THE HELTON-HOWE TRACE FORMULA FOR SUBMODULES

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Abstract. We consider a class of submodules $\mathcal{R}$ of the Bergman module $L^2_a(\mathbf{B})$ that are associated with analytic sets $\tilde{M} \subset \mathbf{C}^n$ with $\dim_{\mathbf{C}}\tilde{M} = d$. In analogue to the usual Toeplitz operator on $L^2_a(\mathbf{B})$, we have the "Toeplitz operator for the submodule" R_φ on $\mathcal{R}$. We show that the Helton-Howe trace formula holds for the antisymmetric sum $[R_{f_1}, R_{f_2}, \dots, R_{f_{2n}}]$, $f_1, f_2, \dots, f_{2n} \in \mathbf{C}[z_1, \bar{z}_1, \dots, z_n, \bar{z}_n]$.

1. Introduction

Given any bounded operators $A_1, \dots, A_k$ on a Hilbert space $\mathcal{H}$, one has the antisymmetric sum

$$[A_1, \dots, A_k] = \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) A_{\sigma(1)} \cdots A_{\sigma(k)},$$

which naturally generalizes the notion of commutator. This was first introduced by Helton and Howe in [14], and has since become an important part of non-commutative geometry. See [5,10] in particular. As it turns out, operators on reproducing-kernel Hilbert spaces provide some of the particularly interesting examples of antisymmetric sums.

Let $\mathbf{B}$ be the unit ball $\{z \in \mathbf{C}^n : |z| < 1\}$ in $\mathbf{C}^n$. As usual, we write $L^2_a(\mathbf{B})$ for the Bergman space. Let $P : L^2(\mathbf{B}) \to L^2_a(\mathbf{B})$ be the orthogonal projection. Given a $\varphi \in L^\infty(\mathbf{B})$, we have the familiar Toeplitz operator T_φ defined by the formula

$$T_\varphi f = P(\varphi f), \quad f \in L^2_a(\mathbf{B}). \tag{1.1}$$

This paper is mainly motivated by the following classic result of Helton and Howe:

Theorem 1.1. [14, Theorem 7.2] *For $f_1, f_2, \dots, f_{2n} \in \mathbf{C}[z_1, \bar{z}_1, \dots, z_n, \bar{z}_n]$, the antisymmetric sum $[T_{f_1}, T_{f_2}, \dots, T_{f_{2n}}]$ is in the trace class. Moreover,*

$$\operatorname{tr}[T_{f_1}, T_{f_2}, \dots, T_{f_{2n}}] = \frac{n!}{(2\pi i)^n} \int_{\mathbf{B}} df_1 \wedge df_2 \wedge \cdots \wedge df_{2n}. \tag{1.2}$$

See Roger Howe's retrospective [19, pages 1678-1679] for the historical context for this result. More than four decades later, this result still commands considerable interest, for several reasons.

One of these reasons is the Arveson-Douglas Conjecture [1,2,6,7], which has been the subject of very active research [3,8,9,11,12,15]. Indeed advances on the Arveson-Douglas

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Keywords: Submodule, antisymmetric sum, trace formula.
[1]Supported in part by National Science Foundation grant DMS-1900076.

Conjecture have made it possible for one to consider the analogue of Theorem 1.1 on submodules or even quotient modules. A first step in that direction was taken in [18]. Although it was not explicitly mentioned in [18], even a casual reader would notice that [18] left an obvious question unanswered: does the analogue of trace formula (1.2) hold on the *submodules* considered in that paper? The purpose of this paper is to report the affirmative answer to this question.

At this point, it will be helpful to recall the setting in [18]. First of all, we will use the same notations as those in [18] whenever possible. Let $\tilde{M}$ be an analytic set in Assumption 1.1 in [18]. Throughout the paper, we denote $d = \dim_{\mathbf{C}} \tilde{M}$. As in [18], we always assume $1 \leq d \leq n-1$. Denote $M = \tilde{M} \cap \mathbf{B}$. Then

$$\mathcal{R} = \{f \in L^2_a(\mathbf{B}) : f = 0 \text{ on } M\}$$

is a submodule of the Bergman module $L^2_a(\mathbf{B})$. Let $R : L^2(\mathbf{B}) \to \mathcal{R}$ be the orthogonal projection. Mimicking (1.1), given a $\varphi \in L^\infty(\mathbf{B})$ we define

$$R_\varphi f = R(\varphi f), \quad f \in \mathcal{R}.$$

We think of R_φ as a "Toeplitz operator for the submodule $\mathcal{R}$". It was proved in [18] that for any $f_1, f_2, \dots, f_{2n} \in \mathbf{C}[z_1, \bar{z}_1, \dots, z_n, \bar{z}_n]$, the antisymmetric sum $[R_{f_1}, R_{f_2}, \dots, R_{f_{2n}}]$ is in the trace class. The question that was not explicitly stated in [18], but was obvious to even the casual reader, was

Question 1.2. Does the trace formula

$$\operatorname{tr}[R_{f_1}, R_{f_2}, \dots, R_{f_{2n}}] = \frac{n!}{(2\pi i)^n} \int_{\mathbf{B}} df_1 \wedge df_2 \wedge \cdots \wedge df_{2n},$$

$f_1, f_2, \dots, f_{2n} \in \mathbf{C}[z_1, \bar{z}_1, \dots, z_n, \bar{z}_n]$, hold on the submodule $\mathcal{R}$?

We were not able to compute $\operatorname{tr}[R_{f_1}, R_{f_2}, \dots, R_{f_{2n}}]$ in [18], but, after many attempts, we are finally able to do this computation now, which is the main result of the paper:

Theorem 1.3. *The trace formula*

$$\operatorname{tr}[R_{f_1}, R_{f_2}, \dots, R_{f_{2n}}] = \frac{n!}{(2\pi i)^n} \int_{\mathbf{B}} df_1 \wedge df_2 \wedge \cdots \wedge df_{2n}$$

holds for all $f_1, f_2, \dots, f_{2n} \in \mathbf{C}[z_1, \bar{z}_1, \dots, z_n, \bar{z}_n]$.

Although Question 1.2 is about the submodule $\mathcal{R}$, it is the corresponding *quotient module* that holds the key to the answer. Recall that

$$\mathcal{Q} = L^2_a(\mathbf{B}) \ominus \mathcal{R}$$

is the quotient module that corresponds to $\mathcal{R}$. Let $Q : L^2(\mathbf{B}) \to \mathcal{Q}$ be the orthogonal projection. Recall that Q was the focus of both [11] and [18]. Mimicking (1.1), we have "Toeplitz operators for the quotient module $\mathcal{Q}$". That is, for each $\varphi \in L^\infty(\mathbf{B})$, we define

$$Q_\varphi f = Q(\varphi f), \quad f \in \mathcal{Q}.$$

For $f_1, f_2, \ldots, f_{2n} \in \mathbf{C}[z_1, \bar{z}_1, \ldots, z_n, \bar{z}_n]$, the relation $P = R + Q$ leads to

$$[T_{f_1}, T_{f_2}, \ldots, T_{f_{2n}}] = [R_{f_1}, R_{f_2}, \ldots, R_{f_{2n}}] + [Q_{f_1}, Q_{f_2}, \ldots, Q_{f_{2n}}] + G. \tag{1.3}$$

We already know that the antisymmetric sum $[Q_{f_1}, Q_{f_2}, \ldots, Q_{f_{2n}}]$ is in the trace class with *zero trace* [18, Theorem 1.8]. Thus to show that Question 1.2 has an affirmative answer, it suffices to prove that G has zero trace.

The fact that $\mathrm{tr}(G) = 0$ can be directly verified in some particular cases, and direct verification was how we began our effort to answer Question 1.2. For example, if we either assume $d < (2/3)n$ or $n \in \{2, 3, 4\}$, then we can directly verify that $\mathrm{tr}(G) = 0$, although such verifications are *very* tedious. Tedious though direct verifications may be, the results do suggest that we should try to answer Question 1.2 in the affirmative in general. On the other hand, the direct-verification method has its limitations. For example, if one tries to directly verify that $\mathrm{tr}(G) = 0$ in the case where $n = 5$ and $d = 4$, one runs into what seem to be insurmountable obstacles.

We took these obstacles as a hint that, perhaps, the general result $\mathrm{tr}(G) = 0$ needs to be proved in a round-about way, not by a "frontal assault" in the way of direct verification. After various attempts, we managed to find a way to show $\mathrm{tr}(G) = 0$ by an indirect route. The main idea is this: we approximate the orthogonal projection Q by a sequence of finite-rank positive operators in a very specific way. These finite-rank approximations are then "dilated" to orthogonal projections on $L^2(\mathbf{B}) \oplus L^2(\mathbf{B})$. These dilations and the orthogonal projection $R' = R \oplus 0$ give rise to various "deformed Toeplitz operators". Analysis of these operators eventually leads to the desired result,

$$\mathrm{tr}[R_{f_1}, R_{f_2}, \ldots, R_{f_{2n}}] = \mathrm{tr}[T_{f_1}, T_{f_2}, \ldots, T_{f_{2n}}]$$

for $f_1, f_2, \ldots, f_{2n} \in \mathbf{C}[z_1, \bar{z}_1, \ldots, z_n, \bar{z}_n]$.

The rest of the paper is organized as follows. First of all, Section 2 collects a number of necessary preliminaries for the rest of the paper. Then, with the efforts in Sections 3 and 4, we construct a sequence $\{A_m\}_{m=1}^{\infty}$ of finite-rank operators such that

$$0 \le A_m \le Q$$

and such that they have the right approximation property:

$$\lim_{m\to\infty} \|[Q - A_m, M_f]\|_p = 0$$

for $2d < p < \infty$ and $f \in \mathbf{C}[z_1, \bar{z}_1, \ldots, z_n, \bar{z}_n]$, and more. In Section 5, these operators are dilated to orthogonal projections $Q^{(m)}$ on $L^2(\mathbf{B}) \oplus L^2(\mathbf{B})$. With the projections $P^{(m)} = R' + Q^{(m)}$, we have "deformed Toeplitz operators" $T_f^{(m)}$. In Section 6, we first show that these "deformed Toeplitz operators" approximate the real ones in the way that matters:

$$\lim_{m\to\infty} \mathrm{tr}[T_{f_1}^{(m)}, T_{f_2}^{(m)}, \ldots, T_{f_{2n}}^{(m)}] = \mathrm{tr}[T_{f_1}, T_{f_2}, \ldots, T_{f_{2n}}]$$

for $f_1, f_2, \ldots, f_{2n} \in \mathbf{C}[z_1, \bar{z}_1, \ldots, z_n, \bar{z}_n]$. The punch line of Section 6 is that in the "deformed version" of (1.3), the G does have zero trace, which finishes the proof of Theorem 1.3.

