ESSENTIAL NORMALITY OF POLYNOMIAL-GENERATED
SUBMODULES: HARDY SPACE AND BEYOND

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Abstract. Recently, Douglas and Wang proved that for each polynomial \( q \), the submodule \([q]\) of the Bergman module generated by \( q \) is essentially normal [9]. Using improved techniques, we show that the Hardy-space analogue of this result holds, and more.

1. Introduction

Let \( B \) be the unit ball in \( C^n \). Throughout the paper, the complex dimension \( n \) is always assumed to be greater than or equal to 2. Recall that the Drury-Arveson space \( H_n^2 \) is the Hilbert space of analytic functions on \( B \) with \((1 - \langle \zeta, z \rangle)^{-1}\) as its reproducing kernel. The space \( H_n^2 \) is naturally considered as a Hilbert module over the polynomial ring \( C[z_1, \ldots, z_n] \). In [3-6], Arveson raised the question of whether graded submodules \( M \) of \( H_n^2 \) are essentially normal. That is, for the restricted operators

\[
Z_{M,j} = M z_j, 1 \leq j \leq n,
\]

on \( M \), do commutators \([Z_{M,j}^*, Z_{M,i}]\) belong to the Schatten class \( C_p \) for \( p > n \)? This problem is commonly referred to as the Arveson conjecture.

Numerous papers have been written on this problem [4,6,7,10,13,14]. In particular, Guo and Wang showed that the answer to the above question is affirmative if \( M \) is generated by a homogeneous polynomial [14]. In [8], Douglas proposed analogous essential normality problems for submodules of the Bergman module \( L_a^2(B, dv) \).

As it turns out, the Bergman space case is more tractable. In fact, the Bergman space version of the problem was recently solved by Douglas and Wang in [9] for arbitrary polynomials. In that paper, Douglas and Wang showed that for any polynomial \( q \in C[z_1, \ldots, z_n] \), the submodule \([q]\) of the Bergman module generated by \( q \) is \( p \)-essentially normal for \( p > n \). What is especially remarkable is that [9] contains many novel ideas.

The present paper grew out of a remark in [9]. Toward the end of [9], Douglas and Wang commented

"It seems likely that the argument in this paper can be generalized to obtain the same result for the Hardy and the Drury-Arveson spaces. However, while we believe that both results hold, perhaps techniques from [9,8] may be needed to complete the proofs."

While the Drury-Arveson space case is out of reach at the moment, in this paper we will settle the Hardy space case mentioned above, and we will go a little farther than that.

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The key realization is that Bergman space, Hardy space and Drury-Arveson space are all members of a family of reproducing-kernel Hilbert spaces of analytic functions on $B$ parametrized by a real-valued parameter $-n \leq t < \infty$. In fact, the spaces corresponding to the values $t \in \mathbb{Z}_+$ were used in an essential way in the proofs in [9]. Our main observation is that if one considers other values of $t$, then one will see how to extend the techniques in [9] beyond the Bergman space case. In short, in this paper we establish the analogue of the main result in [9] for spaces with parameter $-2 < t < \infty$. Before stating the result, let us first introduce these spaces.

For each real number $-n \leq t < \infty$, let $H(t)$ be the Hilbert space of analytic functions on $B$ with the reproducing kernel

$$\frac{1}{(1 - \langle \zeta, z \rangle)^{n+1+t}}.$$ 

Alternately, one can describe $H(t)$ as the completion of $\mathbb{C}[z_1, \ldots, z_n]$ with respect to the norm $\| \cdot \|_t$ arising from the inner product $\langle \cdot, \cdot \rangle_t$ defined according to the following rules:

$$\langle z^\alpha, z^\beta \rangle_t = \frac{\alpha!}{\prod_{j=1}^{\|\alpha\|} (n + t + j)}$$

if $\alpha \in \mathbb{Z}^n_+ \setminus \{0\}$, and $\langle 1, 1 \rangle_t = 1$. Here and throughout the paper, we use the conventional multi-index notation [15, page 3].

Obviously, $H(0)$ is the Bergman space $L^2_a(B, dv)$. One can view the Bergman space $H(0) = L^2_a(B, dv)$ as a benchmark, against which the other spaces in the family should be compared. Note that for each $-1 < t < \infty$, $H(t)$ is a weighted Bergman space.

Let $S$ denote the unit sphere $\{ z \in \mathbb{C}^n : |z| = 1 \}$ in $\mathbb{C}^n$. Let $\sigma$ be the positive, regular Borel measure on $S$ that is invariant under the orthogonal group $O(2n)$, i.e., the group of isometries on $\mathbb{C}^n \cong \mathbb{R}^{2n}$ that fix 0. We take the usual normalization $\sigma(S) = 1$. Recall that the Hardy space $H^2(S)$ is the closure of $\mathbb{C}[z_1, \ldots, z_n]$ in $L^2(S, d\sigma)$.

Obviously, $H(-1)$ is just the Hardy space $H^2(S)$. Moreover, $H(-n)$ is none other than the Drury-Arveson space $H^2_n$.

It is well known that for each $-n \leq t < -1$, the tuple of multiplication operators $(M_{z_1}, \ldots, M_{z_n})$ is not jointly subnormal on $H(t)$ [1, Theorem 3.9]. In other words, if $-n \leq t < -1$, then $H(t)$ is more like the Drury-Arveson space than the Hardy space. The practical consequence of this is that it is difficult to do estimates on $H(t)$ if $-n \leq t < -1$.

Let $q \in \mathbb{C}[z_1, \ldots, z_n]$. For each $-n \leq t < \infty$, let $[q](t)$ denote the closure of

$$\{ qf : f \in \mathbb{C}[z_1, \ldots, z_n] \}$$

in $H(t)$. Since $H(t)$ is a Hilbert module over $\mathbb{C}[z_1, \ldots, z_n]$, $[q](t)$ is a submodule. For each $j \in \{1, \ldots, n\}$, define submodule operator

$$Z_{q,j}^{(t)} = M_{z_j} [q](t).$$
Recall that the submodule \([q]^{(t)}\) is said to be \(p\)-essentially normal if the commutators 
\([Z_{q,i}^{(t)} , Z_{q,j}^{(t)}], i,j \in \{1,\ldots,n\}\), all belong to the Schatten class \(C_p\). With the foregoing preparation, we are now ready to state our result.

**Theorem 1.1.** Let \(q\) be an arbitrary polynomial in \(C[z_1,\ldots,z_n]\). Then for each real number \(-2 < t < \infty\), the submodule \([q]^{(t)}\) of \(\mathcal{H}^{(t)}\) is \(p\)-essentially normal for every \(p > n\).

Clearly, the Hardy-space case mentioned in [9] is settled by applying Theorem 1.1 to the special case \(t = -1\).

On the other hand, it is a real pity that the requirement \(t > -2\) in Theorem 1.1 does not allow us to capture any Drury-Arveson space in dimensions \(n \geq 2\). But as a consolation, Theorem 1.1 does cover spaces \(\mathcal{H}^{(t)}\) for \(-2 < t < -1\), which, as we mentioned, are more Drury-Arveson-like than Hardy-like.

On the technical side, this paper does offer some improvement over [9]. As the authors of [9] stated, the key step in the proof of their result rests on weighted norm estimates given in Section 3 in that paper. At the core of their weighted estimates is an argument using a covering lemma. This is where we offer the most significant improvement. In this paper, the covering-lemma argument of [9] is done away with entirely. In its place, we use a much simpler argument based on Fubini’s theorem.

