ON THE ESSENTIAL NORMALITY OF PRINCIPAL SUBMODULES OF THE DRURY-ARVESON MODULE

Quanlei Fang and Jingbo Xia

Abstract. Continuing our earlier investigation [15] of the essential normality of submodules generated by polynomials, the emphasis of this paper is on submodules of the Drury-Arveson module H^2_n. In the case of two complex variables, we show that for every polynomial $q \in \mathbf{C}[z_1, z_2]$, the submodule $[q]$ of H^2_2 is p-essentially normal for $p > 2$. In the case of three complex variables, we show that there is a significant class of $q \in \mathbf{C}[z_1, z_2, z_3]$ for which the submodule $[q]$ of H^2_3 is p-essentially for $p > 3$. The difficulties involved in the proofs of these results are determined by the weight t $(-n \le t < \infty)$ of the space involved. Our earlier paper [15] covered the range $-2 < t < \infty$, which was enough to settle the problem for all polynomial-generated submodes of the Hardy module $H^2(S)$. In this paper we first solve the problem unconditionally for the weight range $-3 < t \le -2$, a consequence of which is the H^2_2-result mentioned above. We then consider the weight $t = -3$, which requires a substantial amount of additional work. At the moment we are only able to solve the $t = -3$ problem under a technical restriction on q, giving us the partial H^2_3-result mentioned above.

1. Introduction

This paper is a continuation of the investigation [15]. Here we pay particular attention to the case of the Drury-Arveson space H^2_n, a case that was left untouched in [15].

Let $\mathbf{B}$ be the open unit ball in $\mathbf{C}^n$. Throughout the paper, the complex dimension n is always assumed to be greater than or equal to 2. Recall that the Drury-Arveson space H^2_n is naturally a Hilbert module over the polynomial ring $\mathbf{C}[z_1, \dots, z_n]$. A decade ago, Arveson raised the question of whether graded submodules $\mathcal{M}$ of H^2_n are essentially normal [2,4,5,8], which is now called the Arveson conjecture. That is, for the restricted operators

$$Z_{\mathcal{M},j} = M_{z_j}|\mathcal{M}, \quad 1 \le j \le n,$$

on $\mathcal{M}$, do commutators $[Z^*_{\mathcal{M},j}, Z_{\mathcal{M},i}]$ belong to the Schatten class $\mathcal{C}_p$ for $p > n$? Later in [10], Douglas proposed analogous, but more refined essential normality problems for submodules of the Bergman module $L^2_a(\mathbf{B}, dv)$. Ever since, these essential normality problems have become a very active area of research interest [3,9,11,13,14,18-20,23].

In a breakthrough [12], Douglas and Wang showed that for every $q \in \mathbf{C}[z_1, \dots, z_n]$, the submodule $[q]$ of the Bergman module $L^2_a(\mathbf{B}, dv)$ is p-essentially normal for $p > n$. What is remarkable about this result is that it is *unconditional* in the respect that it makes

Keywords: Essential normality, submodule, polynomial.

no assumptions about the polynomial q. This led to our earlier paper [15], where we showed that the analogous essential normality also holds for every polynomial-generated submodule $[q]$ of the Hardy module $H^2(S)$.

One of the things that we figured out in [15] is that there is a real-valued "weight" t, $-n \leq t < \infty$, that is naturally associated with the essential normality problem. In fact, the integer-valued weights $t \in \mathbf{Z}_+$ were already involved in an essential way in [12]. It is the value of t that actually determines the level of difficulty of the problem: the more negative the value of t, the harder it is to solve the corresponding problem. In fact, we will see in this paper that the case $t = -3$ is much, much harder than the case $t > -3$. The complex dimension n affects the level of difficulty only in the sense that in order to solve the problem for the Drury-Arveson space H^2_n, one must deal with the weight $t = -n$.

Recall that for each real number $-n \leq t < \infty$, in [15] we introduced the Hilbert space $\mathcal{H}^{(t)}$ of analytic functions on $\mathbf{B}$ with the reproducing kernel

$$(1.1) \qquad \frac{1}{(1 - \langle \zeta, z \rangle)^{n+1+t}}.$$

Expanding (1.1), one can describe $\mathcal{H}^{(t)}$ as the completion of $\mathbf{C}[z_1, \ldots, z_n]$ with respect to the norm $\|\cdot\|_t$ arising from the inner product $\langle \cdot, \cdot \rangle_t$ defined according to the following rules: $\langle z^\alpha, z^\beta \rangle_t = 0$ whenever $\alpha \neq \beta$,

$$\langle z^\alpha, z^\alpha \rangle_t = \frac{\alpha!}{\prod_{j=1}^{|\alpha|}(n + t + j)}$$

