/2)(1 - |z|)u)$ and $f_z(u) = f(z + (1/2)(1 - |z|)u)$ for each $z \in D$. Then $2^{-k}(1 - |z|)^k (\partial^k g)(z) = (\partial^k g_z)(0)$ and $f(z) = f_z(0)$. By Lemma 2.1,

$$\begin{aligned} |(\partial^k g)(z)f(z)|^2 &= \frac{2^{2k} |(\partial^k g_z)(0)f_z(0)|^2}{(1-|z|)^{2k}} \le \frac{2^{4k+2} (K!)^2}{(1-|z|)^{2k}} \int |g_z(u)f_z(u)|^2 dA(u) \\ &= \frac{2^{4k+2} (K!)^2}{(1-|z|)^{2k}} \cdot \frac{1}{A(D(z))} \int_{D(z)} |g(w)f(w)|^2 dA(w). \end{aligned}$$

Therefore, if t - 2k > -1, then

$$\begin{split} \int |(\partial^k g)(z)f(z)|^2 (1-|z|^2)^t dA(z) \\ &\leq 2^{6k+2} (K!)^2 \int \frac{(1-|z|^2)^{t-2k}}{A(D(z))} \left(\int_{D(z)} |g(w)f(w)|^2 dA(w) \right) dA(z) \\ &= 2^{6k+2} (K!)^2 \int \left\{ \int \frac{(1-|z|^2)^{t-2k}}{A(D(z))} \chi_{D(z)}(w) dA(z) \right\} |g(w)f(w)|^2 dA(w) \\ &\leq 2^{6k+2\max\{t-2k,0\}+7} (K!)^2 \int (1-|w|^2)^{t-2k} |g(w)f(w)|^2 dA(w), \end{split}$$

where the last step is an application of Lemma 2.2. This completes the proof. \Box

3. Derivatives on the Ball

Recall that there is a constant $A_0 \in (2^{-n}, \infty)$ such that

(3.1)
$$2^{-n}r^n \le \sigma(\{\xi \in S : |1 - \langle u, \xi \rangle| < r\}) \le A_0 r^n$$

for all $u \in S$ and $0 < r \le 2$ [15,Proposition 5.1.4]. For each $z \in \mathbf{B}$, define the subset

$$T(z) = \{ w \in \mathbf{B} : |1 - \langle w, z \rangle| < 2(1 - |z|^2), \ 1 - |w|^2 > (1/2)(1 - |z|^2) \}$$

of the unit ball. We begin our estimates with the properties of the set T(z).

Let dv be the volume measure on **B** with the normalization $v(\mathbf{B}) = 1$.

Lemma 3.1. There is a constant $0 < C_{3,1} < \infty$ such that for all $\zeta \in \mathbf{B}$ and $x \in (-1, \infty)$,

$$\int (1-|z|^2)^{x-n-1} \chi_{T(z)}(\zeta) dv(z) \le C_{3.1} 2^{\max\{x,0\}} (1-|\zeta|^2)^x.$$

Proof. Let $\zeta, z \in \mathbf{B}$ be such that $\zeta \in T(z)$. Then we have $1 - |\zeta|^2 \leq 2(1 - |\zeta|) \leq 2|1 - \langle \zeta, z \rangle| < 4(1 - |z|^2)$. Combining this with the condition $1 - |\zeta|^2 > (1/2)(1 - |z|^2)$, we have

(3.2)
$$(1/4)(1-|\zeta|^2) \le 1-|z|^2 \le 2(1-|\zeta|^2).$$

Therefore, for $x \in (-1, \infty)$ we have

$$(1 - |z|^2)^x \le \begin{cases} 2^x (1 - |\zeta|^2)^x & \text{if } 0 \le x < \infty \\ \\ 4(1 - |\zeta|^2)^x & \text{if } -1 < x < 0 \end{cases}$$

Thus, to complete the proof, it suffices to show that there is a $0 < C < \infty$ such that

(3.3)
$$\int \frac{\chi_{T(z)}(\zeta)}{(1-|z|^2)^{n+1}} dv(z) \le C$$

for every $\zeta \in \mathbf{B}$. Given a $\zeta \in \mathbf{B}$, consider the set $\Omega(\zeta) = \{z \in \mathbf{B} : \zeta \in T(z)\}$. Write $\zeta = |\zeta|\eta$ with $\eta \in S$. If $z = |z|\xi \in \Omega(\zeta)$, where $\xi \in S$, then $|1 - \langle \eta, \xi \rangle| \le 2|1 - \langle \zeta, z \rangle| < 4(1 - |z|^2) < 8(1 - |\zeta|^2)$. Also, $1 - |z| \le |1 - \langle \zeta, z \rangle| < 4(1 - |\zeta|^2)$ if $z \in \Omega(\zeta)$. Hence

$$\Omega(\zeta) \subset \{r\xi : 0 < 1 - r < 4(1 - |\zeta|^2); \ \xi \in S, \ |1 - \langle \eta, \xi \rangle| < 8(1 - |\zeta|^2)\}.$$

By (3.1) and the decomposition $dv = 2nr^{2n-1}drd\sigma$, there is a $0 < C_1 < \infty$ such that $v(\Omega(\zeta)) \leq C_1(1-|\zeta|^2)^{n+1}$ for every $\zeta \in \mathbf{B}$. By (3.2), $(1-|z|^2)^{-n-1} \leq 4^{n+1}(1-|\zeta|^2)^{-n-1}$ when $z \in \Omega(\zeta)$. Clearly, (3.3) follows from these two inequalities. \Box

Lemma 3.2. There is a constant $0 < \epsilon < 1$ such that for each $0 \le a < 1$, the set $T((a, 0, \ldots, 0))$ contains the polydisc