In fact, using Fubini’s theorem-based argument in place of covering-lemma argument is a situation with which we are quite familiar. See, for example, the proofs of Proposition 2.6 and Lemma 5.2 in [11].

There are many technical contributions made in [9]. Perhaps, the most important among these is Lemma 3.2 in that paper. This lemma will again be the basis for analysis here. The reader will see that with the combination of [9,Lemma 3.2] and our Fubini’s theorem-based argument, the analysis part of the proof is actually easy.

As it was the case in [9], an essential role in the proof is played by the number operator \(N\) introduced by Arveson in [2]. Recall that, for a polynomial \(f(z) = \sum \alpha c_\alpha z^\alpha\),

\[(Nf)(z) = \sum_\alpha c_\alpha |\alpha| z^\alpha.\]

Here as well as in [9], the proof boils down to the estimate of an operator series where the \(k\)-th term has the operator 
\[(N + 1 + n + t)^{-k-1}\]
as a factor, \(k \geq 0\). Douglas and Wang’s idea is to factor the above in the form

\[(N + 1 + n + t)^{-k} = (N + 1 + n + t)^{-1/2} \cdot (N + 1 + n + t)^{-k-(1/2)},\]

“reserve” the factor \((N + 1 + n + t)^{-1/2}\) for establishing the requisite Schatten-class membership, and use the other factor, \((N + 1 + n + t)^{-k-(1/2)}\), to boost the weight of the space. This is another place where [9] and the present paper differ. Instead of factoring, we will apply the whole of \((N + 1 + n + t)^{-k-1}\) to boost weight. Proposition 4.2 below allows us
to recover an equivalent of $(N + 1 + n + t)^{-1/2}$ at the end of the estimate. This is why we are able to push $t$ below $-1$.

The rest of the paper is organized as follows. Since the analysis part of the proof is now easy, we will take care of that first, in Sections 2 and 3. Section 4 contains a brief discussion of the relation between the natural embedding $\mathcal{H}^{(t)} \rightarrow \mathcal{H}^{(t+1)}$ and norm ideals. Section 5, which mirrors Section 2 in [9], contains the proof of our result.

### 2. Derivative on the Disc

Write $D$ for the open unit disc $\{z \in \mathbb{C} : |z| < 1\}$ in the complex plane. Let $dA$ be the area measure on $D$ with the normalization $A(D) = 1$. The unit circle $\{\tau \in \mathbb{C} : |\tau| = 1\}$ will be denote by $T$. Furthermore, let $dm$ be the Lebesgue measure on $T$ with the normalization $m(T) = 1$. For convenience, we write $\partial$ for the one-variable differentiation $d/dz$ on $\mathbb{C}$.

Our first lemma is basically a restatement of Lemma 3.2 in [9].

**Lemma 2.1.** Suppose that $g$ is a one-variable polynomial of degree $K \geq 1$, and that $f$ is analytic on $D$. Then for each $k \in \mathbb{N}$ we have

$$|(\partial^k g)(0)f(0)|^2 \leq 2^{2k+2}(K!)^2 \int |gf|^2 dA.$$ 

**Proof.** For each $0 \leq r < 1$, let $f_r(z) = g(rz)$ and $f_r(0) = f(r)$. We only need to consider the case $1 \leq k \leq K$. For such a $k$, Lemma 3.2 in [9] tells us that $|(\partial^k g_r)(0)f_r(0)| \leq K! \int_T |g_r f_r|dm$. Since $(\partial^k g_r)(0) = r^k(\partial^k g)(0)$ and $f_r(0) = f(0)$, we have

$$|(\partial^k g)(0)f(0)| = 2 \int_{1/2}^1 r^{-k} |(\partial^k g_r)(0)f_r(0)| dr \leq 2K! \int_{1/2}^1 r^{-k} \int_T |g_r(\tau)f_r(\tau)| dm(\tau)dr$$ 

$$\leq 2^{k+1}K! \int_{1/2}^1 2r \int_T |g(\tau)f(\tau)| dm(\tau)dr = 2^{k+1}K! \int_T |gf|^2 dA.$$ 

Squaring both sides and applying the Cauchy-Schwarz inequality, the lemma follows. ☐

For each $z \in D$, define the disc $D(z) = \{w \in D : |w - z| < (1/2)(1 - |z|)\}$.

**Lemma 2.2.** For all $w \in D$ and $x \in (-1, \infty)$, we have

$$\int \frac{(1 - |z|^2)^x}{A(D(z))} \chi_{D(z)}(w)dA(z) \leq 2^{2\max\{x,0\}+5}(1 - |w|^2)^x.$$ 

**Proof.** Let $w \in D$, and let $z \in D$ be such that $w \in D(z)$. Then we have $1 - |w| \leq 1 - |z| + |z - w| < (3/2)(1 - |z|)$. Also, $1 - |z| \leq 1 - |w| + |w - z| \leq 1 - |w| + (1/2)(1 - |z|)$. After cancellation, we find $(1/2)(1 - |z|) \leq 1 - |w|$. Thus

$$(2.1) \quad (2/3)(1 - |w|) \leq 1 - |z| \leq 2(1 - |w|) \quad \text{whenever} \quad w \in D(z).$$
Thus, to complete the proof, it suffices to show that 

\[(1 - |z|^2)^x \leq \begin{cases} 
2^{2x}(1 - |w|^2)^x & \text{if } 0 \leq x < \infty \\
3(1 - |w|^2)^x & \text{if } -1 < x < 0
\end{cases}.
\]

Thus, to complete the proof, it suffices to show that

\[(2.2) \quad \int \frac{\chi_{D(z)}(w)}{A(D(z))} dA(z) \leq 9
\]

for every \(w \in D\). For each \(w \in D\), let \(G(w) = \{z \in D : w \in D(z)\}\). If \(z \in G(w)\), then \(|z - w| \leq (1/2)(1 - |z|) \leq 1 - |w|\) by (2.1). Hence \(A(G(w)) \leq (1 - |w|)^2\). On the other hand, if \(z \in G(w)\), then \(A(D(z)) = (1/4)(1 - |z|)^2 \geq (1/3)^2(1 - |w|)^2\); also by (2.1). Clearly, (2.2) follows from these two inequalities. □

**Proposition 2.3.** Suppose that \(g\) is a one-variable polynomial of degree \(K \geq 1\), and that \(f\) is analytic on \(D\). Then for all \(k \in \mathbb{N}\) and \(t \in (0, \infty)\) satisfying the condition \(t - 2k > -1\),

\[
\int |(\partial^k g)(z)f(z)|^2(1 - |z|^2)^t dA(z) 
\leq 2^{6k+2}\max\{t-2k,0\} + 7(K!)^2 \int |g(w)f(w)|^2(1 - |w|^2)^{t-2k} dA(w).
\]