2. Preliminaries

First, let us recall the precise definition of the submodules and quotient modules that we consider in this paper.

Definition 2.1. [4,16] Let Ω be a complex manifold. A set $A \subset \Omega$ is called a *complex analytic subset* of Ω if for each point $a \in \Omega$ there are a neighborhood U of a and functions $f_1, \cdots, f_N$ analytic in this neighborhood such that

$$A \cap U = \{z \in U : f_1(z) = \cdots = f_N(z) = 0\}.$$

A point $a \in A$ is called *regular* if there is a neighborhood U of a in Ω such that $A \cap U$ is a complex submanifold of Ω. A point $a \in A$ is called a *singular point* of A if it is not regular.

Definition 2.2. Let Y be a manifold and let X, Z be submanifolds of Y. We say that the submanifolds X and Z *intersect transversely* if for every $x \in X \cap Z$, $T_x(X) + T_x(Z) = T_x(Y)$.

Assumption 2.3. [18, Assumption 1.1] Let $\tilde{M}$ be an analytic subset of an open neighborhood of the closed ball $\overline{\mathbf{B}}$. Furthermore, $\tilde{M}$ satisfies the following conditions:

(1) $\tilde{M}$ intersects $\partial\mathbf{B}$ transversely.
(2) $\tilde{M}$ has no singular points on $\partial\mathbf{B}$.
(3) $\dim_{\mathbf{C}} \tilde{M} = d$, where $1 \le d \le n-1$.

We emphasize that Assumption 2.3 will always be in force for the rest of the paper. Given such an $\tilde{M}$, we fix M, $\mathcal{R}$, R, $\mathcal{Q}$ and Q as follows.

Notation 2.4 [18, Notation 1.2] (a) Let $M = \tilde{M} \cap \mathbf{B}$.
(b) Denote $\mathcal{R} = \{f \in L^2_a(\mathbf{B}) : f = 0 \text{ on } M\}$.
(c) Let R be the orthogonal projection from $L^2(\mathbf{B})$ onto $\mathcal{R}$.
(d) Denote $\mathcal{Q} = L^2_a(\mathbf{B}) \ominus \mathcal{R}$.
(e) Let Q be the orthogonal projection from $L^2(\mathbf{B})$ onto $\mathcal{Q}$.

As we have already mentioned, the key to what we do in this paper is the projection Q. We need to use everything we know about Q, starting with

Theorem 2.5. [11, Theorem 4.3] *There exist a measure μ on M and $0 < c \le C < \infty$ such that*

$$c\|f\|^2 \le \int_M |f(w)|^2 d\mu(w) \le C\|f\|^2 \tag{2.1}$$

for every $f \in \mathcal{Q}$.

For the rest of the paper, the symbol μ will be used exclusively for the measure given in Theorem 2.5. On $L^2_a(\mathbf{B})$, this μ defines a Toeplitz operator via the formula

$$(T_\mu f)(z) = \int_M \frac{f(w)}{(1 - \langle z, w\rangle)^{n+1}} d\mu(w). \tag{2.2}$$

The two bounds in Theorem 2.5 translate to the operator inequality

$$cQ \le T_\mu \le CQ \tag{2.3}$$

on $L^2_a(\mathbf{B})$. Write $K_w(z) = (1 - \langle z, w\rangle)^{-n-1}$, the reproducing kernel for the Bergman space $L^2_a(\mathbf{B})$. Then straightforward verification gives us the formula

$$T_\mu = \int_M K_w \otimes K_w d\mu(w) \tag{2.4}$$

on $L^2_a(\mathbf{B})$. By regarding each K_w as an element in $L^2(\mathbf{B})$, this automatically extends the Toeplitz operator T_μ to an operator on the big space $L^2(\mathbf{B})$, by exactly the same integral formula! Moreover, if we consider both Q and T_μ as operators on $L^2(\mathbf{B})$, then operator inequality (2.3) holds on the big space $L^2(\mathbf{B})$.

Next, we review the operator ideals that will be involved in this paper. Given an operator A, let $s_1(A), \dots, s_k(A), \dots$ denote its s-numbers [13]. As usual, for $1 \le p < \infty$, we write $\mathcal{C}_p$ for the Schatten p-class. That is, $\mathcal{C}_p = \{A : \|A\|_p < \infty\}$, where $\|A\|_p = (\sum_{k=1}^\infty \{s_k(A)\}^p)^{1/p}$.

Compared to the Schatten classes, there is another family of ideals that are more user-friendly. These are the ideals $\mathcal{C}_p^+$, which are defined as follows. For each $1 \le p < \infty$, the formula

$$\|A\|_p^+ = \sup_{k \ge 1} \frac{s_1(A) + s_2(A) + \cdots + s_k(A)}{1^{-1/p} + 2^{-1/p} + \cdots + k^{-1/p}}$$

defines a symmetric norm for operators. On a Hilbert space $\mathcal{H}$, the set

$$\mathcal{C}_p^+ = \{A \in \mathcal{B}(\mathcal{H}) : \|A\|_p^+ < \infty\}$$

is a norm ideal. See Sections III.2 and III.14 in [13]. It is well known that $\mathcal{C}_p^+$ contains the Schatten class $\mathcal{C}_p$ and that $\mathcal{C}_p^+ \ne \mathcal{C}_p$. Moreover, we have $\mathcal{C}_p^+ \subset \mathcal{C}_{p'}$ for all $1 \le p < p' < \infty$.

Lemma 2.6. [18, Lemma 2.8] *Suppose T is in the weak operator closure of a set of operators $\{T_\alpha\}_{\alpha \in I}$. Assume $T_\alpha \in \mathcal{C}_p^+$ and*

$$\sup_{\alpha \in I} \|T_\alpha\|_p^+ \le C < \infty.$$

Then $T \in \mathcal{C}_p^+$ and $\|T\|_p^+ \le C$.

The reason why the $\mathcal{C}_p^+$'s are the preferred ideals in the study of the Arveson-Douglas conjecture is that norm estimates in these ideals are particularly easy:

Lemma 2.7. [18, Lemma 2.9] *Given any positive numbers $0 < a \le b < \infty$, there is a constant $0 < B(a, b) < \infty$ such that the following holds true: Let $\mathcal{H}$ be a Hilbert space, and suppose that $F_0, F_1, \dots, F_k, \dots$ are operators on $\mathcal{H}$ such that the following two conditions are satisfied for every k:*

(1) $\|F_k\| \le 2^{-ak}$,

(2) $\mathrm{rank}(F_k) \le 2^{bk}$.
Then the operator $F = \sum_{k=0}^{\infty} F_k$ *satisfies the estimate* $\|F\|_{b/a}^{+} \le B(a,b)$. *In particular,* $F \in \mathcal{C}_{b/a}^{+}$.

Lemma 2.8. *Given any* $1 \le r < p < \infty$, *there is a constant* $C_{2.8} = C_{2.8}(r,p)$ *such that*

$$\|A\|_p \le C_{2.8}\|A\|^{(p-r)/(4p)}\big(\|A\|_r^{+}\big)^{(3p+r)/(4p)}$$

for every operator A.

Proof. Given any $1 \le r < p < \infty$, we set $s = (p+r)/2$ and $t = (p+s)/2 = (3/4)p + (1/4)r$. Then $1 \le r < s < t < p$. For any operator A and any $k \in \mathbf{N}$, it follows from the definition of $\|\cdot\|_s^{+}$ that

$$ks_k(A) \le s_1(A) + \cdots + s_k(A) \le \|A\|_s^{+}(1^{-1/s} + \cdots + k^{-1/s}) \le C_1\|A\|_s^{+}k^{1-(1/s)}.$$

Since $\|A\|_s^{+} \le \|A\|_r^{+}$, this implies

$$s_k(A) \le C_1\|A\|_r^{+}k^{-1/s}$$

for every $k \in \mathbf{N}$. Consequently,

$$\sum_{k=1}^{\infty} s_k^p(A) \le \|A\|^{p-t}\sum_{k=1}^{\infty} s_k^t(A) \le \|A\|^{p-t}\big(C_1\|A\|_r^{+}\big)^t\sum_{k=1}^{\infty} k^{-t/s} = C_2\|A\|^{p-t}\big(\|A\|_r^{+}\big)^t.$$

Raising both sides to the power $1/p$, the lemma is proved. □

We will need the following basic properties of Schatten classes.

Lemma 2.9. *Let* $A \in \mathcal{C}_{p_1}$ *and* $B \in \mathcal{C}_{p_2}$, *where* $p_1, p_2 \in [1,\infty)$. *If* $p_1p_2/(p_1+p_2) \ge 1$, *then* $AB \in \mathcal{C}_{p_1p_2/(p_1+p_2)}$ *with*

$$\|AB\|_{p_1p_2/(p_1+p_2)} \le \|A\|_{p_1}\|B\|_{p_2}.$$

If $p_1p_2/(p_1+p_2) < 1$, *then* $AB \in \mathcal{C}_1$.

Proof. By (7.9) on page 63 in [13], for every $k \in \mathbf{N}$, the inequality

$$s_1(AB) + \cdots + s_k(AB) \le s_1(A)s_1(B) + \cdots + s_k(A)s_k(B)$$

holds. By [13, Lemma III.3.1], this implies that for each $1 \le p < \infty$, we have

$$\sum_{j=1}^{\infty}\{s_j(AB)\}^p \le \sum_{j=1}^{\infty}\{s_j(A)s_j(B)\}^p.$$

Then an application of appropriate Hölder's inequality completes the proof. □

Our proof of Theorems 1.3 relies on the following vanishing principle for trace:

Lemma 2.10. [14, Lemma 1.3] *Suppose that X is a self-adjoint operator and C is a compact operator. If $[X, C]$ is in the trace class, then* $\mathrm{tr}[X, C] = 0$.