(3.4)
$$P_a = \{ (a+u, \zeta_2, \dots, \zeta_n) : |u| < \epsilon(1-a^2), \ |\zeta_j| < \epsilon\sqrt{1-a^2}, \ 2 \le j \le n \}.$$

Proof. Given an *a* ∈ [0, 1), write *α* = (*a*, 0, ..., 0). Let 0 < *ε* < 1, and suppose that *u* and *ζ*₂, ..., *ζ*_n satisfy the conditions $|u| < ε(1 - a^2)$ and $|ζ_j| < ε\sqrt{1 - a^2}$, $2 \le j \le n$. Then consider the vector $w = (a + u, ζ_2, ..., ζ_n)$. We have $|1 - \langle w, \alpha \rangle| = |1 - a^2 - au| < (1 + ε)(1 - a^2)$. Moreover, $1 - |w|^2 = 1 - |a + u|^2 - (|ζ_2|^2 + \dots + |ζ_n|^2) \ge 1 - |a + u|^2 - (n - 1)ε^2(1 - a^2)$. On the other hand, $1 - |a + u|^2 = 1 - (a^2 + 2\text{Re}(au) + |u|^2) \ge 1 - a^2 - 3|u| \ge (1 - 3ε)(1 - a^2)$. Hence $1 - |w|^2 \ge (1 - (n + 2)ε)(1 - a^2)$. Thus $ε = \{3(n + 2)\}^{-1}$ suffices for our purpose. □

As usual, write $\partial_1, \ldots, \partial_n$ for the differentiations with respect to the complex variables z_1, \ldots, z_n . For each vector $b = (b_1, \ldots, b_n) \in \mathbb{C}^n$, define the directional derivative

$$\partial_b = b_1 \partial_1 + \dots + b_n \partial_n.$$

Lemma 3.3. There is a constant $0 < C_{3,3} < \infty$ such that the following estimate holds: Suppose that $q \in \mathbb{C}[z_1, \ldots, z_n]$ and that $\deg(q) = K \ge 1$. Let $f \in \mathbb{C}[z_1, \ldots, z_n]$. If z and b are vectors in $\mathbb{B}\setminus\{0\}$ satisfying the relation $\langle b, z \rangle = 0$, then

$$|(\partial_b q)(z)f(z)|^2 \le \frac{C_{3,3}(K!)^2}{(1-|z|^2)^{n+2}} \int_{T(z)} |qf|^2 dv.$$

Proof. Consider the special case where $z = \alpha = (a, 0, ..., 0)$ for some 0 < a < 1. Let ϵ be the constant provided by Lemma 3.2. Define the polydisc

$$Y = \{ (a+u, 0, \zeta_3, \dots, \zeta_n) : |u| < \epsilon(1-a^2), \ |\zeta_j| < \epsilon\sqrt{1-a^2}, \ 3 \le j \le n \}.$$

For each $y \in Y$, we define the one-varible polynomial $q_y(w) = q(y + \epsilon \sqrt{1 - a^2}we_2)$, where $e_2 = (0, 1, 0, \dots, 0)$. Similarly, define $f_y(w) = f(y + \epsilon \sqrt{1 - a^2}we_2)$ on D. Since $(\partial q_y)(0) = \epsilon \sqrt{1 - a^2}(\partial_2 q)(y)$ and $f_y(0) = f(y)$, we apply Lemma 2.1 to obtain

$$|(\partial_2 q)(y)f(y)|^2 = \frac{|(\partial q_y)(0)f_y(0)|^2}{\epsilon^2(1-a^2)} \le \frac{16(K!)^2}{\epsilon^2(1-a^2)} \int |q_y(w)f_y(w)|^2 dA(w) + \frac{16(K!)^2}{\epsilon^2(1-a^2)} \int |q_y(w)f_y(w)f_y(w)|^2 dA(w) + \frac{16(K!)^2}{\epsilon^2(1-a^2)} \int |q_y(w)f_y(w)f_y(w)f_y(w)|^2 dA(w) + \frac{16(K!)^2}{\epsilon^2(1-a^2)} \int |q_y(w)f_y(w)f_y(w)f_y(w)|^2 dA(w) + \frac{16(K!)^2}{\epsilon^2(1-a^2)} \int |q_y(w)f_y(w)f_y(w)f_y(w)|^2 dA(w) + \frac{16(K!)^2}{\epsilon^2(1-a^2)} \int |q_y(w)f_y(w)f_y(w)f_y(w)f_y(w)|^2 dA(w) + \frac{16(K!)^2}{\epsilon^2(1-a^2)} \int |q_y(w)f_y(w)f_y(w)f_y(w)f_y(w)f_y(w) + \frac{16(K!)^2}{\epsilon^2(1-a^2)}$$

Making the substitution $\zeta_2 = \epsilon \sqrt{1 - a^2} w$, we find that

$$|(\partial_2 q)(y)f(y)|^2 \le \frac{16(K!)^2}{\epsilon^4(1-a^2)^2} \int_{|\zeta_2| < \epsilon\sqrt{1-a^2}} |q(y+\zeta_2 e_2)f(y+\zeta_2 e_2)|^2 dA(\zeta_2).$$

Now, integrating both sides over Y, we see that

$$\epsilon^{2n-2}(1-a^2)^n |(\partial_2 q)(\alpha)f(\alpha)|^2 \le \int_Y |(\partial_2 q)(y)f(y)|^2 dy \le \frac{16C(K!)^2}{\epsilon^4(1-a^2)^2} \int_{P_a} |qf|^2 dv,$$

where P_a is given by (3.4) and C accounts for the normalization constants for the measures involved. Since Lemma 3.2 tells us that $P_a \subset T(\alpha)$, we have

$$|(\partial_2 q)(\alpha)f(\alpha)|^2 \le \frac{16\epsilon^{-(2n+2)}C(K!)^2}{(1-a^2)^{n+2}} \int_{T(\alpha)} |qf|^2 dv.$$