**Proof.** Define \(g_z(u) = g(z + (1/2)(1 - |z|)u)\) and \(f_z(u) = f(z + (1/2)(1 - |z|)u)\) for each \(z \in D\). Then \(2^{-k}(1 - |z|)^k(\partial^k g)(z) = (\partial^k g_z)(0)\) and \(f(z) = f_z(0)\). By Lemma 2.1,

\[
|(\partial^k g)(z)f(z)|^2 = \frac{2^{2k}|(\partial^k g_z)(0)f_z(0)|^2}{(1 - |z|)^{2k}} \leq \frac{2^{4k+2}(K!)^2}{(1 - |z|)^{2k}} \int |g_z(u)f_z(u)|^2 dA(u) 
= \frac{2^{4k+2}(K!)^2}{(1 - |z|)^{2k}} \cdot \frac{1}{A(D(z))} \int_{D(z)} |g(w)f(w)|^2 dA(w).
\]

Therefore, if \(t - 2k > -1\), then

\[
\int |(\partial^k g)(z)f(z)|^2(1 - |z|^2)^t dA(z) 
\leq 2^{6k+2}(K!)^2 \int \frac{(1 - |z|^2)^{t-2k}}{A(D(z))} \left( \int_{D(z)} |g(w)f(w)|^2 dA(w) \right) dA(z) 
= 2^{6k+2}(K!)^2 \int \left\{ \int \frac{(1 - |z|^2)^{t-2k}}{A(D(z))} \chi_{D(z)}(w) dA(z) \right\} |g(w)f(w)|^2 dA(w) 
\leq 2^{6k+2}\max\{t-2k,0\} + 7(K!)^2 \int (1 - |w|^2)^{t-2k} |g(w)f(w)|^2 dA(w),
\]

where the last step is an application of Lemma 2.2. This completes the proof. □
3. Derivatives on the Ball

Recall that there is a constant $A_0 \in (2^{-n}, \infty)$ such that

\[(3.1) \quad 2^{-n}r^n \leq \sigma(\{x \in S : |1-\langle u, \xi \rangle| < r\}) \leq A_0 r^n \]

for all $u \in S$ and $0 < r \leq 2$ [15, Proposition 5.1.4]. For each $z \in B$, define the subset

\[ T(z) = \{w \in B : |1 - \langle w, z \rangle| < 2(1 - |z|^2), \ 1 - |w|^2 > (1/2)(1 - |z|^2)\} \]

of the unit ball. We begin our estimates with the properties of the set $T(z)$.

Let $dv$ be the volume measure on $B$ with the normalization $v(B) = 1$.

**Lemma 3.1.** There is a constant $0 < C_{3.1} < \infty$ such that for all $\zeta \in B$ and $x \in (-1, \infty)$,

\[
\int (1 - |\zeta|^2)^{x-n-1} \chi_{T(z)}(\zeta) dv(z) \leq C_{3.1} 2^{\max\{x,0\}} (1 - |\zeta|^2)^x.
\]

**Proof.** Let $\zeta, z \in B$ be such that $\zeta \in T(z)$. Then we have $1 - |\zeta|^2 \leq 2(1 - |\zeta|) \leq 2|1 - \langle \zeta, z \rangle| < 4(1 - |z|^2)$. Combining this with the condition $1 - |\zeta|^2 > (1/2)(1 - |z|^2)$, we have

\[(3.2) \quad (1/4)(1 - |\zeta|^2) \leq 1 - |z|^2 \leq 2(1 - |\zeta|^2).
\]

Therefore, for $x \in (-1, \infty)$ we have

\[
(1 - |z|^2)^x \leq \begin{cases} 
2^x (1 - |\zeta|^2)^x & \text{if } 0 \leq x < \infty \\
4(1 - |\zeta|^2)^x & \text{if } -1 < x < 0 
\end{cases}.
\]

Thus, to complete the proof, it suffices to show that there is a $0 < C < \infty$ such that

\[(3.3) \quad \int \frac{\chi_{T(z)}(\zeta)}{(1 - |\zeta|^2)^{n+1}} dv(z) \leq C
\]

for every $\zeta \in B$. Given a $\zeta \in B$, consider the set $\Omega(\zeta) = \{z \in B : \zeta \in T(z)\}$. Write $\zeta = |\zeta| \eta$ with $\eta \in S$. If $z = |z|\xi \in \Omega(\zeta)$, where $\xi \in S$, then $|1 - \langle \eta, \xi \rangle| \leq 2|1 - \langle \zeta, z \rangle| < 4(1 - |z|^2) < 8(1 - |\zeta|^2)$. Also, $1 - |z| \leq |1 - \langle \zeta, z \rangle| < 4(1 - |\zeta|^2)$ if $z \in \Omega(\zeta)$. Hence

\[
\Omega(\zeta) \subset \{r\xi : 0 < 1 - r < 4(1 - |\zeta|^2); \ \xi \in S; \ |1 - \langle \eta, \xi \rangle| < 8(1 - |\zeta|^2)\}.
\]

By (3.1) and the decomposition $dv = 2n r^{2n-1} dr d\sigma$, there is a $0 < C_1 < \infty$ such that $v(\Omega(\zeta)) \leq C_1 (1 - |\zeta|^2)^{n+1}$ for every $\zeta \in B$. By (3.2), $(1 - |z|^2)^{-n-1} \leq 4^{n+1}(1 - |\zeta|^2)^{-n-1}$ when $z \in \Omega(\zeta)$. Clearly, (3.3) follows from these two inequalities. $\Box$

**Lemma 3.2.** There is a constant $0 < \epsilon < 1$ such that for each $0 \leq a < 1$, the set $T((a, 0, \ldots, 0))$ contains the polydisc

\[(3.4) \quad P_a = \{(a + u, \zeta_2, \ldots, \zeta_n) : |u| < \epsilon(1 - a^2), \ |\zeta_j| < \epsilon \sqrt{1 - a^2}, \ 2 \leq j \leq n\}.
\]
Proof. Given an \( a \in [0,1) \), write \( \alpha = (a,0,\ldots,0) \). Let \( 0 < \epsilon < 1 \), and suppose that \( u \) and \( \zeta_2, \ldots, \zeta_n \) satisfy the conditions \( |u| < \epsilon(1-a^2) \) and \( |\zeta_j| < \epsilon\sqrt{1-a^2}, \ 2 \leq j \leq n \). Then consider the vector \( w = (a+u,\zeta_2,\ldots,\zeta_n) \). We have \( |1-(w,\alpha)| = |1-a^2-au| < (1+\epsilon)(1-a^2) \). Moreover, \( 1-|w|^2 = 1-|a+u|^2 - (|\zeta_2|^2+\cdots+|\zeta_n|^2) \geq 1-|a+u|^2 - (n-1)\epsilon^2(1-a^2) \).