Recall the following Schatten-class memberships for commutators:

Theorem 2.11. [18, Theorem 3.1] *For each $f \in \mathbf{C}[z_1, \bar{z}_1, \ldots, z_n, \bar{z}_n]$, the commutator $[M_f, T_\mu]$ is in the Schatten class $\mathcal{C}_p$ for every $p > 2d$.*

Theorem 2.12. [18, Theorem 3.6] *For each $f \in \mathbf{C}[z_1, \bar{z}_1, \ldots, z_n, \bar{z}_n]$, the commutator $[M_f, Q]$ is in the Schatten class $\mathcal{C}_p$ for every $p > 2d$.*

As in [18], we also have to deal with double commutators:

Proposition 2.13. [18, Proposition 5.11] *For $f, g \in \mathbf{C}[z_1, \bar{z}_1, \ldots, z_n, \bar{z}_n]$, the double commutator $[M_f, [M_g, Q]]$ belongs to $\mathcal{C}^+_{2d/(1+\epsilon)}$ for every $0 < \epsilon < 1/n$.*

3. Restriction of the measure μ to subsets of M

For each natural number $m \in \mathbf{N}$, we define the subset

$$M^{(m)} = \{z \in M : 1 - 2^{-2m} \le |z| < 1\}$$

of M. Recall that the measure μ in Theorem 2.5 is concentrated on M. For each $m \in \mathbf{N}$, by restricting μ to $M^{(m)}$ and $M \backslash M^{(m)}$ we obtain two measures. That is, we define the measures $\mu^{(m)}$ and $\lambda^{(m)}$ by the formulas

$$\mu^{(m)}(E) = \mu(E \cap M^{(m)}) \quad \text{and} \quad \lambda^{(m)}(E) = \mu(E \cap \{M \backslash M^{(m)}\}) \tag{3.1}$$

for Borel sets E. We have, of course, $\mu = \mu^{(m)} + \lambda^{(m)}$ for each m. The measures $\mu^{(m)}$ and $\lambda^{(m)}$ give rise to Toeplitz operators $T_{\mu^{(m)}}$ and $T_{\lambda^{(m)}}$. More precisely, we have

$$(T_{\mu^{(m)}} f)(z) = \int \frac{f(w)}{(1 - \langle z, w\rangle)^{n+1}} d\mu^{(m)}(w) \quad \text{and}$$
$$(T_{\lambda^{(m)}} f)(z) = \int \frac{f(w)}{(1 - \langle z, w\rangle)^{n+1}} d\lambda^{(m)}(w)$$

for $f \in L^2_a(\mathbf{B})$. In this section we will focus on $T_{\mu^{(m)}}$, and in the next section we will consider the functional calculus of $T_{\lambda^{(m)}}$.

Recall that the proof of Theorem 2.11 relied on the discretization scheme in [18, Section 3]. We will now use the same scheme to deal with the commutator $[M_f, T_{\mu^{(m)}}]$ for $f \in \mathbf{C}[z_1, \bar{z}_1, \ldots, z_n, \bar{z}_n]$ and $m \in \mathbf{N}$. More specifically, we need to know how this commutator behaves as m increases. Fortunately, we can use the techniques from [18, Section 3] to find the answer.

Let β denote the Bergman metric on $\mathbf{B}$. That is,

$$\beta(z, w) = \frac{1}{2} \log \frac{1 + |\varphi_w(z)|}{1 - |\varphi_w(z)|}, \quad z, w \in \mathbf{B},$$

where φ_w is the Möbius transform of $\mathbf{B}$ [17, Section 2.2]. For each $z \in \mathbf{B}$ and each $a > 0$, we define the corresponding β-ball $D(z, a) = \{w \in \mathbf{B} : \beta(z, w) < a\}$.

We choose a subset $\mathcal{L} \subset M$ that is maximal with respect to the property that

$$D(z, 1) \cap D(w, 1) = \emptyset \quad \text{for all} \quad z \neq w \quad \text{in} \quad \mathcal{L}.$$

As in [18], we write it in the form $\mathcal{L} = \{z_i\}_{i=1}^{\infty}$. It follows from the maximality of $\mathcal{L}$ that

$$\bigcup_{i=1}^{\infty} D(z_i, 2) \supset M.$$

There exist Borel sets $\Delta_1, \Delta_2, \ldots, \Delta_i, \ldots$ in $\mathbf{B}$ satisfying the following three requirements:

(1) $D(z_i, 1) \subset \Delta_i \subset D(z_i, 2)$ for every i.

(2) $\Delta_i \cap \Delta_{i'} = \emptyset$ for $i \neq i'$.

(3) $\cup_{i=1}^{\infty} \Delta_i = \cup_{i=1}^{\infty} D(z_i, 2) \supset M$.

As usual, we write

$$k_w(z) = \frac{(1 - |w|^2)^{(n+1)/2}}{(1 - \langle z, w \rangle)^{n+1}}, \quad z, w \in \mathbf{B},$$

which is the normalized reproducing kernel for the Bergman space $L_a^2(\mathbf{B})$.

Let $m \in \mathbf{N}$. Similar to the c_i defined on bottom of page 1073 in [18], we now define

$$c_i^{(m)} = \int_{\Delta_i} (1 - |w|^2)^{-(n+1)} d\mu^{(m)}(w),$$

$i \geq 1$. As was explained in [18], because μ is a Carleson measure for $L_a^2(\mathbf{B})$, there is a constant $0 < C < \infty$ such that $c_i^{(m)} \leq C$ for all i and m.

Define $N^{(m)} = \{i \in \mathbf{N} : \mu^{(m)}(\Delta_i) \neq 0\} = \{i \in \mathbf{N} : c_i^{(m)} > 0\}$. For each $i \in N^{(m)}$, we define the measure $d\mu_i^{(m)}$ to be the restriction of the measure $(c_i^{(m)})^{-1}(1-|w|^2)^{-(n+1)}d\mu^{(m)}$ to the set Δ_i. Obviously, $\mu_i^{(m)}(\Delta_i) = 1$ if $i \in N^{(m)}$. Observe that

$$\begin{aligned} T_{\mu^{(m)}} &= \int K_w \otimes K_w d\mu^{(m)}(w) = \sum_{i=1}^{\infty} \int_{\Delta_i} K_w \otimes K_w d\mu^{(m)}(w) \\ &= \sum_{i \in N^{(m)}} c_i^{(m)} \int_{\Delta_i} k_w \otimes k_w d\mu_i^{(m)}(w). \end{aligned}$$

Since $\mu^{(m)}$ is supported on $M^{(m)}$, for each $i \in N^{(m)}$, the probability measure $\mu_i^{(m)}$ can be approximated in the weak-* topology by measures of the form $\frac{1}{k} \sum_{j=1}^{k} \delta_{w_j}$, where $w_j \in \Delta_i \cap M^{(m)}$. Therefore each operator $\int_{\Delta_i} k_w \otimes k_w d\mu_i^{(m)}(w)$ can be approximated in the weak operator topology by operators of the form

$$\frac{1}{k} \sum_{j=1}^{k} k_{w_j} \otimes k_{w_j}, \quad w_j \in \Delta_i \cap M^{(m)}.$$

Hence $T_{\mu^{(m)}}$ can be weakly approximated by operators of the form

$$\sum_{i\in F} c_i^{(m)} \frac{1}{k}\sum_{j=1}^{k} k_{w_{i,j}} \otimes k_{w_{i,j}} = \frac{1}{k}\sum_{j=1}^{k}\sum_{i\in F} c_i^{(m)} k_{w_{i,j}} \otimes k_{w_{i,j}},$$

where F is a finite subset of $N^{(m)}$, $k \in \mathbf{N}$, and $w_{i,j} \in \Delta_i \cap M^{(m)}$. We summarize the above analysis as follows, which is just [18, Lemma 3.2] restated for the present situation:

Lemma 3.1. *For each $m \in \mathbf{N}$, the Toeplitz operator $T_{\mu^{(m)}}$ is in the weak closure of the convex hull of operators of the form*

$$\sum_{i\in F} c_i^{(m)} k_{w_i} \otimes k_{w_i},$$

where F is any finite subset of $N^{(m)} = \{i \in \mathbf{N} : \mu^{(m)}(\Delta_i) \neq 0\}$, $w_i \in \Delta_i \cap M^{(m)}$ and $0 < c_i^{(m)} \le C$. Moreover, the finite bound C depends only on μ.

Proposition 3.2. *Given any $0 < \epsilon < 1/2$, there are constants $0 < C' < \infty$ and $0 < C'' < \infty$ such that the following holds true: Let $m \in \mathbf{N}$ and consider any finite subset F of $N^{(m)} = \{i \in \mathbf{N} : \mu^{(m)}(\Delta_i) \neq 0\}$. Suppose that for each $i \in F$, we have $w_i \in \Delta_i \cap M^{(m)}$ and $0 \le c_i \le C$, where C is the constant in Lemma 3.1. Define $\nu = \sum_{i\in F} c_i(1-|w_i|^2)^{n+1}\delta_{w_i}$ and*

$$T_\nu = \sum_{i\in F} c_i k_{w_i} \otimes k_{w_i}.$$

Then we have $\|[T_\nu, M_{z_j}]\|^+_{2d/(1-2\epsilon)} \le C'$ and $\|[T_\nu, M_{z_j}]\| \le C'' 2^{-(1-2\epsilon)m}$ for every $j \in \{1, \dots, n\}$.