Obviously, the above inequality also holds if we replace ∂_2 by ∂_j for any $2 \leq j \leq n$. Applying these and the Cauchy-Schwarz inequality, we see that

$$|(\partial_b q)(\alpha)f(\alpha)|^2 \le (n-1)\frac{16\epsilon^{-(2n+2)}C(K!)^2}{(1-a^2)^{n+2}}\int_{T(\alpha)}|qf|^2dv \quad \text{if } \langle b,\alpha\rangle = 0, \ b\in\mathbf{B}.$$

This proves the lemma in the special case where $z = \alpha = (a, 0, ..., 0), 0 < a < 1$. The general case follows from this special case and the following easily-verified relations: If U is any unitary transformation on \mathbf{C}^n and $w, b \in \mathbf{B}$, then UT(w) = T(Uw) and $(\partial_b(q \circ U))(w) = (\partial_{Ub}q)(Uw)$. \Box

Following [9], for each pair of $i \neq j$ in $\{1, \ldots, n\}$ we define $L_{i,j} = \bar{z}_j \partial_i - \bar{z}_i \partial_j$.

Proposition 3.4. There is a constant $1 \leq C_{3,4} < \infty$ such that the following estimate holds: Suppose that $q \in \mathbf{C}[z_1, \ldots, z_n]$ and that $\deg(q) = K \geq 1$. Let $f \in \mathbf{C}[z_1, \ldots, z_n]$. Then for every positive number t > 0 and all integers $i \neq j$ in $\{1, \ldots, n\}$, we have

$$\int |(L_{i,j}q)(z)f(z)|^2 (1-|z|^2)^t dv(z) \le C_{3,4} 2^t (K!)^2 \int |q(\zeta)f(\zeta)|^2 (1-|\zeta|^2)^{t-1} dv(\zeta) dv(\zeta)$$

Proof. It follows from Lemma 3.3 that

$$|(L_{i,j}q)(z)f(z)|^2 \le \frac{C_{3,3}(K!)^2}{(1-|z|^2)^{n+2}} \int_{T(z)} |q(\zeta)f(\zeta)|^2 dv(\zeta),$$

 $z \in \mathbf{B}$. Multiplying both sides by $(1 - |z|^2)^t$ and integrating, we find that

$$\int |(L_{i,j}q)(z)f(z)|^2 (1-|z|^2)^t dv(z)$$

$$\leq C_{3.3}(K!)^2 \int \left((1-|z|^2)^{t-n-2} \int_{T(z)} |q(\zeta)f(\zeta)|^2 dv(\zeta) \right) dv(z)$$

$$= C_{3.3}(K!)^2 \int \left\{ \int (1-|z|^2)^{t-n-2} \chi_{T(z)}(\zeta) dv(z) \right\} |q(\zeta)f(\zeta)|^2 dv(\zeta)$$

Applying Lemma 3.1 with x = t - 1 to the $\{\cdots\}$ above, the proposition follows. \Box

Write $R = z_1 \partial_1 + \cdots + z_n \partial_n$, the radial derivative in *n* variables. We will denote the one-variable radial derivative by \mathcal{R} . For each polynomial *h* and each $\xi \in S$, define the "slice" function $h_{\xi}(z) = h(z\xi), z \in D$. If *q* is a polynomial in *n* variables, then for every $\xi \in S$ we have the relation $(\mathcal{R}q_{\xi})(z) = (Rq)_{\xi}(z)$.

Proposition 3.5. There is a constant $1 \leq C_{3.5} < \infty$ such that the following estimate holds: Suppose that $q \in \mathbf{C}[z_1, \ldots, z_n]$ and that $\deg(q) = K \geq 1$. Let $f \in \mathbf{C}[z_1, \ldots, z_n]$. Then for each pair of $k \in \mathbf{N}$ and $t \in (0, \infty)$ satisfying the condition t - 2k > -1,

$$\int |(R^k q)(\zeta) f(\zeta)|^2 (1 - |\zeta|^2)^t dv(\zeta) \le C_{3.5}^{K(k+t)} (K!)^2 \int |q(\zeta) f(\zeta)|^2 (1 - |\zeta|^2)^{t-2k} dv(\zeta).$$

Proof. As in [9], we need the following relation between dv, $d\sigma$ and dA: Since $dv = 2nr^{2n-1}drd\sigma$, dA = 2rdrdm, and $d\sigma$ is invariant under rotation, we have

(3.5)
$$\int g dv = n \int \left(\int g(z\xi) |z|^{2n-2} dA(z) \right) d\sigma(\xi)$$

By Lemma 3.6 in [9], for each $k \in \mathbf{N}$,

(3.6)
$$\mathcal{R}^{k} = \sum_{j=1}^{k} a_{j}^{(k)} z^{j} \partial^{j} \quad \text{with} \ |a_{j}^{(k)}| < (j+1)^{k}$$

Since the degree of q equals K, for each $\xi \in S$ we have

$$(R^{k}q)_{\xi}(z) = (\mathcal{R}^{k}q_{\xi})(z) = \sum_{j=1}^{\min\{k,K\}} a_{j}^{(k)} z^{j} (\partial^{j}q_{\xi})(z).$$