On the other hand, \( 1-|a+u|^2 = 1-(a^2+2Re(au)+|u|^2) \geq 1-a^2-3|u| \geq (1-3\epsilon)(1-a^2) \). Hence \( 1-|w|^2 \geq (1-(n+2)\epsilon)(1-a^2) \). Thus \( \epsilon = (3(n+2))^{-1} \) suffices for our purpose. \( \square \)

As usual, write \( \partial_1, \ldots, \partial_n \) for the differentiations with respect to the complex variables \( z_1, \ldots, z_n \). For each vector \( b = (b_1, \ldots, b_n) \in \mathbb{C}^n \), define the directional derivative

\[
\partial_b = b_1 \partial_1 + \cdots + b_n \partial_n.
\]

Lemma 3.3. There is a constant \( 0 < C_{3,3} < \infty \) such that the following estimate holds: Suppose that \( q \in \mathbb{C}[z_1, \ldots, z_n] \) and that \( \deg(q) = K \geq 1 \). Let \( f \in \mathbb{C}[z_1, \ldots, z_n] \). If \( z \) and \( b \) are vectors in \( B \setminus \{0\} \) satisfying the relation \( \langle b, z \rangle = 0 \), then

\[
|(\partial_b q)(z)f(z)|^2 \leq \frac{C_{3,3}(K!)^2}{(1-|z|^2)^{n+2}} \int_{T(z)} |qf|^2 dv.
\]

Proof. Consider the special case where \( z = \alpha = (a,0,\ldots,0) \) for some \( 0 < a < 1 \). Let \( \epsilon \) be the constant provided by Lemma 3.2. Define the polydisc

\[
Y = \{(a+u,0,\zeta_3,\ldots,\zeta_n) : |u| < \epsilon(1-a^2), \ |\zeta_j| < \epsilon\sqrt{1-a^2}, \ 3 \leq j \leq n \}.
\]

For each \( y \in Y \), we define the one-variable polynomial \( q_y(w) = q(y + \epsilon\sqrt{1-a^2}we_2) \), where \( e_2 = (0,1,0,\ldots,0) \). Similarly, define \( f_y(w) = f(y + \epsilon\sqrt{1-a^2}we_2) \) on \( D \). Since \( (\partial y_q)(0) = \epsilon\sqrt{1-a^2}(\partial_2 q)(y) \) and \( f_y(0) = f(y) \), we apply Lemma 2.1 to obtain

\[
|(\partial_2 q)(y)f(y)|^2 \leq \frac{16(K!)^2}{\epsilon^2(1-a^2)} \int |q_y(w)f_y(w)|^2 dA(w).
\]

Making the substitution \( \zeta_2 = \epsilon\sqrt{1-a^2}w \), we find that

\[
|(\partial_2 q)(y)f(y)|^2 \leq \frac{16(K!)^2}{\epsilon^2(1-a^2)^2} \int_{|\zeta_2| < \epsilon\sqrt{1-a^2}} |q(y + \zeta_2 e_2)f(y + \zeta_2 e_2)|^2 dA(\zeta_2).
\]

Now, integrating both sides over \( Y \), we see that

\[
\epsilon^{2n-2}(1-a^2)^n |(\partial_2 q)(\alpha)f(\alpha)|^2 \leq \int_Y |(\partial_2 q)(y)f(y)|^2 dy \leq \frac{16C(K!)^2}{\epsilon^4(1-a^2)^2} \int_{P_{\alpha}} |qf|^2 dv,
\]

where \( P_{\alpha} \) is given by (3.4) and \( C \) accounts for the normalization constants for the measures involved. Since Lemma 3.2 tells us that \( P_{\alpha} \subset T(\alpha) \), we have

\[
|(\partial_2 q)(\alpha)f(\alpha)|^2 \leq \frac{16\epsilon^{2(n+2)}C(K!)^2}{(1-a^2)^{n+2}} \int_{T(\alpha)} |qf|^2 dv.
\]
Obviously, the above inequality also holds if we replace \( \partial_2 \) by \( \partial_j \) for any \( 2 \leq j \leq n \). Applying these and the Cauchy-Schwarz inequality, we see that

\[
|(\partial_b q)(\alpha)f(\alpha)|^2 \leq (n-1) \frac{16\epsilon^{-(2n+2)}C(K!)}{(1-a^2)^{n+2}} \int_{T(\alpha)} |qf|^2dv \quad \text{if } b, \alpha = 0, b \in \mathbf{B}.
\]

This proves the lemma in the special case where \( z = \alpha = (a,0,\ldots,0) \), \( 0 < a < 1 \). The general case follows from this special case and the following easily-verified relations: If \( U \) is any unitary transformation on \( \mathbf{C}^n \) and \( w,b \in \mathbf{B} \), then \( UT(w) = T(Uw) \) and \( (\partial_b (q \circ U))(w) = (\partial_{Ub}q)(Uw) \). \( \square \)

Following [9], for each pair of \( i \neq j \) in \( \{1, \ldots, n\} \) we define \( L_{i,j} = \bar{z}_j \partial_i - \bar{z}_i \partial_j \).

**Proposition 3.4.** There is a constant \( 1 \leq C_{3.4} < \infty \) such that the following estimate holds: Suppose that \( q \in \mathbf{C}[z_1, \ldots, z_n] \) and that \( \deg(q) = K \geq 1 \). Let \( f \in \mathbf{C}[z_1, \ldots, z_n] \). Then for every positive number \( t > 0 \) and all integers \( i \neq j \) in \( \{1, \ldots, n\} \), we have

\[
\int |(L_{i,j}q)(z)f(z)|^2(1-|z|^2)^t dv(z) \leq C_{3.4} 2^t (K!)^2 \int |q(\zeta)f(\zeta)|^2(1-|\zeta|^2)^{t-1} dv(\zeta).
\]

**Proof.** It follows from Lemma 3.3 that

\[
|(L_{i,j}q)(z)f(z)|^2 \leq \frac{C_{3.3}(K!)^2}{(1-|z|^2)^{n+2}} \int_{T(z)} |q(\zeta)f(\zeta)|^2 dv(\zeta),
\]

\( z \in \mathbf{B} \). Multiplying both sides by \( (1-|z|^2)^t \) and integrating, we find that

\[
\int |(L_{i,j}q)(z)f(z)|^2(1-|z|^2)^t dv(z)
\]

\[
\leq C_{3.3}(K!)^2 \int \left( (1-|z|^2)^{t-n-2} \int_{T(z)} |q(\zeta)f(\zeta)|^2 dv(\zeta) \right) dv(z)
\]

\[
= C_{3.3}(K!)^2 \int \left\{ \left( \int (1-|z|^2)^{t-n-2} \chi_{T(z)}(\zeta) dv(z) \right) \right\} |q(\zeta)f(\zeta)|^2 dv(\zeta).
\]

Applying Lemma 3.1 with \( x = t-1 \) to the \( \{\cdots\} \) above, the proposition follows. \( \square \)

Write \( R = z_1 \partial_1 + \cdots + z_n \partial_n \), the radial derivative in \( n \) variables. We will denote the one-variable radial derivative by \( R \). For each polynomial \( h \) and each \( \xi \in S \), define the “slice” function \( h_\xi(z) = h(z\xi) \), \( z \in D \). If \( q \) is a polynomial in \( n \) variables, then for every \( \xi \in S \) we have the relation \( (Rq_\xi)(z) = (Rq)_\xi(z) \).