Proof. As in [18], for each $k \ge 0$ we define

$$M_k = \{z \in M : 1 - 2^{-2k} \le |z| < 1 - 2^{-2(k+1)}\}$$

and we define ν_k to be the restriction of ν to M_k. That is,

$$\nu_k = \sum_{i\in F, w_i \in M_k} c_i(1-|w_i|^2)^{n+1}\delta_{w_i}.$$

Also, write

$$F_k = [T_{\nu_k}, M_{z_j}] = \sum_{i\in F, w_i\in M_k} c_i[k_{w_i} \otimes k_{w_i}, M_{z_j}]$$

for $k \ge 0$. Let $0 < \epsilon < 1/2$ be given. As was shown in [18] (see (3.4) and (3.5) in that paper), there are constants C_1 and C_2 such that

$$\|F_k\| \le C_1 2^{-(1-2\epsilon)k} \tag{3.2}$$

and

$$\operatorname{rank}(F_k) \le C_2 2^{2dk}$$

for every $k \geq 0$. Since $w_i \in \Delta_i \cap M^{(m)}$ for each $i \in F$, if $0 \leq k \leq m-1$, then $\nu_k = 0$ and, consequently, $T_{\nu_k} = 0$. Thus $F_0 = \cdots = F_{m-1} = 0$ and $\sum_{k=m}^{\infty} F_k = [T_\nu, M_{z_j}]$. It follows from the above two estimates and Lemma 2.7 that

$$\|[T_\nu, M_{z_j}]\|_{2d/(1-2\epsilon)}^{+} \leq C_1(1+C_2)B(1-2\epsilon, 2d).$$

Thus the constant $C' = C_1(1+C_2)B(1-2\epsilon, 2d)$ suffices for the $\|\cdot\|_{2d/(1-2\epsilon)}^{+}$-bound. The bound on $\|[T_\nu, M_{z_j}]\|$ follows from the inequality $\|[T_\nu, M_{z_j}]\| \leq \sum_{k=m}^{\infty} \|F_k\|$ and (3.2). $\square$

Proposition 3.3. *For all $2d < p < \infty$ and $f \in \mathbf{C}[z_1, \bar{z}_1, \ldots, z_n, \bar{z}_n]$, we have*

$$\lim_{m\to\infty} \|[T_{\mu^{(m)}}, M_f]\|_p = 0. \tag{3.3}$$

Proof. Given any $p \in (2d, \infty)$, we pick an $\epsilon \in (0, 1/2)$ such that $2d < 2d/(1-2\epsilon) < p$. For each $m \in \mathbf{N}$, Lemma 3.1 tells us that $T_{\mu^{(m)}}$ is in the weak closure of the convex hull of the operators T_ν given in Proposition 3.2. Therefore by Lemma 2.6 and Proposition 3.2, for each $j \in \{1, \ldots, n\}$ we have

$$\|[T_{\mu^{(m)}}, M_{z_j}]\|_{2d/(1-2\epsilon)}^{+} \leq C' \quad \text{and} \quad \|[T_{\mu^{(m)}}, M_{z_j}]\| \leq C'' 2^{-(1-2\epsilon)m}.$$

Applying Lemma 2.8 with $r = 2d/(1-2\epsilon)$, we find that

$$\|[T_{\mu^{(m)}}, M_{z_j}]\|_p \leq C_{2.8}\big(C'' 2^{-(1-2\epsilon)m}\big)^{(p-\{2d/(1-2\epsilon)\})/(4p)}\big(C'\big)^{(3p+\{2d/(1-2\epsilon)\})/(4p)}.$$

Therefore we have

$$\lim_{m\to\infty} \|[T_{\mu^{(m)}}, M_{z_j}]\|_p = 0 \tag{3.4}$$

for each $j \in \{1, \ldots, n\}$. Obviously, (3.3) follows from (3.4) and the "product rule" for commutators, $[T_{\mu^{(m)}}, AB] = [T_{\mu^{(m)}}, A]B + A[T_{\mu^{(m)}}, B]$. $\square$

4. Functional calculus

Let us recall the standard smooth functional calculus for self-adjoint operators. Suppose that $\varphi \in C_c^\infty(\mathbf{R})$. Then the Fourier inversion formula reads

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} \int \hat{\varphi}(y) e^{iyx} dy, \quad x \in \mathbf{R},$$

where $\hat{\varphi}$ is the Fourier transform of φ. Let T be a self-adjoint operator. It follows from the above formula and the spectral decomposition of T that

$$\varphi(T) = \frac{1}{\sqrt{2\pi}} \int \hat{\varphi}(y) e^{iyT} dy.$$

Thus if X is any bounded operator, then

$$[\varphi(T), X] = \frac{1}{\sqrt{2\pi}} \int \hat{\varphi}(y)[e^{iyT}, X]dy.$$

Since $[e^{iyT}, X] = (e^{iyT}Xe^{-iyT} - X)e^{iyT}$, an application of the fundamental theorem of calculus leads to

$$[\varphi(T), X] = \frac{i}{\sqrt{2\pi}} \int \hat{\varphi}(y)y \int_0^1 e^{ityT}[T, X]e^{i(1-t)yT} dtdy. \tag{4.1}$$

Recall that for each $m \in \mathbf{N}$, the measure $\lambda^{(m)}$ was given in (3.1).

Proposition 4.1. *Let $\varphi \in C_c^\infty(\mathbf{R})$. Then for every $f \in \mathbf{C}[z_1, \bar{z}_1, \ldots, z_n, \bar{z}_n]$ and every $2d < p < \infty$ we have*

$$\lim_{m\to\infty} \|[\varphi(T_\mu), M_f] - [\varphi(T_{\lambda^{(m)}}), M_f]\|_p = 0.$$

Proof. From (4.1) we obtain the decomposition

$$[\varphi(T_\mu), M_f] - [\varphi(T_{\lambda^{(m)}}), M_f] = G_m + H_m,$$

where

$$G_m = \frac{i}{\sqrt{2\pi}} \int \hat{\varphi}(y)y \int_0^1 \left(e^{ityT_\mu}[T_\mu, M_f]e^{i(1-t)yT_\mu} - e^{ityT_{\lambda^{(m)}}}[T_\mu, M_f]e^{i(1-t)yT_{\lambda^{(m)}}}\right) dtdy,$$
$$H_m = \frac{i}{\sqrt{2\pi}} \int \hat{\varphi}(y)y \int_0^1 e^{ityT_{\lambda^{(m)}}}[T_\mu - T_{\lambda^{(m)}}, M_f]e^{i(1-t)yT_{\lambda^{(m)}}} dtdy.$$

By (3.1) we have $T_\mu - T_{\lambda^{(m)}} = T_{\mu^{(m)}}$. Therefore $\|H_m\|_p \le C_1\|[T_{\mu^{(m)}}, M_f]\|_p$. Since we assume $2d < p < \infty$, it follows from Proposition 3.3 that $\|H_m\|_p \to 0$ as $m \to \infty$.

From the definition of $M^{(m)}$ it is clear that $T_{\mu^{(m)}} \to 0$ strongly as $m \to \infty$. That is,

$$\lim_{m\to\infty} T_{\lambda^{(m)}} = T_\mu \tag{4.2}$$

in the strong operator topology. This implies the strong convergence

$$\lim_{m\to\infty} e^{isT_{\lambda^{(m)}}} = e^{isT_\mu}$$

for every $s \in \mathbf{R}$. Since $[T_\mu, M_f] \in \mathcal{C}_p$ (cf. Theorem 2.11), this strong convergence leads to

$$\lim_{m\to\infty} \|e^{ityT_\mu}[T_\mu, M_f]e^{i(1-t)yT_\mu} - e^{ityT_{\lambda^{(m)}}}[T_\mu, M_f]e^{i(1-t)yT_{\lambda^{(m)}}}\|_p = 0$$

for each $y \in \mathbf{R}$ and each $t \in [0,1]$. Then there is the obvious bound

$$\|e^{ityT_\mu}[T_\mu, M_f]e^{i(1-t)yT_\mu} - e^{ityT_{\lambda^{(m)}}}[T_\mu, M_f]e^{i(1-t)yT_{\lambda^{(m)}}}\|_p \leq 2\|[T_\mu, M_f]\|_p.$$

Since

$$\|G_m\|_p \leq \int |\hat{\varphi}(y)y| \int_0^1 \|e^{ityT_\mu}[T_\mu, M_f]e^{i(1-t)yT_\mu} - e^{ityT_{\lambda^{(m)}}}[T_\mu, M_f]e^{i(1-t)yT_{\lambda^{(m)}}}\|_p dtdy,$$

it now follows from the dominated convergence theorem that $\|G_m\|_p \to 0$ as $m \to \infty$. This completes the proof. □

For the rest of the paper, c and C will denote the two scalars in (2.3). It follows from (2.3) that the spectrum of T_μ is contained in $\{0\} \cup [c, C]$, and that the spectral projection of T_μ corresponding to the interval $[c, C]$ equals Q. Thus we can write $Q = h(T_\mu)$ for some smooth function h. Obviously, there are many smooth functions h on $\mathbf{R}$ that achieve the same result, $h(T_\mu) = Q$. But for the purpose of this paper, a casually picked h will not do; we must choose our h carefully.

To find the desired h, we begin with a C^∞ function η on $\mathbf{R}$ satisfying the following three conditions:

(1) $0 \leq \eta \leq 1$ on $\mathbf{R}$.

(2) $\eta = 1$ on $(-\infty, c/3] \cup [C+2, \infty)$.

(3) $\eta = 0$ on $[c/2, C+1]$.

With this η so chosen, we define

$$h = 1 - \eta^2. \tag{4.3}$$

Obviously, the C^∞ function h has the following properties:

(α) $0 \leq h \leq 1$ on $\mathbf{R}$.

(β) $h = 1$ on $[c/2, C+1]$.

(γ) $h = 0$ on $(-\infty, c/3] \cup [C+2, \infty)$.