if $\alpha \in \mathbf{Z}^n_+ \backslash \{0\}$, and $\langle 1, 1 \rangle_t = 1$. Here and throughout the paper, we use the conventional multi-index notation [22,page 3]. Accordingly, the standard orthonormal basis for $\mathcal{H}^{(t)}$ is $\{e^{(t)}_\alpha : \alpha \in \mathbf{Z}^n_+\}$, where

$$(1.2) \qquad e^{(t)}_\alpha(\zeta) = \left(\frac{1}{\alpha!} \prod_{j=1}^{|\alpha|}(n + t + j) \right)^{1/2} \zeta^\alpha, \quad \alpha \neq 0,$$

and $e^{(t)}_0(\zeta) = 1$.

The main interest of this paper, the Drury-Arveson space H^2_n, is none other than $\mathcal{H}^{(-n)}$. In comparison, $\mathcal{H}^{(0)}$ is just the Bergman space $L^2_a(\mathbf{B}, dv)$, and $\mathcal{H}^{(-1)}$ is the Hardy space $H^2(S)$. More generally, for each $t > -1$, $\mathcal{H}^{(t)}$ is a weighted Bergman space.

We emphasize that the weight t and the complex dimension n will always satisfy the relation $t \geq -n$ in this paper. Thus, for example, if the weight of the space is assumed to satisfy the condition $t < -2$, then the complex dimension n is automatically required to be at least 3.

Let $q \in \mathbf{C}[z_1, \ldots, z_n]$. For each $-n \leq t < \infty$, let $[q]^{(t)}$ denote the closure of

$$\{qf : f \in \mathbf{C}[z_1, \ldots, z_n]\}$$

in $\mathcal{H}^{(t)}$. Since $\mathcal{H}^{(t)}$ is a Hilbert module over $\mathbf{C}[z_1, \dots, z_n]$, $[q]^{(t)}$ is a submodule. For each $j \in \{1, \dots, n\}$, define submodule operator

$$Z^{(t)}_{q,j} = M_{z_j}|[q]^{(t)}.$$

Recall that the submodule $[q]^{(t)}$ is said to be p-essentially normal if the commutators $[Z^{(t)*}_{q,j}, Z^{(t)}_{q,i}]$, $i, j \in \{1, \dots, n\}$, all belong to the Schatten class $\mathcal{C}_p$.

To motivate what we will do in this paper, let us recall

Theorem 1.1. [15,Theorem 1.1] *Let q be an arbitrary polynomial in $\mathbf{C}[z_1, \dots, z_n]$. Then for each real number $-2 < t < \infty$, the submodule $[q]^{(t)}$ of $\mathcal{H}^{(t)}$ is p-essentially normal for every $p > n$.*

For this paper, the first order of business is to extend the above theorem to the weight range $-3 < t \le -2$:

Theorem 1.2. *Let q be an arbitrary polynomial in $\mathbf{C}[z_1, \dots, z_n]$. Then for each real number $-3 < t \le -2$, the submodule $[q]^{(t)}$ of $\mathcal{H}^{(t)}$ is p-essentially normal for every $p > n$.*

Applying Theorem 1.2 to the case where $n = 2$ and $t = -2$, we obtain the first unconditional essential normality in a Drury-Arveson space case:

Corollary 1.3. *For every $q \in \mathbf{C}[z_1, z_2]$, the submodule $[q]$ of the two-variable Drury-Arveson module H^2_2 is p-essentially normal for every $p > 2$.*

Once we have Corollary 1.3, the obvious question is, what about the polynomial-generated submodules of H^2_n for $n \ge 3$? This obviously forces us to deal with weights $t \le -3$. In this paper we will only consider the case $t = -3$, and even for this case we need to impose technical conditions on the polynomials involved.

For each $q \in \mathbf{C}[z_1, \dots, z_n]$, we write $\mathcal{Z}(q)$ for its zero locus. That is,

$$\mathcal{Z}(q) = \{z \in \mathbf{C}^n : q(z) = 0\}.$$

As usual, we write $\partial_1, \dots, \partial_n$ for the differentiations with respect to the complex variables $z_1, \dots, z_n$. Furthermore, we write R for the radial derivative in n variables, i.e,

$$R = z_1\partial_1 + \cdots + z_n\partial_n. \tag{1.3}$$

Let S denote the unit sphere $S = \{\xi \in \mathbf{C}^n : |\xi| = 1\}$ in $\mathbf{C}^n$.