**Proposition 3.5.** There is a constant \( 1 \leq C_{3.5} < \infty \) such that the following estimate holds: Suppose that \( q \in \mathbf{C}[z_1, \ldots, z_n] \) and that \( \deg(q) = K \geq 1 \). Let \( f \in \mathbf{C}[z_1, \ldots, z_n] \). Then for each pair of \( k \in \mathbf{N} \) and \( t \in (0,\infty) \) satisfying the condition \( t - 2k > -1 \),

\[
\int |(R^k q)(\zeta)f(\zeta)|^2(1-|\zeta|^2)^t dv(\zeta) \leq C_{3.5}^{K(k+t)} (K!)^2 \int |q(\zeta)f(\zeta)|^2(1-|\zeta|^2)^{t-2k} dv(\zeta).
\]
Proof. As in [9], we need the following relation between $dv$, $d\sigma$ and $dA$: Since $dv = 2nr^{2n-1}drd\sigma$, $dA = 2rdrdm$, and $d\sigma$ is invariant under rotation, we have

$$\int gdv = n\int \left( \int g(z\xi)|z|^{2n-2}dA(z) \right) d\sigma(\xi). \quad (3.5)$$

By Lemma 3.6 in [9], for each $k \in \mathbb{N}$,

$$\mathcal{R}^k = \sum_{j=1}^{k} a_j^{(k)} z^j \partial^j \quad \text{with} \quad |a_j^{(k)}| < (j+1)^k. \quad (3.6)$$

Since the degree of $q$ equals $K$, for each $\xi \in S$ we have

$$(R^k q)_{\xi}(z) = (\mathcal{R}^k q_{\xi})(z) = \sum_{j=1}^{\min\{k,K\}} a_j^{(k)} z^j (\partial^j q_{\xi})(z).$$

Given $f$, for each $\xi \in S$ we define the “rigged” slice function $f(\xi)(z) = z^{n-1} f(z\xi)$, $z \in D$. Applying first (3.6) and then Proposition 2.3, when $t - 2k > -1$, we have

$$\int |(R^k q_{\xi})(z)f(\xi)(z)|^2(1 - |z|^2)^t dA(z) \leq K(K + 1)^{2k} \sum_{j=1}^{\min\{k,K\}} 2^{6j+2\max\{t-2j,0\}+7}(K!)^2 \int |q_{\xi}(z)f(\xi)(z)|^2(1 - |z|^2)^{t-2j} dA(z) \leq K^2(K + 1)^{2k} 2^{6k+2t+7}(K!)^2 \int |q_{\xi}(z)f(\xi)(z)|^2(1 - |z|^2)^{t-2k} dA(z) \leq C_{3.5}^{K(K+t)}(K!)^2 \int |q_{\xi}(z)f(\xi)(z)|^2(1 - |z|^2)^{t-2k} dA(z).$$

By the relations $(\mathcal{R}^k q_{\xi})(z) = (R^k q)_{\xi}(z)$, $f(\xi)(z) = z^{n-1} f(z\xi)$ and $|z| = |z\xi|$, we now have

$$\int |(R^k q)(z\xi)f(z\xi)|^2(1 - |z\xi|^2)^t|z|^{2n-2} dA(z) \leq C_{3.5}^{K(K+t)}(K!)^2 \int |q(z\xi)f(z\xi)|^2(1 - |z\xi|^2)^{t-2k}|z|^{2n-2} dA(z).$$

Integrating both sides with respect to the measure $d\sigma$ on $S$ and applying (3.5), the proposition follows. $\square$

Proposition 3.6. There is a constant $1 \leq C_{3.6} < \infty$ such that the following estimate holds: Suppose that $q \in C[z_1, \ldots, z_n]$ and that deg($q$) = $K \geq 1$. Let $f \in C[z_1, \ldots, z_n]$. Then for each $t \in (1, \infty)$ and each $j \in \{1, \ldots, n\}$, we have

$$\int |(\partial_j q)(\zeta)f(\zeta)|^2(1 - |\zeta|^2)^tdv(\zeta) \leq C_{3.6}^{K(j)}(K!)^2 \int |q(\zeta)f(\zeta)|^2(1 - |\zeta|^2)^{t-2} dv(\zeta). \quad (3.7)$$
Proof. There is a $C$ such that for every analytic function $h$ on $\mathcal{B}$ and every $t > 0$, we have

\[
(3.8) \quad \int_{|\zeta|<1/2} |h(\zeta)|^2 (1 - |\zeta|^2)^t dv(\zeta) \leq C \left( \frac{16}{7} \right)^t \int_{1/2 \leq |\zeta|<3/4} |h(\zeta)|^2 (1 - |\zeta|^2)^t dv(\zeta).
\]

Now apply Proposition 3.4 and the case $k = 1$ in Proposition 3.5: by the identity $|z|^2 \partial_j = \bar{z_j} R + \sum_{i \neq j} z_i L_{j,i}$, (3.7) obviously holds if $(\partial_j q)(\zeta)$ is replaced by $|\zeta|^2 (\partial_j q)(\zeta)$ on the right-hand side. The extra factor $|\zeta|^2$ is then removed by using (3.8). \(\square\)

4. Embedding and Norm Ideals

For a bounded operator $A$, we write its $s$-numbers as $s_1(A), \ldots, s_k(A), \ldots$ as usual. Recall that, for each $1 \leq p < \infty$, the formula

\[
(4.1) \quad \|A\|_p^+ = \sup_{k \geq 1} \frac{s_1(A) + s_2(A) + \cdots + s_k(A)}{1^{-1/p} + 2^{-1/p} + \cdots + k^{-1/p}}
\]

defines a symmetric norm for operators [12, Section III.14]. On any Hilbert space $\mathcal{H}$, the set $\mathcal{C}_p^+ = \{ A \in B(\mathcal{H}) : \|A\|_p^+ < \infty \}$ is a norm ideal [12, Section III.2]. It is well known that if $p < p'$, then $\mathcal{C}_p^+$ is contained in the Schatten class $\mathcal{C}_{p'}^+$.

For a non-increasing sequence of non-negative numbers $\{a_1, \ldots, a_k, \ldots\}$, if $a_1 + \cdots + a_k \leq C(1^{-1/p} + \cdots + k^{-1/p})$, then $ka_k \leq C(1^{-1/p} + \cdots + k^{-1/p})$. It follows that if $p > 1$ and if $T \in \mathcal{C}_p^+$, then there is a $0 < C(T) < \infty$ such that $s_k(T) \leq C(T) k^{-1/p}$ for every $k \in \mathbb{N}$. Thus if $p > 1$ and if $B$ is a bounded operator such that $B^* B \in \mathcal{C}_p^+$, then $B \in \mathcal{C}_{2p}^+$.

Proposition 4.1. For each $t \geq -n$, let $I^{(t)} : \mathcal{H}^{(t)} \to \mathcal{H}^{(t+1)}$ be the natural embedding. Then $I^{(t)*} I^{(t)} \in \mathcal{C}_n^+$.

Proof. Expanding the reproducing kernel $(1 - \langle \zeta, z \rangle)^{-n-1+t}$, we see that the standard orthonormal basis for $\mathcal{H}^{(t)}$ is $\{ e_{\alpha}^{(t)} : \alpha \in \mathbb{Z}_n^+ \}$, where

\[
(4.2) \quad e_{\alpha}^{(t)}(\zeta) = \left( \frac{1}{\alpha!} \prod_{j=1}^{|\alpha|} (n + t + j) \right)^{1/2} \zeta^\alpha, \quad \alpha \neq 0,
\]

and $e_0^{(t)}(\zeta) = 1$. Given these orthonormal bases, it is straightforward to verify that

\[
I^{(t)*} I^{(t)} e_{\alpha}^{(t)} = \frac{n + 1 + t}{n + 1 + |\alpha| + t} e_{\alpha}^{(t)}, \quad \alpha \in \mathbb{Z}_n^+.
\]

This formula gives us all the $s$-numbers for $I^{(t)*} I^{(t)}$. By (4.1), $I^{(t)*} I^{(t)} \in \mathcal{C}_n^+$. \(\square\)

Proposition 4.2. Suppose that $E$ is a linear subspace of $\mathbb{C}[z_1, \ldots, z_n]$ and that $t \geq -n$. Let $E^{(t)}$ be the closure of $E$ in $\mathcal{H}^{(t)}$, and let $E^{(t)}$ be the orthogonal projection from $\mathcal{H}^{(t)}$ to $E^{(t)}$. Suppose that $A \in B(\mathcal{H}^{(t)})$, and suppose that there is a $C$ such that

\[
(4.3) \quad \|Ag\|_t \leq C\|g\|_{t+1}
\]
for every $g \in E$. Then $A\mathcal{E}^{(t)} \in C_{2n}^+$. 