Given f, for each $\xi \in S$ we define the "rigged" slice function $f^{(\xi)}(z) = z^{n-1}f(z\xi), z \in D$. Applying first (3.6) and then Proposition 2.3, when t - 2k > -1, we have

$$\begin{split} &\int |(\mathcal{R}^{k}q_{\xi})(z)f^{(\xi)}(z)|^{2}(1-|z|^{2})^{t}dA(z) \\ &\leq K(K+1)^{2k}\sum_{j=1}^{\min\{k,K\}}\int |(\partial^{j}q_{\xi})(z)f^{(\xi)}(z)|^{2}(1-|z|^{2})^{t}dA(z) \\ &\leq K(K+1)^{2k}\sum_{j=1}^{\min\{k,K\}}2^{6j+2\max\{t-2j,0\}+7}(K!)^{2}\int |q_{\xi}(z)f^{(\xi)}(z)|^{2}(1-|z|^{2})^{t-2j}dA(z) \\ &\leq K^{2}(K+1)^{2k}2^{6k+2t+7}(K!)^{2}\int |q_{\xi}(z)f^{(\xi)}(z)|^{2}(1-|z|^{2})^{t-2k}dA(z) \\ &\leq C_{3.5}^{K(k+t)}(K!)^{2}\int |q_{\xi}(z)f^{(\xi)}(z)|^{2}(1-|z|^{2})^{t-2k}dA(z). \end{split}$$

By the relations $(\mathcal{R}^k q_{\xi})(z) = (\mathcal{R}^k q)_{\xi}(z), f^{(\xi)}(z) = z^{n-1}f(z\xi)$ and $|z| = |z\xi|$, we now have

$$\int |(R^{k}q)(z\xi)f(z\xi)|^{2}(1-|z\xi|^{2})^{t}|z|^{2n-2}dA(z)$$

$$\leq C_{3.5}^{K(k+t)}(K!)^{2}\int |q(z\xi)f(z\xi)|^{2}(1-|z\xi|^{2})^{t-2k}|z|^{2n-2}dA(z).$$

Integrating both sides with respect to the measure $d\sigma$ on S and applying (3.5), the proposition follows. \Box

Proposition 3.6. There is a constant $1 \leq C_{3.6} < \infty$ such that the following estimate holds: Suppose that $q \in \mathbf{C}[z_1, \ldots, z_n]$ and that $\deg(q) = K \geq 1$. Let $f \in \mathbf{C}[z_1, \ldots, z_n]$. Then for each $t \in (1, \infty)$ and each $j \in \{1, \ldots, n\}$, we have

(3.7)
$$\int |(\partial_j q)(\zeta) f(\zeta)|^2 (1 - |\zeta|^2)^t dv(\zeta) \le C_{3.6}^{Kt} (K!)^2 \int |q(\zeta) f(\zeta)|^2 (1 - |\zeta|^2)^{t-2} dv(\zeta).$$

Proof. There is a C such that for every analytic function h on **B** and every t > 0, we have

(3.8)
$$\int_{|\zeta|<1/2} |h(\zeta)|^2 (1-|\zeta|^2)^t dv(\zeta) \le C \left(\frac{16}{7}\right)^t \int_{1/2 \le |\zeta|<3/4} |h(\zeta)|^2 (1-|\zeta|^2)^t dv(\zeta).$$

Now apply Proposition 3.4 and the case k = 1 in Proposition 3.5: by the identity $|z|^2 \partial_j = \bar{z}_j R + \sum_{i \neq j} z_i L_{j,i}$, (3.7) obviously holds if $(\partial_j q)(\zeta)$ is replaced by $|\zeta|^2 (\partial_j q)(\zeta)$ on the right-hand side. The extra factor $|\zeta|^2$ is then removed by using (3.8). \Box

4. Embedding and Norm Ideals

For a bounded operator A, we write its s-numbers as $s_1(A), \ldots, s_k(A), \ldots$ as usual. Recall that, for each $1 \le p < \infty$, the formula

(4.1)
$$\|A\|_p^+ = \sup_{k \ge 1} \frac{s_1(A) + s_2(A) + \dots + s_k(A)}{1^{-1/p} + 2^{-1/p} + \dots + k^{-1/p}}$$

defines a symmetric norm for operators [12,Section III.14]. On any Hilbert space \mathcal{H} , the set $\mathcal{C}_p^+ = \{A \in \mathcal{B}(\mathcal{H}) : ||A||_p^+ < \infty\}$ is a norm ideal [12,Section III.2]. It is well known that if p < p', then \mathcal{C}_p^+ is contained in the Schatten class $\mathcal{C}_{p'}$.

For a non-increasing sequence of non-negative numbers $\{a_1, \ldots, a_k, \ldots\}$, if $a_1 + \cdots + a_k \leq C(1^{-1/p} + \cdots + k^{-1/p})$, then $ka_k \leq C(1^{-1/p} + \cdots + k^{-1/p})$. It follows that if p > 1 and if $T \in \mathcal{C}_p^+$, then there is a $0 < C(T) < \infty$ such that $s_k(T) \leq C(T)k^{-1/p}$ for every $k \in \mathbb{N}$. Thus if p > 1 and if B is a bounded operator such that $B^*B \in \mathcal{C}_p^+$, then $B \in \mathcal{C}_{2p}^+$.

Proposition 4.1. For each $t \ge -n$, let $I^{(t)} : \mathcal{H}^{(t)} \to \mathcal{H}^{(t+1)}$ be the natural embedding. Then $I^{(t)*}I^{(t)} \in \mathcal{C}_n^+$.