Definition 1.4. Let $\mathcal{G}_n$ be the collection of polynomials $q \in \mathbf{C}[z_1, \dots, z_n]$ satisfying the following two conditions:
(a) The radial derivative Rq does not vanish on the set $\mathcal{Z}(q) \cap S$.
(b) The zero locus $\mathcal{Z}(q)$ intersects the unit sphere S transversely.

Note that condition (a) implies that the analytic gradient $\partial q = (\partial_1 q, \dots, \partial_n q)$ does not vanish on the set $\mathcal{Z}(q) \cap S$, which ensures that (b) makes sense. At every point in S, the (real) co-dimension of the tangent space to S is 1. Thus condition (b) is simply equivalent to the condition that if $\xi \in \mathcal{Z}(q) \cap S$, then the tangent space to $\mathcal{Z}(q)$ at ξ is not contained in the tangent space to S at ξ. In Section 7 we will show that the combination of these two conditions is equivalent to a very simple inequality. From this simple inequality we will see that the membership $q \in \mathcal{G}_n$ is stable under small perturbation. Consequently, there are plenty of polynomials in $\mathcal{G}_n$.

Here is what we can prove in the case $t = -3$:

Theorem 1.5. *If $q \in \mathcal{G}_n$, $n \geq 3$, then the submodule $[q]^{(-3)}$ of $\mathcal{H}^{(-3)}$ is p-essentially normal for every $p > n$.*

In the case $n = 3$, we have $\mathcal{H}^{(-3)} = H_3^2$, the Drury-Arveson space in three variables. Therefore the above implies

Corollary 1.6. *If $q \in \mathcal{G}_3$, then the submodule $[q]$ of H_3^2 is p-essentially normal for every $p > 3$.*

Next let us explain the strategy for the proofs of Theorems 1.2 and 1.5, and give a brief outline of the organization of the paper. The reader will see that, as the weight t goes down the negative scale, more and more analysis is required to prove the desired essential normality. In particular, all the analysis from [15] will be needed in this paper. For that reason, we begin our proofs by recalling the necessary propositions in Section 2.

Given the material in Section 2, the central issue in the proofs of Theorems 1.2 and 1.5 revolves around just one single inequality. The best way to explain this is to introduce

Definition 1.7. Suppose that $-n \leq t < \infty$ and $0 \leq \epsilon < 1$. Then a polynomial $q \in \mathbf{C}[z_1, \ldots, z_n]$ is said to be in the class $\mathcal{P}_n(t; \epsilon)$ if there is a $0 < C = C(q) < \infty$ such that

$$\|fRq\|_{t+3} \leq C\|qf\|_{t+1-\epsilon} \tag{1.4}$$

for every $f \in \mathbf{C}[z_1, \ldots, z_n]$.

In Sections 3 and 4, we reduce the proofs of Theorems 1.2 and 1.5 to the proof of (1.4). More precisely, the work in Sections 3 and 4 culminates in Proposition 4.4, which tells us that for weights $t \geq -3$, the membership $q \in \mathcal{P}_n(t; \epsilon)$ implies the desired essential normality for the submodule $[q]^{(t)}$.

Then in Section 5, we show that for $-3 < t \leq -2$, we have $\mathcal{P}_n(t; 0) = \mathbf{C}[z_1, \ldots, z_n]$. This equality together with Proposition 4.4 imply Theorem 1.2. Obviously, we would like to establish the equality

$$\mathcal{P}_n(-3; 0) = \mathbf{C}[z_1, \ldots, z_n].$$

But this we are not able to do as of this writing. Instead, it takes the efforts of Sections 6, 7 and 8 to just show that $\mathcal{G}_n \subset \mathcal{P}_n(-3; \epsilon)$, $0 < \epsilon < 1/2$, giving us Theorem 1.5.

Finally, in Section 9 we show that there are plenty of polynomials in the class $\mathcal{G}_n$.

2. Some known facts

This section serves two purposes. First, we collect a number of standard notations. Second, we recall several propositions from [15] that will be needed in this paper.

Following [12,15], for each pair of $i \neq j$ in $\{1, \dots, n\}$ we define

$$L_{i,j} = \bar{z}_j \partial_i - \bar{z}_i \partial_j.$$

In addition to the n-variable radial derivative R given by (1.3), we will also need the one-variable radial derivative. We will denote the one-variable radial derivative by $\mathcal{R}$.

An alternate representation of the n-variable radial derivative is the number operator N introduced by Arveson in [1]. Recall that, for a polynomial $f(z) = \sum_\alpha c_\alpha z^\alpha$,

$$(Nf)(z) = \sum_\alpha c_\alpha |\alpha| z^\alpha.$$

Thus, in fact we have $Nf = Rf$ for every $f \in \mathbf{C}[z_1, \dots, z_n]$. But historically, both N and R are standard fixtures in papers about the Drury-Arveson space, and that will be the case for this paper.