Proof. By (4.3), for each $g \in E$ we have
\[
(A^*Ag, g)_t = \|Ag\|_t^2 \leq C\|g\|_{t+1}^2 = C^2\|I^{(t)}g\|_{t+1}^2 = C^2\langle f^{(t)}g, I^{(t)}g\rangle_{t+1} = C^2\langle f^{(t)*}f^{(t)}g, g\rangle_t.
\]
That is, the operator inequality $(A\mathcal{E}^{(t)})^*A\mathcal{E}^{(t)} \leq C^2\mathcal{E}^{(t)}f^{(t)*}f^{(t)}\mathcal{E}^{(t)}$ holds on $\mathcal{H}^{(t)}$. Thus $s_j((A\mathcal{E}^{(t)})^*A\mathcal{E}^{(t)}) \leq s_j(C^2\mathcal{E}^{(t)}I^{(t)*}I^{(t)}\mathcal{E}^{(t)})$ for each $j \in \mathbb{N} [12, \text{Lemma II.1.1}].$ By Proposition 4.1, $(A\mathcal{E}^{(t)})^*A\mathcal{E}^{(t)} \in C_n^+$. Since $n \geq 2$, this implies $A\mathcal{E}^{(t)} \in C_{2n}^+$. $\Box$

5. Proof of Theorem 1.1

For each $t \geq -n$ and each polynomial $q$, we write $M_q^{(t)}$ for the operator of multiplication by $q$ on the space $\mathcal{H}^{(t)}$. Keep in mind that the notation "*" is $t$-specific: $M_q^{(t)*}$ means the adjoint of $M_q^{(t)}$ with respect to the inner product $\langle \cdot, \cdot \rangle_t$.

Proposition 5.1. Let $q \in \mathbb{C}[z_1, \ldots, z_n]$, $1 \leq j \leq n$ and $t \geq -n$. For $f \in \mathbb{C}[z_1, \ldots, z_n]$ satisfying the condition $f(0) = 0$, we have

\[
M_{z_j}^{(t)*}M_q^{(t)}f - M_q^{(t)}M_{z_j}^{(t)*}f = \sum_{k=0}^{\infty} (N + 1 + n + t)^{-k-1}(M_{\partial_q}^{(t)}M_{z_j}^{(t)*} - M_{z_j}^{(t)*}M_{\partial_q}^{(t)}M_{z_j}^{(t)*})f.
\]

Proof. The main idea is that both sides are linear with respect to both $q$ and $f$. Therefore the proof is a matter of straightforward verification in the special case of $q = z^\alpha$ and $f = z^\beta$, $\beta \neq 0$, using (4.2). The details of the verification are similar to the Bergman space case (see the proof of Proposition 2.1 in [9]). $\Box$

Proposition 5.2. Let $t \geq -n$ and $\ell \in \mathbb{N}$. (1) For each $f \in \mathbb{C}[z_1, \ldots, z_n]$ satisfying the condition $(\partial^\alpha f)(0) = 0$ for $|\alpha| \leq \ell$ and each non-negative integer $k$, we have
\[
\|(N + 1 + n + t)^{-k-1}f\|_t^2 \leq \frac{(n + 2k + 2 + t + \ell)^\ell}{(\ell + 1 + n + t)^{2k+2}} \|f\|_{2k+2+t}^2.
\]
(2) For each $f \in \mathbb{C}[z_1, \ldots, z_n]$ satisfying the condition $(\partial^\alpha f)(0) = 0$ for $|\alpha| \leq \ell + 1$, each non-negative integer $k$ and each $1 \leq j \leq n$, we have
\[
\|(N + 1 + n + t)^{-k-1}(M_{z_j}^{(t)*} - M_{z_j}^{(t+2k+2)*})f\|_t^2 \leq (2k + 4)^4 \frac{(n + 2k + 4 + t + \ell)^{2k}}{(\ell + 1 + n + t)^{2k+2}} \|f\|_{2k+4+t}^2.
\]

Proof. For (1), it suffices to consider the case where $f$ is a homogeneous polynomial of degree $m \geq \ell$, as it was the case for the corresponding part in [9]. For such an $f$,
\[
\|(N + 1 + n + t)^{-k-1}f\|_t^2 = \frac{\|f\|_t^2}{(m + 1 + n + t)^{2k+2}} = \frac{\|f\|_{2k+2+t}^2}{(m + 1 + n + t)^{2k+2}} \prod_{j=1}^{m} \frac{n + 2k + 2 + t + j}{n + t + j},
\]
(see (4.2))
\[
= \frac{\|f\|_{2k+2+t}^2}{(m + 1 + n + t)^{2k+2}} \prod_{j=1}^{2k+2} \frac{n + m + t + j}{n + t + j}.
\]
where the last equality is obtained by considering $\prod_{j=1}^{2k+2+m} (n+t+j)$. Since $m \geq \ell$, for each $j \geq 1$ we have $(n+m+t+j)/(m+1+n+t) \leq (n+\ell+t+j)/(\ell+1+n+t)$. Hence

$$
\|(N + 1 + n + t)^{-k-1} f\|^2_t \leq \|f\|^2_{2k+2+t} \prod_{j=1}^{2k+2} \frac{n + \ell + t + j}{(\ell + 1 + n + t)(n + t + j)}
$$

$$
= \frac{\|f\|^2_{2k+2+t}}{(\ell + 1 + n + t)^{2k+2}} \prod_{j=1}^{\ell} \frac{n + 2k + 2 + t + j}{n + t + j} \leq \frac{(n + 2k + 2 + t + \ell)\ell}{(\ell + 1 + n + t)^{2k+2}} \|f\|^2_{2k+2+t}.
$$

This proves (1).