Proof. Expanding the reproducing kernel $(1 - \langle \zeta, z \rangle)^{-(n+1+t)}$, we see that the standard orthonormal basis for $\mathcal{H}^{(t)}$ is $\{e_{\alpha}^{(t)} : \alpha \in \mathbb{Z}_{+}^{n}\}$, where

(4.2)
$$e_{\alpha}^{(t)}(\zeta) = \left(\frac{1}{\alpha!} \prod_{j=1}^{|\alpha|} (n+t+j)\right)^{1/2} \zeta^{\alpha}, \quad \alpha \neq 0,$$

and $e_0^{(t)}(\zeta) = 1$. Given these orthonormal bases, it is straightforward to verify that

$$I^{(t)*}I^{(t)}e^{(t)}_{\alpha} = \frac{n+1+t}{n+1+|\alpha|+t}e^{(t)}_{\alpha}, \quad \alpha \in \mathbf{Z}^{n}_{+}.$$

This formula gives us all the s-numbers for $I^{(t)*}I^{(t)}$. By (4.1), $I^{(t)*}I^{(t)} \in \mathcal{C}_n^+$.

Proposition 4.2. Suppose that E is a linear subspace of $\mathbf{C}[z_1, \ldots, z_n]$ and that $t \geq -n$. Let $E^{(t)}$ be the closure of E in $\mathcal{H}^{(t)}$, and let $\mathcal{E}^{(t)}$ be the orthogonal projection from $\mathcal{H}^{(t)}$ to $E^{(t)}$. Suppose that $A \in \mathcal{B}(\mathcal{H}^{(t)})$, and suppose that there is a C such that

(4.3)
$$||Ag||_t \le C||g||_{t+1}$$

for every $g \in E$. Then $A\mathcal{E}^{(t)} \in \mathcal{C}_{2n}^+$.

Proof. By (4.3), for each $g \in E$ we have

 $\langle A^* Ag, g \rangle_t = \|Ag\|_t^2 \le C \|g\|_{t+1}^2 = C^2 \|I^{(t)}g\|_{t+1}^2 = C^2 \langle I^{(t)}g, I^{(t)}g \rangle_{t+1} = C^2 \langle I^{(t)*}I^{(t)}g, g \rangle_t.$

That is, the operator inequality $(A\mathcal{E}^{(t)})^*A\mathcal{E}^{(t)} \leq C^2\mathcal{E}^{(t)}I^{(t)*}I^{(t)}\mathcal{E}^{(t)}$ holds on $\mathcal{H}^{(t)}$. Thus $s_j((A\mathcal{E}^{(t)})^*A\mathcal{E}^{(t)}) \leq s_j(C^2\mathcal{E}^{(t)}I^{(t)*}I^{(t)}\mathcal{E}^{(t)})$ for each $j \in \mathbb{N}$ [12,Lemma II.1.1]. By Proposition 4.1, $(A\mathcal{E}^{(t)})^*A\mathcal{E}^{(t)} \in \mathcal{C}_n^+$. Since $n \geq 2$, this implies $A\mathcal{E}^{(t)} \in \mathcal{C}_{2n}^+$. \Box

5. Proof of Theorem 1.1

For each $t \geq -n$ and each polynomial q, we write $M_q^{(t)}$ for the operator of multiplication by q on the space $\mathcal{H}^{(t)}$. Keep in mind that the notation "*" is t-specific: $M_q^{(t)*}$ means the adjoint of $M_q^{(t)}$ with respect to the inner product $\langle \cdot, \cdot \rangle_t$.

Proposition 5.1. Let $q \in \mathbb{C}[z_1, \ldots, z_n]$, $1 \leq j \leq n$ and $t \geq -n$. For $f \in \mathbb{C}[z_1, \ldots, z_n]$ satisfying the condition f(0) = 0, we have

$$M_{z_j}^{(t)*}M_q^{(t)}f - M_q^{(t)}M_{z_j}^{(t)*}f = \sum_{k=0}^{\infty} (N+1+n+t)^{-k-1} (M_{\partial_j R^k q}^{(t)} - M_{z_j}^{(t)*}M_{R^{k+1}q}^{(t)})f.$$

Proof. The main idea is that both sides are linear with respect to both q and f. Therefore the proof is a matter of straightforward verification in the special case of $q = z^{\alpha}$ and $f = z^{\beta}, \beta \neq 0$, using (4.2). The details of the verification are similar to the Bergman space case (see the proof of Proposition 2.1 in [9]). \Box

Proposition 5.2. Let $t \ge -n$ and $\ell \in \mathbf{N}$. (1) For each $f \in \mathbf{C}[z_1, \ldots, z_n]$ satisfying the condition $(\partial^{\alpha} f)(0) = 0$ for $|\alpha| < \ell$ and each non-negative integer k, we have

$$\|(N+1+n+t)^{-k-1}f\|_t^2 \le \frac{(n+2k+2+t+\ell)^\ell}{(\ell+1+n+t)^{2k+2}} \|f\|_{2k+2+t}^2$$

(2) For each $f \in \mathbf{C}[z_1, \ldots, z_n]$ satisfying the condition $(\partial^{\alpha} f)(0) = 0$ for $|\alpha| < \ell + 1$, each non-negative integer k and each $1 \leq j \leq n$, we have

$$\|(N+1+n+t)^{-k-1}(M_{z_j}^{(t)*}-M_{z_j}^{(t+2k+2)*})f\|_t^2 \le (2k+4)^4 \frac{(n+2k+4+t+\ell)^{2\ell}}{(\ell+1+n+t)^{2k+2}} \|f\|_{2k+4+t+\ell}^2$$