Combining (β) and (γ) with (2.3), we have $h(T_\mu) = Q$. Furthermore, (γ) tells us that $h \in C_c^\infty(\mathbf{R})$. The above care was taken so that the following holds true:

Lemma 4.2. *For the h defined above, we have $(1-h^2)^{1/2}h \in C_c^\infty(\mathbf{R})$.*

Proof. By (γ), what needs to be proved is that $(1-h^2)^{1/2}$ is a C^∞ function on $\mathbf{R}$. But $1 - h^2 = (1+h)(1-h) = (1+h)\eta^2$. Hence $(1-h^2)^{1/2} = (1+h)^{1/2}\eta$, whose smoothness is ensured by the fact that $h \geq 0$ on $\mathbf{R}$. □

With the function h given by (4.3), we now define

$$A_m = h(T_{\lambda^{(m)}}), \tag{4.4}$$

$m \in \mathbf{N}$. Since the measure $\lambda^{(m)}$ is concentrated on $M \backslash M^{(m)}$, the Toeplitz operator $T_{\lambda^{(m)}}$ is compact. Since $h = 0$ on $(-\infty, c/3]$, we conclude that

$$\operatorname{rank}(A_m) < \infty \tag{4.5}$$

for every $m \in \mathbf{N}$. We have $0 \le T_{\lambda^{(m)}} \le T_\mu \le CQ$. Since h vanishes on a neighborhood of 0, $h(T_{\lambda^{(m)}})$ is the limit in operator norm of operators of the form $T_{\lambda^{(m)}} q(T_{\lambda^{(m)}})$, where q are polynomials. Hence for each $m \in \mathbf{N}$, the range of A_m is contained in the quotient module $\mathcal{Q} = QL_a^2(\mathbf{B})$. In particular, we have $A_m Q = A_m = QA_m$ and $A_m R = 0 = RA_m$. These facts and (4.5) will be important for the next two sections.

Since the range of A_m is contained in $\mathcal{Q}$ and since $0 \le h \le 1$ on $\mathbf{R}$, we have the operator inequality

$$0 \le A_m \le Q \tag{4.6}$$

for every $m \in \mathbf{N}$. Thus we have a sequence of positive, finite-rank operators $\{A_m\}_{m=1}^\infty$ that approximate the projection Q in the sense of Proposition 4.1.

5. Projections galore

If the finite-rank operators A_m defined by (4.4) were *projections*, then the proof of Theorem 1.3 would be much more straightforward. Instead, the best we can construct are the finite-rank positive contractions A_m that approximate Q from below. It is even possible that there are *obstructions* to the existence of finite-rank projections that satisfy (4.6) and still approximate Q in the sense of Proposition 4.1. In any case, we do not have at our disposal the kind of finite-rank projections on $\mathcal{Q}$ that we wish we had.

Then, came the idea that, perhaps, we can take the operators that are at our disposal and *dilate* them to projections on a bigger space. The hope is that that can also lead to a proof of Theorem 1.3. As luck would have it, this idea works!

First of all, it is easy to dilate a positive contraction to an orthogonal projection. Observe that for any $0 \le x \le 1$, the 2×2 matrix

$$\begin{bmatrix} x^2 & (1-x^2)^{1/2}x \\ (1-x^2)^{1/2}x & 1-x^2 \end{bmatrix}$$

is an idempotent. As mentioned earlier, $QA_m = A_m = A_m Q$ for every $m \in \mathbf{N}$. Thus by the spectral decomposition of A_m, the operator

$$Q^{(m)} = \begin{bmatrix} A_m^2 & (1-A_m^2)^{1/2}A_m \\ (1-A_m^2)^{1/2}A_m & Q - A_m^2 \end{bmatrix}$$

is an orthogonal projection on $L^2(\mathbf{B}) \oplus L^2(\mathbf{B})$. Alternately, we can write

$$Q^{(m)} = \begin{bmatrix} h^2(T_{\lambda^{(m)}}) & \psi(T_{\lambda^{(m)}}) \\ \psi(T_{\lambda^{(m)}}) & Q - h^2(T_{\lambda^{(m)}}) \end{bmatrix}, \tag{5.1}$$

where $\psi = (1-h^2)^{1/2}h$, which is in $C_c^\infty(\mathbf{R})$ according to Lemma 4.2.

Our approach involves more projections. On $L^2(\mathbf{B}) \oplus L^2(\mathbf{B})$, we define

$$R' = \begin{bmatrix} R & 0 \\ 0 & 0 \end{bmatrix}, \quad Q' = \begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix}, \quad P' = \begin{bmatrix} P & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad Q'' = \begin{bmatrix} 0 & 0 \\ 0 & Q \end{bmatrix}.$$

Since $P = R + Q$, we have $P' = R' + Q'$. Note that the ranges of R' and $Q^{(m)}$ are orthogonal to each other. Thus for each $m \in \mathbf{N}$, we define the orthogonal projection

$$P^{(m)} = R' + Q^{(m)}.$$

For each $f \in \mathbf{C}[z_1, \bar{z}_1, \ldots, z_n, \bar{z}_n]$, we define

$$D_f = \begin{bmatrix} M_f & 0 \\ 0 & M_f \end{bmatrix}$$

on $L^2(\mathbf{B}) \oplus L^2(\mathbf{B})$. We can alternatively write $R' = R \oplus 0$, $Q' = Q \oplus 0$, $P' = P \oplus 0$, $Q'' = 0 \oplus Q$ and $D_f = M_f \oplus M_f$. Since $\operatorname{rank}(A_m) < \infty$, we have

$$Q^{(m)} = Q'' + L_m \quad \text{with} \quad \operatorname{rank}(L_m) < \infty. \tag{5.2}$$

Lemma 5.1. *For all $m \in \mathbf{N}$ and $f \in \mathbf{C}[z_1, \bar{z}_1, \ldots, z_n, \bar{z}_n]$, the ranks of $R'D_fQ^{(m)}$ and $Q^{(m)}D_fR'$ are finite.*

Proof. It is obvious that $R'D_fQ'' = 0$ and $Q''D_fR' = 0$ for every $f \in \mathbf{C}[z_1, \bar{z}_1, \ldots, z_n, \bar{z}_n]$. The lemma follows from this fact and (5.2). □

Lemma 5.2. (1) *For all $m \in \mathbf{N}$ and $f \in \mathbf{C}[z_1, \bar{z}_1, \ldots, z_n, \bar{z}_n]$, we have $[D_f, Q^{(m)}] \in \mathcal{C}_p$ for every $p > 2d$.*
(2) *For all $m \in \mathbf{N}$ and $f, g \in \mathbf{C}[z_1, \bar{z}_1, \ldots, z_n, \bar{z}_n]$, we have $[D_f, [D_g, Q^{(m)}]] \in \mathcal{C}^+_{2d/(1+\epsilon)}$ for every $0 < \epsilon < 1/n$.*

Proof. By (5.2), $[D_f, Q^{(m)}]$ is a finite-rank perturbation of $[D_f, Q''] = 0 \oplus [M_f, Q]$. Conclusion (1) follows from this fact and Theorem 2.12. Similarly, (5.2) tells us that $[D_f, [D_g, Q^{(m)}]]$ is a finite-rank perturbation of $[D_f, [D_g, Q'']] = 0 \oplus [M_f, [M_g, Q]]$. Conclusion (2) follows from this fact and Proposition 2.13. □

Proposition 5.3. *For all $f \in \mathbf{C}[z_1, \bar{z}_1, \ldots, z_n, \bar{z}_n]$ and $2d < p < \infty$, we have*

$$\lim_{m\to\infty} \|[Q', D_f] - [Q^{(m)}, D_f]\|_p = 0 \quad \textit{and} \quad \lim_{m\to\infty} \|[P', D_f] - [P^{(m)}, D_f]\|_p = 0.$$

Proof. By Lemma 4.2, the function $\psi = (1 - h^2)^{1/2}h$ is in $C_c^\infty(\mathbf{R})$. Using (5.1), it follows from Proposition 4.1 that if $f \in \mathbf{C}[z_1, \bar{z}_1, \ldots, z_n, \bar{z}_n]$ and $2d < p < \infty$, then

$$\lim_{m\to\infty} \|[X, D_f] - [Q^{(m)}, D_f]\|_p = 0,$$

where

$$X = \begin{bmatrix} h^2(T_\mu) & \psi(T_\mu) \\ \psi(T_\mu) & Q - h^2(T_\mu) \end{bmatrix}. \tag{5.3}$$

Since $h(T_\mu) = Q$ and $Q^2 = Q$, we have $h^2(T_\mu) = Q$. By (β) and (γ) in Section 4, the function $\psi = (1-h^2)^{1/2}h$ vanishes on $(-\infty, c/3] \cup [c/2, C+1]$. As we explained in Section 4, the spectrum of T_μ is contained in $\{0\} \cup [c, C]$. Hence $\psi(T_\mu) = 0$. That is, $X = Q'$, which proves the first limit. For the second limit, observe that $P' - P^{(m)} = Q' - Q^{(m)}$. $\square$

Lemma 5.4. *We have*

$$\lim_{m\to\infty} Q^{(m)} = Q'$$

in the strong operator topology.

Proof. By (4.2) and (5.1), we have the strong limit

$$\lim_{m\to\infty} Q^{(m)} = X,$$

where X is given by (5.3). But we showed in the preceding proof that $X = Q'$. $\square$

With the projections P', R', $P^{(m)}$ and $Q^{(m)}$, we now define more "Toeplitz operators". That is, given any $f \in \mathbf{C}[z_1, \bar{z}_1, \ldots, z_n, \bar{z}_n]$, we define

$$T'_f = P'D_fP', \quad R'_f = R'D_fR', \quad T_f^{(m)} = P^{(m)}D_fP^{(m)} \quad \text{and} \quad Q_f^{(m)} = Q^{(m)}D_fQ^{(m)},$$

$m \in \mathbf{N}$. It is obvious that $T'_f = T_f \oplus 0$ and $R'_f = R_f \oplus 0$. We think of $Q_f^{(m)}$ and $T_f^{(m)}$ as "deformed versions" of $Q_f \oplus 0$ and $T_f \oplus 0$ respectively.