Let $e_j$ be the element in $\mathbb{Z}_n^\ell$ whose $j$-th component is 1 and whose other components are 0. To prove (2), first note that (4.2) gives us

$$
M_{z_j^*}^{(t)*} z^\alpha = \frac{\alpha_j}{n + t + |\alpha|} z^{\alpha - e_j}
$$

whenever the $j$-th component $\alpha_j$ of $\alpha$ is greater than 0. Hence

$$
(M_{z_j^*}^{(t)*} - M_{z_j^*}^{(2k+2+t)*}) z^\alpha = \frac{\alpha_j (2k + 2)}{(n + t + |\alpha|)(n + 2k + 2 + t + |\alpha|)} z^{\alpha - e_j}
$$

$$
= \frac{2k + 2}{n + t + |\alpha|} M_{z_j^*}^{(2k+2+t)*} z^{\alpha - e_j}.
$$

(5.1)

For $f \in \mathbb{C}[z_1, \ldots, z_n]$ with $f(0) = 0$, $(N + n + t)^{-1} f$ is well defined. Thus we can define

$$
f_{t,k} = (N + 1 + n + 2k + 2 + t)(N + n + t)^{-1} f.
$$

We have $\|f_{t,k}\|_{\tau} \leq (2k + 4) \|f\|_{\tau}$ for every $\tau \geq -n$. Obviously, (5.1) implies

$$
(M_{z_j^*}^{(t)*} - M_{z_j^*}^{(2k+2+t)*}) f = (2k + 2) M_{z_j^*}^{(2k+2+t)*} (N + 1 + n + 2k + 2 + t)^{-1} f_{t,k}
$$

Now suppose that $(\partial^\alpha f)(0) = 0$ for $|\alpha| < \ell + 1$. Applying (1) twice, we have

$$
\|(N + 1 + n + t)^{-k-1} (M_{z_j^*}^{(t)*} - M_{z_j^*}^{(2k+2+t)*}) f\|^2_t
$$

$$
\leq (2k + 2)^2 \|(N + 1 + n + t)^{-k-1} M_{z_j^*}^{(2k+2+t)*} (N + 1 + n + 2k + 2 + t)^{-1} f_{t,k}\|^2_t
$$

$$
\leq (2k + 2)^2 \frac{(n + 2k + 2 + t + \ell)\ell}{(\ell + 1 + n + t)^{2k+2}} \|M_{z_j^*}^{(2k+2+t)*} (N + 1 + n + 2k + 2 + t)^{-1} f_{t,k}\|^2_{2k+2+t}
$$

$$
\leq (2k + 2)^2 \frac{(n + 2k + 2 + t + \ell)\ell}{(\ell + 1 + n + t)^{2k+2}} \|(N + 1 + n + 2k + 2 + t)^{-1} f_{t,k}\|^2_{2k+2+t}
$$

$$
\leq (2k + 2)^2 \frac{(n + 2k + 4 + t + \ell)\ell}{(\ell + 1 + n + 2k + 2 + t)^2} \|f_{t,k}\|^2_{2k+4+t}
$$

$$
\leq (2k + 2)^2 \frac{(n + 2k + 4 + t + \ell)^2\ell}{(\ell + 1 + n + t)^{2k+2}} \|f\|^2_{2k+4+t}.
$$
This completes the proof of (2). □

For each real number \( t > -1 \), define

\[
a_{n,t} = \frac{1}{n!} \prod_{j=1}^{n} (t + j).
\]

Using (4.2) and [15, Proposition 1.4.9], it is straightforward to verify that

\[
\langle f, g \rangle_t = a_{n,t} \int f(\zeta) \overline{g(\zeta)}(1 - |\zeta|^2)^t \, dv(\zeta)
\]

for \( f, g \in \mathcal{H}(t), \ t > -1 \). In other words, if \( t > -1 \), then \( \mathcal{H}(t) \) is the weighted Bergman space \( L^2_{\alpha}(B, a_{n,t}(1 - |\zeta|^2)^t \, dv(\zeta)) \).

Our next step requires the assumption that \( t > -2 \).

**Proposition 5.3.** Let real number \( t > -2 \) and integer \( K \geq 1 \) be given. Then there is a constant \( C_{5.3} = C_{5.3}(n, K, t) \) such that the following estimate holds: Let \( q \in \mathbb{C}[z_1, \ldots, z_n] \) be such that \( \deg(q) = K \). Suppose that \( f \in \mathbb{C}[z_1, \ldots, z_n] \) satisfies the condition \( (\partial^\alpha f)(0) = 0 \) for \( |\alpha| \leq \ell + 1 \), where \( \ell \in \mathbb{N} \). Then for every integer \( k \geq 0 \) and every \( j \in \{1, \ldots, n\} \),

\[
\|(N + 1 + n + t)^{-k-1}(M_{\partial_j R^k q}^{(t)} - M_{z_j}^{(t)} M_{R^{k+1} q}^{(t)} )f\|_t \leq \frac{(n + 2k + 4 + t + \ell)^{\ell+2}}{(\ell + 1 + n + t)^{k+1}} C_{5.3}^{k+1} \|qf\|_{t+1}.
\]

**Proof.** Since

\[
M_{\partial_j R^k q}^{(t)} - M_{z_j}^{(t)} M_{R^{k+1} q}^{(t)} = (M_{\partial_j R^k q}^{(t)} - M_{z_j}^{(2k+2+t)} M_{R^{k+1} q}^{(t)} ) - (M_{z_j}^{(t)} - M_{z_j}^{(2k+2+t)}) M_{R^{k+1} q}^{(t)},
\]

we have

\[
\|(N + 1 + n + t)^{-k-1}(M_{\partial_j R^k q}^{(t)} - M_{z_j}^{(t)} M_{R^{k+1} q}^{(t)} )f\|_t \leq A + B,
\]

where

\[
A = \|(N + 1 + n + t)^{-k-1}(M_{\partial_j R^k q}^{(t)} - M_{z_j}^{(2k+2+t)} M_{R^{k+1} q}^{(t)} )f\|_t \quad \text{and}\n
B = \|(N + 1 + n + t)^{-k-1}(M_{z_j}^{(t)} - M_{z_j}^{(2k+2+t)}) M_{R^{k+1} q}^{(t)} f\|_t.
\]

We estimate \( A \) and \( B \) separately. For \( A \), we apply Proposition 5.2(1), which gives us

\[
A \leq \frac{(n + 2k + 2 + t + \ell)^{\ell}}{(\ell + 1 + n + t)^{k+1}} \|(M_{\partial_j R^k q}^{(t)} - M_{z_j}^{(2k+2+t)} M_{R^{k+1} q}^{(t)} )f\|_{2k+2+t}
\]

\[
= \frac{(n + 2k + 2 + t + \ell)^{\ell}}{(\ell + 1 + n + t)^{k+1}} \|(M_{\partial_j R^k q}^{(2k+2+t)} - M_{z_j}^{(2k+2+t)} M_{R^{k+1} q}^{(2k+2+t)}) f\|_{2k+2+t}.
\]
Since \( t > -2 \), we have \( 2k + 2 + t > 0 \) for each \( k \geq 0 \). Hence \( H^{(2k+2+t)} \) is a weighted Bergman space. By (5.2), we have

\[
\| (M_{\partial_j}^{2k+2+t} - M_{z_j}^{2k+2+t} M_{R^k+1,q}^{2k+2+t}) f \|_{2k+2+t} \\
\leq a_{n,2k+2+t}^{1/2} \left( \int |\{(\partial_j R^k q)(z) - \bar{z}_j (R^{k+1} q)(z)\} f(z) |^2 (1 - |z|^2)^{2k+2+t} dv(z) \right)^{1/2}.
\]

The identity \( \partial_j - \bar{z}_j R = (1 - |z|^2) \partial_j + \sum_{i \neq j} z_i L_{j,i} \) then leads to

\[
\| (M_{\partial_j}^{2k+2+t} - M_{z_j}^{2k+2+t} M_{R^k+1,q}^{2k+2+t}) f \|_{2k+2+t} \\
\leq a_{n,2k+2+t}^{1/2} \left( \int |(\partial_j R^k q)(z) f(z) |^2 (1 - |z|^2)^{2k+4+t} dv(z) \right)^{1/2} \\
+ a_{n,2k+2+t}^{1/2} \sum_{i \neq j} \left( \int |(L_{j,i} R^k q)(z) f(z) |^2 (1 - |z|^2)^{2k+2+t} dv(z) \right)^{1/2}.
\]