Proof. For (1), it suffices to consider the case where f is a homogeneous polynomial of degree $m \ge \ell$, as it was the case for the corresponding part in [9]. For such an f,

$$\begin{split} \|(N+1+n+t)^{-k-1}f\|_{t}^{2} &= \frac{\|f\|_{t}^{2}}{(m+1+n+t)^{2k+2}} \\ &= \frac{\|f\|_{2k+2+t}^{2}}{(m+1+n+t)^{2k+2}} \prod_{j=1}^{m} \frac{n+2k+2+t+j}{n+t+j} \quad (\text{see } (4.2)) \\ &= \frac{\|f\|_{2k+2+t}^{2}}{(m+1+n+t)^{2k+2}} \prod_{j=1}^{2k+2} \frac{n+m+t+j}{n+t+j}, \end{split}$$

where the last = is obtained by considering $\prod_{j=1}^{2k+2+m}(n+t+j)$. Since $m \ge \ell$, for each $j \ge 1$ we have $(n+m+t+j)/(m+1+n+t) \le (n+\ell+t+j)/(\ell+1+n+t)$. Hence

$$\begin{split} \|(N+1+n+t)^{-k-1}f\|_t^2 &\leq \|f\|_{2k+2+t}^2 \prod_{j=1}^{2k+2} \frac{n+\ell+t+j}{(\ell+1+n+t)(n+t+j)} \\ &= \frac{\|f\|_{2k+2+t}^2}{(\ell+1+n+t)^{2k+2}} \prod_{j=1}^\ell \frac{n+2k+2+t+j}{n+t+j} \leq \frac{(n+2k+2+t+\ell)^\ell}{(\ell+1+n+t)^{2k+2}} \|f\|_{2k+2+t}^2. \end{split}$$

This proves (1).

Let e_j be the element in \mathbb{Z}^n_+ whose *j*-th component is 1 and whose other components are 0. To prove (2), first note that (4.2) gives us

$$M_{z_j}^{(t)*} z^{\alpha} = \frac{\alpha_j}{n+t+|\alpha|} z^{\alpha-e_j}$$

whenever the *j*-th component α_j of α is greater than 0. Hence

(5.1)
$$(M_{z_j}^{(t)*} - M_{z_j}^{(2k+2+t)*})z^{\alpha} = \frac{\alpha_j(2k+2)}{(n+t+|\alpha|)(n+2k+2+t+|\alpha|)}z^{\alpha-e_j} \\ = \frac{2k+2}{n+t+|\alpha|}M_{z_j}^{(2k+2+t)*}z^{\alpha}.$$

For $f \in \mathbf{C}[z_1, \ldots, z_n]$ with f(0) = 0, $(N + n + t)^{-1}f$ is well defined. Thus we can define

$$f_{t,k} = (N+1+n+2k+2+t)(N+n+t)^{-1}f.$$

We have $||f_{t,k}||_{\tau} \leq (2k+4)||f||_{\tau}$ for every $\tau \geq -n$. Obviously, (5.1) implies

$$(M_{z_j}^{(t)*} - M_{z_j}^{(2k+2+t)*})f = (2k+2)M_{z_j}^{(2k+2+t)*}(N+1+n+2k+2+t)^{-1}f_{t,k}$$

Now suppose that $(\partial^{\alpha} f)(0) = 0$ for $|\alpha| < \ell + 1$. Applying (1) twice, we have

$$\begin{split} \|(N+1+n+t)^{-k-1}(M_{z_{j}}^{(t)*}-M_{z_{j}}^{(2k+2+t)*})f\|_{t}^{2} \\ &= (2k+2)^{2}\|(N+1+n+t)^{-k-1}M_{z_{j}}^{(2k+2+t)*}(N+1+n+2k+2+t)^{-1}f_{t,k}\|_{t}^{2} \\ &\leq (2k+2)^{2}\frac{(n+2k+2+t+\ell)^{\ell}}{(\ell+1+n+t)^{2k+2}}\|M_{z_{j}}^{(2k+2+t)*}(N+1+n+2k+2+t)^{-1}f_{t,k}\|_{2k+2+t}^{2} \\ &\leq (2k+2)^{2}\frac{(n+2k+2+t+\ell)^{\ell}}{(\ell+1+n+t)^{2k+2}}\|(N+1+n+2k+2+t)^{-1}f_{t,k}\|_{2k+2+t}^{2} \\ &\leq (2k+2)^{2}\frac{(n+2k+2+t+\ell)^{\ell}}{(\ell+1+n+t)^{2k+2}} \cdot \frac{(n+2k+4+t+\ell)^{\ell}}{(\ell+1+n+2k+2+t)^{2}}\|f_{t,k}\|_{2k+4+t}^{2} \\ &\leq (2k+2)^{2}\frac{(n+2k+4+t+\ell)^{2\ell}}{(\ell+1+n+t)^{2k+2}}(2k+4)^{2}\|f\|_{2k+4+t}^{2}. \end{split}$$

This completes the proof of (2). \Box

For each real number t > -1, define

$$a_{n,t} = \frac{1}{n!} \prod_{j=1}^{n} (t+j).$$

Using (4.2) and [15, Proposition 1.4.9], it is straightforward to verify that

(5.2)
$$\langle f,g\rangle_t = a_{n,t} \int f(\zeta)\overline{g(\zeta)}(1-|\zeta|^2)^t dv(\zeta)$$

for $f, g \in \mathcal{H}^{(t)}$, t > -1. In other words, if t > -1, then $\mathcal{H}^{(t)}$ is the weighted Bergman space $L^2_a(\mathbf{B}, a_{n,t}(1 - |\zeta|^2)^t dv(\zeta))$.

Our next step requires the assumption that t > -2.