Proposition 5.5. *For all $f, g \in \mathbf{C}[z_1, \bar{z}_1, \ldots, z_n, \bar{z}_n]$ and $2nd/(n+d) < p < \infty$, we have*

$$\lim_{m\to\infty} \|[T'_f, T'_g] - [T_f^{(m)}, T_g^{(m)}]\|_p = 0.$$

Proof. By elementary algebra,

$$\begin{aligned} [T'_f, T'_g] &= P'D_g(1-P')D_fP' - P'D_f(1-P')D_gP' \\ &= [P', D_f](1-P')[P', D_g] - [P', D_g](1-P')[P', D_f]. \end{aligned} \tag{5.4}$$

Similarly,

$$[T_f^{(m)}, T_g^{(m)}] = [P^{(m)}, D_f](1-P^{(m)})[P^{(m)}, D_g] - [P^{(m)}, D_g](1-P^{(m)})[P^{(m)}, D_f].$$

Thus it suffices to prove that for $2nd/(n+d) < p < \infty$ and $f, g \in \mathbf{C}[z_1, \bar{z}_1, \ldots, z_n, \bar{z}_n]$,

$$\lim_{m\to\infty} \|[P', D_f](1-P')[P', D_g] - [P^{(m)}, D_f](1-P^{(m)})[P^{(m)}, D_g]\|_p = 0. \tag{5.5}$$

We will prove this through a number of reduction steps.

First, observe that since $p > 2nd/(n+d)$, there are $s > 2d$ and $t > 2n$ such that $st/(s+t) < p$. We further require that $s < t$. By Proposition 5.3, we have

$$\lim_{m\to\infty} \|[P', D_f] - [P^{(m)}, D_f]\|_s = 0. \tag{5.6}$$

It is well known that $[P, M_g] \in \mathcal{C}_t$. Thus $[P', D_g] = [P, M_g] \oplus 0 \in \mathcal{C}_t$. Combining (5.6) with Lemma 2.9, we obtain

$$\lim_{m\to\infty} \|[P', D_f](1 - P^{(m)})[P', D_g] - [P^{(m)}, D_f](1 - P^{(m)})[P', D_g]\|_{st/(s+t)} = 0. \tag{5.7}$$

Since $t > s$, it follows from (5.6) that there is a C_1 such that $\|[P^{(m)}, D_f]\|_t \le C_1$ for every $m \in \mathbf{N}$. Replacing f by g in (5.6) and applying Lemma 2.9 again, we have

$$\lim_{m\to\infty} \|[P^{(m)}, D_f](1 - P^{(m)})[P', D_g] - [P^{(m)}, D_f](1 - P^{(m)})[P^{(m)}, D_g]\|_{st/(s+t)} = 0. \tag{5.8}$$

Recall that $P' - P^{(m)} = Q' - Q^{(m)}$. Since $p > st/(s+t)$, with (5.7) and (5.8) already established, we see that (5.5) will follow if we can show that

$$\lim_{m\to\infty} \|[P', D_f](Q' - Q^{(m)})[P', D_g]\|_{st/(s+t)} = 0. \tag{5.9}$$

But by the membership $[P', D_g] = [P, M_g] \oplus 0 \in \mathcal{C}_t$ and Lemma 2.9, the proof of (5.9) is further reduced to that of

$$\lim_{m\to\infty} \|[P', D_f](Q' - Q^{(m)})\|_s = 0. \tag{5.10}$$

To prove this, note that $[Q', D_f] = [Q, M_f] \oplus 0 \in \mathcal{C}_s$ (cf. Theorem 2.12). By this membership and Lemma 5.4, we have

$$\lim_{m\to\infty} \|[Q', D_f](Q' - Q^{(m)})\|_s = 0. \tag{5.11}$$

On the other hand, since $R'Q' = 0$ and $R'Q^{(m)} = 0$, we have $[R', D_f](Q' - Q^{(m)}) = R'D_f(Q' - Q^{(m)}) = R'[D_f, Q' - Q^{(m)}]$. Therefore from Proposition 5.3 we obtain

$$\lim_{m\to\infty} \|[R', D_f](Q' - Q^{(m)})\|_s = \lim_{m\to\infty} \|R'[D_f, Q' - Q^{(m)}]\|_s = 0. \tag{5.12}$$

Since $P' = R' + Q'$, (5.10) follows from (5.11) and (5.12). This completes the proof. □

Proposition 5.6. *If $f_1, \dots, f_n, g_1, \dots, g_n \in \mathbf{C}[z_1, \bar{z}_1, \dots, z_n, \bar{z}_n]$, then*

$$\lim_{m\to\infty} \|[T'_{f_1}, T'_{g_1}] \cdots [T'_{f_n}, T'_{g_n}] - [T^{(m)}_{f_1}, T^{(m)}_{g_1}] \cdots [T^{(m)}_{f_n}, T^{(m)}_{g_n}]\|_1 = 0. \tag{5.13}$$

Proof. Since $d \le n-1$, we have $2dn/(n+d) < n$. This allows us to pick a p satisfying the condition

$$2dn/(n+d) < p < n.$$

Since $\{1/p\} + \{(n-1)/n\} > 1$, we can pick an $\epsilon > 0$ such that

$$\frac{1}{p} + \frac{n-1}{n+\epsilon} > 1. \tag{5.14}$$

Let $f, \ldots, f_n, g_1, \ldots, g_n \in \mathbf{C}[z_1, \bar{z}_1, \ldots, z_n, \bar{z}_n]$ be given. By Proposition 5.5, we have

$$\lim_{m\to\infty} \|[T'_{f_j}, T'_{g_j}] - [T^{(m)}_{f_j}, T^{(m)}_{g_j}]\|_p = 0, \tag{5.15}$$

$1 \le j \le n$. It is well known that $[T_{f_j}, T_{g_j}] \in \mathcal{C}_{n+\epsilon}$. Thus $[T'_{f_j}, T'_{g_j}] = [T_{f_j}, T_{g_j}] \oplus 0 \in \mathcal{C}_{n+\epsilon}$. By this membership, it now follows from (5.15), (5.14) and Lemma 2.9 that

$$\lim_{m\to\infty} \|([T'_{f_1}, T'_{g_1}] - [T^{(m)}_{f_1}, T^{(m)}_{g_1}])[T'_{f_2}, T'_{g_2}] \cdots [T'_{f_n}, T'_{g_n}]\|_1 = 0. \tag{5.16}$$

Since $p < n < n+\epsilon$, from (5.15) we obtain a constant C_1 such that $\|[T^{(m)}_{f_j}, T^{(m)}_{g_j}]\|_{n+\epsilon} \le C_1$ for all $m \in \mathbf{N}$ and $1 \le j \le n$. Combining this bound with the membership $[T'_{f_j}, T'_{g_j}] \in \mathcal{C}_{n+\epsilon}$ and with (5.15), (5.14) and Lemma 2.9, we have

$$\begin{aligned}\lim_{m\to\infty} \|[T^{(m)}_{f_1}, T^{(m)}_{g_1}] \cdots [T^{(m)}_{f_{i-1}}, T^{(m)}_{g_{i-1}}]([T'_{f_i}, T'_{g_i}] - [T^{(m)}_{f_i}, T^{(m)}_{g_i}])[T'_{f_{i+1}}, T'_{g_{i+1}}] \cdots [T'_{f_n}, T'_{g_n}]\|_1 \\ = 0\end{aligned} \tag{5.17}$$

if $2 \le i \le n-1$ and

$$\lim_{m\to\infty} \|[T^{(m)}_{f_1}, T^{(m)}_{g_1}] \cdots [T^{(m)}_{f_{n-1}}, T^{(m)}_{g_{n-1}}]([T'_{f_n}, T'_{g_n}] - [T^{(m)}_{f_n}, T^{(m)}_{g_n}])\|_1 = 0. \tag{5.18}$$

By an obvious telescoping sum, (5.13) follows from (5.16), (5.17) and (5.18). □

6. Antisymmetric sums

We will now consider antisymmetric sums that are made of the "Toeplitz operators" defined in Section 5. First, we have the following limit with respect to the norm of the trace class:

Proposition 6.1. *For any $f_1, f_2, \ldots, f_{2n} \in \mathbf{C}[z_1, \bar{z}_1, \ldots, z_n, \bar{z}_n]$, we have*

$$\lim_{m\to\infty} \|[T'_{f_1}, T'_{f_2}, \ldots, T'_{f_{2n}}] - [T^{(m)}_{f_1}, T^{(m)}_{f_2}, \ldots, T^{(m)}_{f_{2n}}]\|_1 = 0. \tag{6.1}$$

Proof. For each $1 \le j \le n$, let $\tau_j : \{1, 2, \ldots, 2n\} \to \{1, 2, \ldots, 2n\}$ be the transposition such that $\tau_j(2j-1) = 2j$, $\tau_j(2j) = 2j-1$ and $\tau_j(k) = k$ for every $k \in \{1, 2, \ldots, 2n\} \backslash \{2j-1, 2j\}$. Let T_{2n} be the subgroup of S_{2n} generated by $\tau_1, \ldots, \tau_n$. Then there is a subset E_{2n} of S_{2n}

such that $S_{2n} = \cup_{\lambda \in E_{2n}} \lambda T_{2n}$ and such that $\lambda T_{2n} \cap \lambda' T_{2n} = \emptyset$ for all $\lambda \neq \lambda'$ in E_{2n}. Given $f_1, f_2, \dots, f_{2n} \in \mathbf{C}[z_1, \bar{z}_1, \dots, z_n, \bar{z}_n]$, we have

$$\begin{aligned}[T'_{f_1}, T'_{f_2}, \dots, T'_{f_{2n}}] &= \sum_{\lambda \in E_{2n}} \operatorname{sgn}(\lambda) \sum_{\sigma \in T_{2n}} \operatorname{sgn}(\sigma) T'_{f_{\lambda(\sigma(1))}} T'_{f_{\lambda(\sigma(2))}} \cdots T'_{f_{\lambda(\sigma(2n))}} \\ &= \sum_{\lambda \in E_{2n}} \operatorname{sgn}(\lambda)[T'_{f_{\lambda(1)}}, T'_{f_{\lambda(2)}}] \cdots [T'_{f_{\lambda(2n-1)}}, T'_{f_{\lambda(2n)}}].\end{aligned}$$

Similarly,

$$[T^{(m)}_{f_1}, T^{(m)}_{f_2}, \dots, T^{(m)}_{f_{2n}}] = \sum_{\lambda \in E_{2n}} \operatorname{sgn}(\lambda)[T^{(m)}_{f_{\lambda(1)}}, T^{(m)}_{f_{\lambda(2)}}] \cdots [T^{(m)}_{f_{\lambda(2n-1)}}, T^{(m)}_{f_{\lambda(2n)}}]$$

for each $m \in \mathbf{N}$. With these identities, (6.1) follows from Proposition 5.6. □