Applying Propositions 3.6 and 3.5, we have

\[
\int |(\partial_j R^k q)(z) f(z) |^2 (1 - |z|^2)^{2k+4+t} dv(z) \\
\leq C_{3,6}^{K(2k+4+t)} (K!)^2 \int |(R^k q)(z) f(z) |^2 (1 - |z|^2)^{2k+2+t} dv(z) \\
\leq (C_{3,6}C_{3,5})^{K(3k+4+t)} (K!)^4 \int |q(z) f(z) |^2 (1 - |z|^2)^{2k+2+t} dv(z).
\]

Since \( 1 + t > -1 \), we can apply Propositions 3.4 and 3.5 to obtain

\[
\int |(L_{j,i} R^k q)(z) f(z) |^2 (1 - |z|^2)^{2k+2+t} dv(z) \\
\leq C_{3,4}^{2k+2+t} (K!)^2 \int |(R^k q)(z) f(z) |^2 (1 - |z|^2)^{2k+1+t} dv(z) \\
\leq C_{3,4}(2C_{3,5})^{K(3k+2+t)} (K!)^4 \int |q(z) f(z) |^2 (1 - |z|^2)^{1+t} dv(z).
\]

By the assumption \( t > -2 \), we have \( a_{n,1+t} \geq (n!)^{-1}(2 + t)^n \). Also note that \( a_{n,2k+2+t} \leq (n!)^{-1}(n + 2k + 2 + t)^n \). Combining (5.5), (5.6), (5.7) and (5.2), we see that there is a \( C_1 \) that depends only on \( n, K \) and \( t > -2 \) such that

\[
\| (M_{\partial_j}^{2k+2+t} - M_{z_j}^{2k+2+t} M_{R^k+1,q}^{2k+2+t}) f \|_{2k+2+t} \leq C_1^{k+1} \| q f \|_{t+1}.
\]

Recalling (5.4), this gives us

\[
A \leq \frac{(n + 2k + 2 + t + \ell)^t}{(t + 1 + n + t)^{k+1}} C_1^{k+1} \| q f \|_{t+1}.
\]
It follows from Proposition 5.2(2) that

\[ B \leq \frac{(n + 2k + 4 + t + \ell)^{\ell+2}}{(\ell + 1 + n + t)^{k+1}} \| M_{R^{k+1}q}^{(t)}f \|_{2k+4+t}. \]

Applying (5.2) and Proposition 3.5, we obtain

\[
\| M_{R^{k+1}q}^{(t)}f \|_{2k+4+t}^2 = a_n,2k+4+t \int \| R^{k+1}q(z)f(z) \|^2 (1 - |z|^2)^{2k+4+t} dv(z) \leq a_n,2k+4+t C_{3.5}^{K(3k+5+t)} (K!)^2 \int |q(z)f(z)|^2 (1 - |z|^2)^{2+t} dv(z).
\]

Thus there is a \( C_2 \) that depends only on \( n, K \) and \( t > -2 \) such that \( \| M_{R^{k+1}q}^{(t)}f \|_{2k+4+t} \leq C_2^{k+1} \| qf \|_{t+1} \). Consequently,

\[ B \leq \frac{(n + 2k + 4 + t + \ell)^{\ell+2}}{(\ell + 1 + n + t)^{k+1}} C_2^{k+1} \| qf \|_{t+1}. \]

Combining this with (5.8) and (5.3), the proof of the proposition is complete. \( \square \)

**Proof of Theorem 1.1.** Let \( q \in C[z_1, \ldots, z_n] \) be such that \( \text{deg}(q) = K, K \geq 1 \). Let \( t > -2 \) also be given. For this pair of \( K \) and \( t \), let \( C_{5.3} = C_{5.3}(n, K, t) \) be the constant provided by Proposition 5.3. Let \( \ell \in \mathbb{N} \) satisfy the condition

\[ \ell + 1 + n + t > 2C_{5.3}. \]  

(5.9)

With this \( \ell \), we now define

\[ E = \{ qf : f \in C[z_1, \ldots, z_n], \ (\partial^{\alpha} f)(0) = 0 \text{ for } |\alpha| \leq \ell + 1 \}. \]

For the given \( q \), let \( Q^{(t)} \) denote the orthogonal projection from \( H^{(t)} \) onto \( H^{(t)} \ominus [q]^{(t)} \). Let \( j \in \{ 1, \ldots, n \} \), and let \( f \in C[z_1, \ldots, z_n] \) be such that \( (\partial^{\alpha} f)(0) = 0 \) for \( |\alpha| \leq \ell + 1 \). Then

\[ Q^{(t)} M_{z_j}^{(t)*} qf = Q^{(t)} M_{z_j}^{(t)*} M_q^{(t)} f = Q^{(t)} (M_{z_j}^{(t)*} M_q^{(t)} - M_q^{(t)} M_{z_j}^{(t)*}) f. \]

Applying Propositions 5.1 and 5.3, we have

\[
\| Q^{(t)} M_{z_j}^{(t)*} qf \|_t \leq \sum_{k=0}^{\infty} \| (N + 1 + n + t)^{-k-1} (M_{\partial_j R^k q}^{(t)} - M_{z_j}^{(t)*} M_{R^{k+1}q}^{(t)}) f \|_t \leq \sum_{k=0}^{\infty} \frac{(n + 2k + 4 + t + \ell)^{\ell+2}}{(\ell + 1 + n + t)^{k+1}} C_{5.3}^{k+1} \| qf \|_{t+1}. \]  

(5.10)

Set

\[ C = \sum_{k=0}^{\infty} \frac{(n + 2k + 4 + t + \ell)^{\ell+2}}{(\ell + 1 + n + t)^{k+1}} C_{5.3}^{k+1}. \]
Then (5.9) ensures that $C < \infty$. Thus (5.10) can be restated as

$$\|Q^{(t)} M^{(t)*}_{z_j} g\|_t \leq C \|g\|_{t+1} \quad \text{for every } g \in E.$$ 

Let $E^{(t)}$ be the closure of $E$ in $\mathcal{H}^{(t)}$, and let $\mathcal{E}^{(t)} : \mathcal{H}^{(t)} \to E^{(t)}$ be the orthogonal projection. By Proposition 4.2, the above implies that

$$Q^{(t)} M^{(t)*}_{z_j} \mathcal{E}^{(t)} \in \mathcal{C}_{2n}^+.$$ 

Obviously, $E^{(t)}$ is a subspace of $[q]^{(t)}$ of finite codimension. That is, if $P^{(t)}$ denotes the orthogonal projection from $\mathcal{H}^{(t)}$ onto $[q]^{(t)}$, then $\text{rank}(P^{(t)} - \mathcal{E}^{(t)}) < \infty$. Therefore

$$Q^{(t)} M^{(t)*}_{z_j} P^{(t)} \in \mathcal{C}_{2n}^+.$$ 

Combining this with the well-known fact that $[M^{(t)*}_{z_j}, M^{(t)}_{z_i}] \in \mathcal{C}_n^+$, it follows from a routine argument that $[Z^{(t)*}_{q,j}, Z^{(t)}_{q,i}] \in \mathcal{C}_n^+$, $i, j \in \{1, \ldots, n\}$. This completes the proof. □

References


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