Proposition 5.3. Let real number t > -2 and integer $K \ge 1$ be given. Then there is a constant $C_{5,3} = C_{5,3}(n, K, t)$ such that the following estimate holds: Let $q \in \mathbf{C}[z_1, \ldots, z_n]$ be such that $\deg(q) = K$. Suppose that $f \in \mathbf{C}[z_1, \ldots, z_n]$ satisfies the condition $(\partial^{\alpha} f)(0) = 0$ for $|\alpha| \le \ell + 1$, where $\ell \in \mathbf{N}$. Then for every integer $k \ge 0$ and every $j \in \{1, \ldots, n\}$,

$$\begin{aligned} \| (N+1+n+t)^{-k-1} (M_{\partial_j R^k q}^{(t)} - M_{z_j}^{(t)*} M_{R^{k+1} q}^{(t)}) f \|_t \\ & \leq \frac{(n+2k+4+t+\ell)^{\ell+2}}{(\ell+1+n+t)^{k+1}} C_{5.3}^{k+1} \| q f \|_{t+1}. \end{aligned}$$

Proof. Since

$$M_{\partial_j R^k q}^{(t)} - M_{z_j}^{(t)*} M_{R^{k+1}q}^{(t)} = (M_{\partial_j R^k q}^{(t)} - M_{z_j}^{(2k+2+t)*} M_{R^{k+1}q}^{(t)}) - (M_{z_j}^{(t)*} - M_{z_j}^{(2k+2+t)*}) M_{R^{k+1}q}^{(t)},$$

we have

(5.3)
$$\| (N+1+n+t)^{-k-1} (M_{\partial_j R^k q}^{(t)} - M_{z_j}^{(t)*} M_{R^{k+1} q}^{(t)}) f \|_t \le A + B,$$

where

$$A = \| (N+1+n+t)^{-k-1} (M_{\partial_j R^k q}^{(t)} - M_{z_j}^{(2k+2+t)*} M_{R^{k+1} q}^{(t)}) f \|_t \quad \text{and}$$

$$B = \| (N+1+n+t)^{-k-1} (M_{z_j}^{(t)*} - M_{z_j}^{(2k+2+t)*}) M_{R^{k+1} q}^{(t)} f \|_t.$$

We estimate A and B separately. For A, we apply Proposition 5.2(1), which gives us

$$A \leq \frac{(n+2k+2+t+\ell)^{\ell}}{(\ell+1+n+t)^{k+1}} \| (M_{\partial_{j}R^{k}q}^{(t)} - M_{z_{j}}^{(2k+2+t)*} M_{R^{k+1}q}^{(t)}) f \|_{2k+2+t} = \frac{(n+2k+2+t+\ell)^{\ell}}{(\ell+1+n+t)^{k+1}} \| (M_{\partial_{j}R^{k}q}^{(2k+2+t)} - M_{z_{j}}^{(2k+2+t)*} M_{R^{k+1}q}^{(2k+2+t)}) f \|_{2k+2+t}.$$

Since t > -2, we have 2k + 2 + t > 0 for each $k \ge 0$. Hence $\mathcal{H}^{(2k+2+t)}$ is a weighted Bergman space. By (5.2), we have

$$\| (M_{\partial_j R^k q}^{(2k+2+t)} - M_{z_j}^{(2k+2+t)*} M_{R^{k+1}q}^{(2k+2+t)}) f \|_{2k+2+t}$$

$$\leq a_{n,2k+2+t}^{1/2} \left(\int |\{ (\partial_j R^k q)(z) - \bar{z}_j (R^{k+1}q)(z) \} f(z)|^2 (1 - |z|^2)^{2k+2+t} dv(z) \right)^{1/2}.$$

The identity $\partial_j - \bar{z}_j R = (1 - |z|^2)\partial_j + \sum_{i \neq j} z_i L_{j,i}$ then leads to

$$\|(M_{\partial_{j}R^{k}q}^{(2k+2+t)} - M_{z_{j}}^{(2k+2+t)*}M_{R^{k+1}q}^{(2k+2+t)})f\|_{2k+2+t} \leq a_{n,2k+2+t}^{1/2} \left(\int |(\partial_{j}R^{k}q)(z)f(z)|^{2}(1-|z|^{2})^{2k+4+t}dv(z)\right)^{1/2} + a_{n,2k+2+t}^{1/2}\sum_{i\neq j} \left(\int |(L_{j,i}R^{k}q)(z)f(z)|^{2}(1-|z|^{2})^{2k+2+t}dv(z)\right)^{1/2}.$$
(5.5)

Applying Propositions 3.6 and 3.5, we have

(5.6)
$$\int |(\partial_j R^k q)(z) f(z)|^2 (1 - |z|^2)^{2k+4+t} dv(z) \\ \leq C_{3.6}^{K(2k+4+t)} (K!)^2 \int |(R^k q)(z) f(z)|^2 (1 - |z|^2)^{2k+2+t} dv(z) \\ \leq (C_{3.6} C_{3.5})^{K(3k+4+t)} (K!)^4 \int |q(z) f(z)|^2 (1 - |z|^2)^{2+t} dv(z).$$

Since 1 + t > -1, we can apply Propositions 3.4 and 3.5 to obtain

(5.7)
$$\int |(L_{j,i}R^{k}q)(z)f(z)|^{2}(1-|z|^{2})^{2k+2+t}dv(z)$$
$$\leq C_{3.4}2^{2k+2+t}(K!)^{2}\int |(R^{k}q)(z)f(z)|^{2}(1-|z|^{2})^{2k+1+t}dv(z)$$
$$\leq C_{3.4}(2C_{3.5})^{K(3k+2+t)}(K!)^{4}\int |q(z)f(z)|^{2}(1-|z|^{2})^{1+t}dv(z).$$