Corollary 6.2. *For any $f_1, f_2, \dots, f_{2n} \in \mathbf{C}[z_1, \bar{z}_1, \dots, z_n, \bar{z}_n]$, we have*

$$\lim_{m \to \infty} \operatorname{tr}[T^{(m)}_{f_1}, T^{(m)}_{f_2}, \dots, T^{(m)}_{f_{2n}}] = \operatorname{tr}[T_{f_1}, T_{f_2}, \dots, T_{f_{2n}}].$$

Proof. Applying Proposition 6.1, we have

$$\begin{aligned}\lim_{m \to \infty} \operatorname{tr}[T^{(m)}_{f_1}, T^{(m)}_{f_2}, \dots, T^{(m)}_{f_{2n}}] = \operatorname{tr}[T'_{f_1}, T'_{f_2}, \dots, T'_{f_{2n}}] &= \operatorname{tr}([T_{f_1}, T_{f_2}, \dots, T_{f_{2n}}] \oplus 0) \\ &= \operatorname{tr}[T_{f_1}, T_{f_2}, \dots, T_{f_{2n}}].\end{aligned}$$

□

Lemma 6.3. (a) *If $p > d$, then $[Q^{(m)}_f, Q^{(m)}_g] \in \mathcal{C}_p$ for all $f, g \in \mathbf{C}[z_1, \bar{z}_1, \dots, z_n, \bar{z}_n]$ and $m \in \mathbf{N}$.*
(b) *Suppose that $d \geq 2$. If $p > 2nd/(2n+1)$, then $[Q^{(m)}_h, [Q^{(m)}_f, Q^{(m)}_g]] \in \mathcal{C}_p$ for all $f, g, h \in \mathbf{C}[z_1, \bar{z}_1, \dots, z_n, \bar{z}_n]$ and $m \in \mathbf{N}$.*
(c) *Suppose that $d = 1$. Then the double commutator $[Q^{(m)}_h, [Q^{(m)}_f, Q^{(m)}_g]]$ is in the trace class for all $f, g, h \in \mathbf{C}[z_1, \bar{z}_1, \dots, z_n, \bar{z}_n]$ and $m \in \mathbf{N}$.*

Proof. The elementary algebra that gave us (5.4), now gives us

$$[Q^{(m)}_f, Q^{(m)}_g] = [Q^{(m)}, D_f](1 - Q^{(m)})[Q^{(m)}, D_g] - [Q^{(m)}, D_g](1 - Q^{(m)})[Q^{(m)}, D_f].$$

Conclusion (a) follows from this identity and Lemmas 5.2(1) and 2.9. Then note that

$$\begin{aligned}[Q^{(m)}_h, [Q^{(m)}_f, Q^{(m)}_g]] &= Q^{(m)}[D_h, [Q^{(m)}_f, Q^{(m)}_g]]Q^{(m)} \\ &= Q^{(m)}[D_h, [Q^{(m)}, D_f](1 - Q^{(m)})[Q^{(m)}, D_g]]Q^{(m)} \\ &- Q^{(m)}[D_h, [Q^{(m)}, D_g](1 - Q^{(m)})[Q^{(m)}, D_f]]Q^{(m)}.\end{aligned}$$

Combining this identity with the "product rule" form commutators, conclusions (b) and (c) now follow from Lemmas 5.2(2), 5.2(1) and 2.9. □

Proposition 6.4. *Let $\nu \geq d$. Then for all $f, g, f_1, f_2, \dots, f_{2\nu} \in \mathbf{C}[z_1, \bar{z}_1, \dots, z_n, \bar{z}_n]$ and $m \in \mathbf{N}$, the operator*

$$[Q_f^{(m)}, Q_g^{(m)}[Q_{f_1}^{(m)}, Q_{f_2}^{(m)}, \dots, Q_{f_{2\nu}}^{(m)}]]$$

is in the trace class with zero trace.

Proof. For convenience, denote

$$Y = [Q_{f_1}^{(m)}, Q_{f_2}^{(m)}, \dots, Q_{f_{2\nu}}^{(m)}].$$

As we saw in the proof of Proposition 6.1, there is a subset $E_{2\nu}$ of $S_{2\nu}$ such that

$$Y = \sum_{\lambda \in E_{2\nu}} \mathrm{sgn}(\lambda)[Q_{f_{\lambda(1)}}^{(m)}, Q_{f_{\lambda(2)}}^{(m)}] \cdots [Q_{f_{\lambda(2\nu-1)}}^{(m)}, Q_{f_{\lambda(2\nu)}}^{(m)}]. \tag{6.2}$$

Since $\nu \geq d$, it follows from Lemma 6.3(a) and Lemma 2.9 that $Y \in \mathcal{C}_p$ for every $p > 1$. Lemma 6.3(a) also tells us that $[Q_f^{(m)}, Q_g^{(m)}] \in \mathcal{C}_{d+\epsilon}$ if $\epsilon > 0$. Hence $[Q_f^{(m)}, Q_g^{(m)}]Y \in \mathcal{C}_1$.

Next we show that $[Q_f^{(m)}, Y] \in \mathcal{C}_1$. If $d = 1$, then this is a direct consequence of (6.2) and Lemma 6.3(c). Suppose that $d \geq 2$. In this case, Lemma 6.3(b) tells us that $[Q_f^{(m)}, [Q_{f_{\lambda(2i-1)}}^{(m)}, Q_{f_{\lambda(2i)}}^{(m)}]] \in \mathcal{C}_p$ for every $p > 2nd/(2n+1)$, where $1 \leq i \leq \nu$ and $\lambda \in E_{2\nu}$. Since $2nd/(2n+1) < d$ and since for every $j \neq i$ we have $[Q_{f_{\lambda(2j-1)}}^{(m)}, Q_{f_{\lambda(2j)}}^{(m)}] \in \mathcal{C}_{d+\epsilon}$ for every $\epsilon > 0$, it follows that $[Q_f^{(m)}, Y] \in \mathcal{C}_1$.

From the last two paragraphs we obtain the membership $[Q_f^{(m)}, Q_g^{(m)}Y] \in \mathcal{C}_1$. Similarly, $[(Q_f^{(m)})^*, Q_g^{(m)}Y] = [Q_{\bar{f}}^{(m)}, Q_g^{(m)}Y] \in \mathcal{C}_1$ since $\bar{f}$ is also in $\mathbf{C}[z_1, \bar{z}_1, \dots, z_n, \bar{z}_n]$. Since Y is compact, it follows from Lemma 2.10 that

$$\mathrm{tr}[Q_f^{(m)} + (Q_f^{(m)})^*, Q_g^{(m)}Y] = 0 = \mathrm{tr}[Q_f^{(m)} - (Q_f^{(m)})^*, Q_g^{(m)}Y].$$

From this we obtain $\mathrm{tr}[Q_f^{(m)}, Q_g^{(m)}Y] = 0$ as promised. □

Thus we have the following analogue of [18, Theorem 1.8]:

Proposition 6.5. *Let $\nu \geq d$. Then for all $f_1, f_2, \dots, f_{2\nu+1}, f_{2\nu+2} \in \mathbf{C}[z_1, \bar{z}_1, \dots, z_n, \bar{z}_n]$ and $m \in \mathbf{N}$, the antisymmetric sum*

$$[Q_{f_1}^{(m)}, Q_{f_2}^{(m)}, \dots, Q_{f_{2\nu+1}}^{(m)}, Q_{f_{2\nu+2}}^{(m)}] \tag{6.3}$$

is in the trace class with zero trace.

Proof. Since $2\nu + 2$ is even, [14, Proposition 1.1] tells us that (6.3) is a linear combination of terms of the form

$$[Q_{f_{\sigma(1)}}^{(m)}, Q_{f_{\sigma(2)}}^{(m)}[Q_{f_{\sigma(3)}}^{(m)}, Q_{f_{\sigma(4)}}^{(m)}, \cdots, Q_{f_{\sigma(2\nu+1)}}^{(m)}, Q_{f_{\sigma(2\nu+2)}}^{(m)}]],$$

where σ runs over a certain subset of the symmetric group $S_{2\nu+2}$. Thus this proposition is a direct consequence of Proposition 6.4. $\square$

Definition 6.6. Given any $m \in \mathbf{N}$, we let $\mathcal{Z}^{(m)}$ denote the collection of $(2n+1)$-tuples $(X_0, \dots, X_{2n})$ satisfying the following two conditions:

(1) For each $j \in \{0, 1, \dots, 2n\}$, X_j is either R' or $Q^{(m)}$.

(2) For each $(X_0, \dots, X_{2n})$, there is at least one $i \in \{0, 1, \dots, 2n\}$ such that $X_i = R'$ and at least one $j \in \{0, 1, \dots, 2n\}$ such that $X_j = Q^{(m)}$.

Lemma 6.7. *Let* $(X_0, \dots, X_{2n}) \in \mathcal{Z}^{(m)}$, $m \in \mathbf{N}$, *and* $f_1, f_2, \dots, f_{2n} \in \mathbf{C}[z_1, \bar{z}_1, \dots, z_n, \bar{z}_n]$ *be given. Then the following hold true*:

(a) *The rank of the operator* $X_0 D_{f_1} X_1 D_{f_2} X_2 \cdots X_{2n-1} D_{f_{2n}} X_{2n}$ *is finite.*

(b) *If* $X_0 = X_{2n}$, *then* $(X_1, X_2, \dots, X_{2n}, X_1) \in \mathcal{Z}^{(m)}$ *and*

$$\operatorname{tr}(X_0 D_{f_1} X_1 D_{f_2} X_2 \cdots X_{2n-1} D_{f_{2n}} X_{2n}) = \operatorname{tr}(X_1 D_{f_2} X_2 \cdots X_{2n-1} D_{f_{2n}} X_{2n} D_{f_1} X_1).$$

(c) *If* $X_0 \neq X_{2n}$, *then* $\operatorname{tr}(X_0 D_{f_1} X_1 D_{f_2} X_2 \cdots X_{2n-1} D_{f_{2n}} X_{2n}) = 0$.

Proof. (a) By (2) in Definition 6.6, there is a $0 \le j \le 2n-1$ such that $X_j \neq X_{j+1}$. Thus by (1) in Definition 6.6, we have either $X_j D_{f_{j+1}} X_{j+1} = R' D_{f_{j+1}} Q^{(m)}$ or $X_j D_{f_{j+1}} X_{j+1} = Q^{(m)} D_{f_{j+1}} R'$. In either case, Lemma 5.1 tells us that $\operatorname{rank}(X_j D_{f_{j+1}} X_{j+1}) < \infty$.

(b) Suppose that $X_0 = X_{2n}$. Then by (2) in Definition 6.6, there is a $1 \le j \le 2n-1$ such that $X_j \neq X_{j+1}$. Hence $(X_1, X_2, \dots, X_{2n}, X_1) \in \mathcal{Z}^{(m)}$. Define

$$A = X_0 D_{f_1} X_1 \quad \text{and} \quad B = X_1 D_{f_2} X_2 \cdots X_{2n-1} D_{f_{2n}} X_{2n}.$$

By the argument in (a), we have $\operatorname{rank}(B) < \infty$. Hence $\operatorname{tr}(AB) = \operatorname{tr}(BA)$. Since

$$\begin{aligned} &X_0 D_{f_1} X_1 D_{f_2} X_2 \cdots X_{2n-1} D_{f_{2n}} X_{2n} = AB \quad \text{whereas} \\ &X_1 D_{f_2} X_2 \cdots X_{2n-1} D_{f_{2n}} X_{2n} D_{f_1} X_1 = BA, \end{aligned}$$

the conclusion follows.

(c) Suppose that $X_0 \neq X_{2n}$. Then X_0 and X_{2n} are orthogonal projections with the property $X_{2n} X_0 = 0$. Thus the conclusion is obvious. $\square$

Proposition 6.8. *For all* $m \in \mathbf{N}$ *and* $f_1, f_2, \dots, f_{2n} \in \mathbf{C}[z_1, \bar{z}_1, \dots, z_n, \bar{z}_n]$, *we have*

$$\operatorname{tr}[T^{(m)}_{f_1}, T^{(m)}_{f_2}, \dots, T^{(m)}_{f_{2n}}] = \operatorname{tr}[R_{f_1}, R_{f_2}, \dots, R_{f_{2n}}].$$

Proof. From the identity $P^{(m)} = R' + Q^{(m)}$ we obtain

$$[T^{(m)}_{f_1}, T^{(m)}_{f_2}, \dots, T^{(m)}_{f_{2n}}] = [R'_{f_1}, R'_{f_2}, \dots, R'_{f_{2n}}] + [Q^{(m)}_{f_1}, Q^{(m)}_{f_2}, \dots, Q^{(m)}_{f_{2n}}] + \mathrm{SOT},$$

where SOT stands for "sum of the other terms". Obviously,

$$\operatorname{tr}[R'_{f_1}, R'_{f_2}, \dots, R'_{f_{2n}}] = \operatorname{tr}([R_{f_1}, R_{f_2}, \dots, R_{f_{2n}}] \oplus 0) = \operatorname{tr}[R_{f_1}, R_{f_2}, \dots, R_{f_{2n}}].$$

Also, Proposition 6.5 tells us that

$$\mathrm{tr}[Q_{f_1}^{(m)}, Q_{f_2}^{(m)}, \ldots, Q_{f_{2n}}^{(m)}] = 0.$$

Therefore the proposition will follow if we can show that $\mathrm{tr}(\mathrm{SOT}) = 0$.

Since $P^{(m)} = R' + Q^{(m)}$, a review of Definition 6.6 tells us that

$$\mathrm{SOT} = \sum_{(X_0,X_1,\ldots,X_{2n})\in\mathcal{Z}^{(m)}} \sum_{\sigma\in S_{2n}} \mathrm{sgn}(\sigma) X_0 D_{f_{\sigma(1)}} X_1 D_{f_{\sigma(2)}} X_2 \cdots X_{2n-1} D_{f_{\sigma(2n)}} X_{2n},$$

which, according to Lemma 6.7(a), is a finite-rank operator. We have the partition

$$\mathcal{Z}^{(m)} = \mathcal{X}^{(m)} \cup \mathcal{Y}^{(m)},$$

where

$$\begin{aligned} \mathcal{X}^{(m)} &= \{(X_0, X_1, \ldots, X_{2n}) \in \mathcal{Z}^{(m)} : X_0 = X_{2n}\} \quad \text{and} \\ \mathcal{Y}^{(m)} &= \{(X_0, X_1, \ldots, X_{2n}) \in \mathcal{Z}^{(m)} : X_0 \neq X_{2n}\}. \end{aligned}$$

Accordingly,

$$\mathrm{SOT} = U + V,$$

where

$$\begin{aligned} U &= \sum_{(X_0,X_1,\ldots,X_{2n})\in\mathcal{X}^{(m)}} \sum_{\sigma\in S_{2n}} \mathrm{sgn}(\sigma) X_0 D_{f_{\sigma(1)}} X_1 D_{f_{\sigma(2)}} X_2 \cdots X_{2n-1} D_{f_{\sigma(2n)}} X_{2n} \quad \text{and} \\ V &= \sum_{(X_0,X_1,\ldots,X_{2n})\in\mathcal{Y}^{(m)}} \sum_{\sigma\in S_{2n}} \mathrm{sgn}(\sigma) X_0 D_{f_{\sigma(1)}} X_1 D_{f_{\sigma(2)}} X_2 \cdots X_{2n-1} D_{f_{\sigma(2n)}} X_{2n}. \end{aligned}$$

Lemma 6.7(c) tells us that $\mathrm{tr}(V) = 0$. Thus what remains is to show that $\mathrm{tr}(U) = 0$.

To do that, we first apply Lemma 6.7(b), which gives us

$$\begin{aligned} \mathrm{tr}(U) &= \sum_{(X_0,X_1,\ldots,X_{2n})\in\mathcal{X}^{(m)}} \sum_{\sigma\in S_{2n}} \mathrm{sgn}(\sigma) \mathrm{tr}(X_0 D_{f_{\sigma(1)}} X_1 D_{f_{\sigma(2)}} X_2 \cdots X_{2n-1} D_{f_{\sigma(2n)}} X_{2n}) \\ &= \sum_{(X_0,X_1,\ldots,X_{2n})\in\mathcal{X}^{(m)}} \sum_{\sigma\in S_{2n}} \mathrm{sgn}(\sigma) \mathrm{tr}(X_1 D_{f_{\sigma(2)}} X_2 \cdots X_{2n-1} D_{f_{\sigma(2n)}} X_{2n} D_{f_{\sigma(1)}} X_1). \end{aligned}$$

Let δ be the cyclic permutation on $\{1, 2, \ldots, 2n\}$ such that $\delta(i) = i + 1$ for $1 \le i \le 2n - 1$ and $\delta(2n) = 1$. Since $2n$ is even, δ is an odd permutation, i.e., $\mathrm{sgn}(\delta) = -1$. Thus

$$\begin{aligned} &\mathrm{tr}(U) \\ &= \sum_{(X_0,X_1,\ldots,X_{2n})\in\mathcal{X}^{(m)}} \sum_{\sigma\in S_{2n}} \mathrm{sgn}(\sigma) \mathrm{tr}(X_1 D_{f_{\sigma\delta(1)}} X_2 \cdots X_{2n-1} D_{f_{\sigma\delta(2n-1)}} X_{2n} D_{f_{\sigma\delta(2n)}} X_1) \\ &= -\sum_{(X_0,X_1,\ldots,X_{2n})\in\mathcal{X}^{(m)}} \sum_{\sigma\in S_{2n}} \mathrm{sgn}(\sigma\delta) \mathrm{tr}(X_1 D_{f_{\sigma\delta(1)}} X_2 \cdots X_{2n-1} D_{f_{\sigma\delta(2n-1)}} X_{2n} D_{f_{\sigma\delta(2n)}} X_1) \end{aligned}$$

$$(6.4) \quad = -\sum_{(X_0,X_1,\ldots,X_{2n})\in\mathcal{X}^{(m)}} \sum_{\sigma\in S_{2n}} \mathrm{sgn}(\sigma) \mathrm{tr}(X_1 D_{f_{\sigma(1)}} X_2 \cdots X_{2n-1} D_{f_{\sigma(2n-1)}} X_{2n} D_{f_{\sigma(2n)}} X_1).$$

Then observe that on the set $\mathcal{X}^{(m)}$, the map

$$(X_0, X_1, \ldots, X_{2n}) \mapsto (X_1, X_2, \ldots, X_{2n}, X_1)$$

is injective, hence surjective. Therefore we can rewrite U in the form

$$U = \sum_{(X_0,X_1,\ldots,X_{2n})\in\mathcal{X}^{(m)}} \sum_{\sigma\in S_{2n}} \mathrm{sgn}(\sigma) X_1 D_{f_{\sigma(1)}} X_2 D_{f_{\sigma(2)}} X_3 \cdots X_{2n} D_{f_{\sigma(2n)}} X_1.$$

Consequently,

$$\text{(6.5)}\quad \mathrm{tr}(U) = \sum_{(X_0,X_1,\ldots,X_{2n})\in\mathcal{X}^{(m)}} \sum_{\sigma\in S_{2n}} \mathrm{sgn}(\sigma)\mathrm{tr}(X_1 D_{f_{\sigma(1)}} X_2 D_{f_{\sigma(2)}} X_3 \cdots X_{2n} D_{f_{\sigma(2n)}} X_1).$$

Comparing (6.5) with (6.4), we see that $\mathrm{tr}(U) = -\mathrm{tr}(U)$. That is, $\mathrm{tr}(U) = 0$ as desired. This completes the proof. □

After so many steps, we finally have the proof of our main result.

Proof of Theorem 1.3. It follows immediately from Corollary 6.2 and Proposition 6.8. □

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