By the assumption t > -2, we have $a_{n,1+t} \ge (n!)^{-1}(2+t)^n$. Also note that $a_{n,2k+2+t} \le (n!)^{-1}(n+2k+2+t)^n$. Combining (5.5), (5.6), (5.7) and (5.2), we see that there is a C_1 that depends only on n, K and t (> -2) such that

$$\|(M_{\partial_j R^k q}^{(2k+2+t)} - M_{z_j}^{(2k+2+t)*} M_{R^{k+1}q}^{(2k+2+t)})f\|_{2k+2+t} \le C_1^{k+1} \|qf\|_{t+1}$$

Recalling (5.4), this gives us

(5.8)
$$A \leq \frac{(n+2k+2+t+\ell)^{\ell}}{(\ell+1+n+t)^{k+1}} C_1^{k+1} \|qf\|_{t+1}.$$

It follows from Proposition 5.2(2) that

$$B \le \frac{(n+2k+4+t+\ell)^{\ell+2}}{(\ell+1+n+t)^{k+1}} \|M_{R^{k+1}q}^{(t)}f\|_{2k+4+t}.$$

Applying (5.2) and Proposition 3.5, we obtain

$$\begin{split} \|M_{R^{k+1}q}^{(t)}f\|_{2k+4+t}^2 &= a_{n,2k+4+t} \int |(R^{k+1}q)(z)f(z)|^2 (1-|z|^2)^{2k+4+t} dv(z) \\ &\leq a_{n,2k+4+t} C_{3.5}^{K(3k+5+t)} (K!)^2 \int |q(z)f(z)|^2 (1-|z|^2)^{2+t} dv(z) \end{split}$$

Thus there is a C_2 that depends only on n, K and t (> -2) such that $\|M_{R^{k+1}q}^{(t)}f\|_{2k+4+t} \leq C_2^{k+1} \|qf\|_{t+1}$. Consequently,

$$B \le \frac{(n+2k+4+t+\ell)^{\ell+2}}{(\ell+1+n+t)^{k+1}} C_2^{k+1} \|qf\|_{t+1}.$$

Combining this with (5.8) and (5.3), the proof of the proposition is complete. \Box

Proof of Theorem 1.1. Let $q \in \mathbf{C}[z_1, \ldots, z_n]$ be such that $\deg(q) = K, K \geq 1$. Let t > -2 also be given. For this pair of K and t, let $C_{5,3} = C_{5,3}(n, K, t)$ be the constant provided by Proposition 5.3. Let $\ell \in \mathbf{N}$ satisfy the condition

(5.9)
$$\ell + 1 + n + t > 2C_{5.3}.$$

With this ℓ , we now define

$$E = \{qf : f \in \mathbf{C}[z_1, \dots, z_n], \ (\partial^{\alpha} f)(0) = 0 \text{ for } |\alpha| \le \ell + 1\}.$$

For the given q, let $Q^{(t)}$ denote the orthogonal projection from $\mathcal{H}^{(t)}$ onto $\mathcal{H}^{(t)} \ominus [q]^{(t)}$. Let $j \in \{1, \ldots, n\}$, and let $f \in \mathbb{C}[z_1, \ldots, z_n]$ be such that $(\partial^{\alpha} f)(0) = 0$ for $|\alpha| \leq \ell + 1$. Then

$$Q^{(t)}M_{z_j}^{(t)*}qf = Q^{(t)}M_{z_j}^{(t)*}M_q^{(t)}f = Q^{(t)}(M_{z_j}^{(t)*}M_q^{(t)} - M_q^{(t)}M_{z_j}^{(t)*})f.$$

Applying Propositions 5.1 and 5.3, we have

(5.10)
$$\begin{aligned} \|Q^{(t)}M_{z_{j}}^{(t)*}qf\|_{t} &\leq \sum_{k=0}^{\infty} \|(N+1+n+t)^{-k-1}(M_{\partial_{j}R^{k}q}^{(t)} - M_{z_{j}}^{(t)*}M_{R^{k+1}q}^{(t)})f\|_{t} \\ &\leq \sum_{k=0}^{\infty} \frac{(n+2k+4+t+\ell)^{\ell+2}}{(\ell+1+n+t)^{k+1}} C_{5.3}^{k+1} \|qf\|_{t+1}. \end{aligned}$$

 Set

$$C = \sum_{k=0}^{\infty} \frac{(n+2k+4+t+\ell)^{\ell+2}}{(\ell+1+n+t)^{k+1}} C_{5.3}^{k+1}.$$

Then (5.9) ensures that $C < \infty$. Thus (5.10) can be restated as

$$||Q^{(t)}M_{z_i}^{(t)*}g||_t \le C||g||_{t+1}$$
 for every $g \in E$.

Let $E^{(t)}$ be the closure of E in $\mathcal{H}^{(t)}$, and let $\mathcal{E}^{(t)} : \mathcal{H}^{(t)} \to E^{(t)}$ be the orthogonal projection. By Proposition 4.2, the above implies that

$$Q^{(t)}M_{z_i}^{(t)*}\mathcal{E}^{(t)} \in \mathcal{C}_{2n}^+.$$

Obviously, $E^{(t)}$ is a subspace of $[q]^{(t)}$ of finite codimension. That is, if $P^{(t)}$ denotes the orthogonal projection from $\mathcal{H}^{(t)}$ onto $[q]^{(t)}$, then $\operatorname{rank}(P^{(t)} - \mathcal{E}^{(t)}) < \infty$. Therefore

$$Q^{(t)}M_{z_i}^{(t)*}P^{(t)} \in \mathcal{C}_{2n}^+$$

Combining this with the well-known fact that $[M_{z_j}^{(t)*}, M_{z_i}^{(t)}] \in \mathcal{C}_n^+$, it follows from a routine argument that $[Z_{q,j}^{(t)*}, Z_{q,i}^{(t)}] \in \mathcal{C}_n^+$, $i, j \in \{1, \ldots, n\}$. This completes the proof. \Box

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