Let dv be the volume measure on $\mathbf{B}$ with the normalization $v(\mathbf{B}) = 1$. For each real number $t > -1$, define

$$a_{n,t} = \frac{1}{n!} \prod_{j=1}^{n} (t + j).$$

Using the reproducing kernel and [22,Proposition 1.4.9], it is straightforward to verify that

$$\langle f, g \rangle_t = a_{n,t} \int f(\zeta) \overline{g(\zeta)} (1 - |\zeta|^2)^t dv(\zeta) \tag{2.1}$$

for $f, g \in \mathcal{H}^{(t)}$, $t > -1$. In other words, if $t > -1$, then $\mathcal{H}^{(t)}$ is the weighted Bergman space $L^2_a(\mathbf{B}, a_{n,t}(1 - |\zeta|^2)^t dv(\zeta))$.

Proposition 2.1. [15,Proposition 3.4] *There is a constant $1 \leq C_{2.1} < \infty$ such that the following estimate holds: Suppose that $q \in \mathbf{C}[z_1, \dots, z_n]$ and that $\deg(q) = K \geq 1$. Let $f \in \mathbf{C}[z_1, \dots, z_n]$. Then for every positive number $t > 0$ and all integers $i \neq j$ in $\{1, \dots, n\}$, we have*

$$\int |(L_{i,j} q)(z) f(z)|^2 (1 - |z|^2)^t dv(z) \leq C_{2.1} 2^t (K!)^2 \int |q(\zeta) f(\zeta)|^2 (1 - |\zeta|^2)^{t-1} dv(\zeta).$$

One can interpret Proposition 2.1 as saying that each $L_{i,j}$ carries a "weight" of -1. In comparison, the "weight" of the radial derivative R is -2:

Proposition 2.2. [15,Proposition 3.5] *There is a constant $1 \leq C_{2.2} < \infty$ such that the following estimate holds: Suppose that $q \in \mathbf{C}[z_1, \dots, z_n]$ and that $\deg(q) = K \geq 1$. Let $f \in \mathbf{C}[z_1, \dots, z_n]$. Then for each pair of $k \in \mathbf{N}$ and $t \in (0, \infty)$ satisfying the condition $t - 2k > -1$,*

$$\int |(R^k q)(\zeta) f(\zeta)|^2 (1 - |\zeta|^2)^t dv(\zeta) \leq C_{2.2}^{K(k+t)} (K!)^2 \int |q(\zeta) f(\zeta)|^2 (1 - |\zeta|^2)^{t-2k} dv(\zeta).$$

Proposition 2.3. [15,Proposition 3.6] *There is a constant* $1 \le C_{2.3} < \infty$ *such that the following estimate holds: Suppose that* $q \in \mathbf{C}[z_1, \dots, z_n]$ *and that* $\deg(q) = K \ge 1$. *Let* $f \in \mathbf{C}[z_1, \dots, z_n]$. *Then for each* $t \in (1, \infty)$ *and each* $j \in \{1, \dots, n\}$, *we have*

$$\int |(\partial_j q)(\zeta) f(\zeta)|^2 (1 - |\zeta|^2)^t dv(\zeta) \le C_{2.3}^{Kt} (K!)^2 \int |q(\zeta) f(\zeta)|^2 (1 - |\zeta|^2)^{t-2} dv(\zeta).$$

For each $t \ge -n$ and each polynomial q, we write $M_q^{(t)}$ for the operator of multiplication by q on the space $\mathcal{H}^{(t)}$. Keep in mind that the operation of taking adjoint "$*$" is t-specific: $M_q^{(t)*}$ means the adjoint of $M_q^{(t)}$ with respect to the inner product $\langle \cdot, \cdot \rangle_t$.

Proposition 2.4. [15,Proposition 5.1] *Let* $q \in \mathbf{C}[z_1, \dots, z_n]$, $1 \le j \le n$ *and* $t \ge -n$. *For* $f \in \mathbf{C}[z_1, \dots, z_n]$ *satisfying the condition* $f(0) = 0$, *we have*

$$M_{z_j}^{(t)*} M_q^{(t)} f - M_q^{(t)} M_{z_j}^{(t)*} f = \sum_{k=0}^{\infty} (N + 1 + n + t)^{-k-1} (M_{\partial_j R^k q}^{(t)} - M_{z_j}^{(t)*} M_{R^{k+1} q}^{(t)}) f.$$

Proposition 2.5. [15,Proposition 5.2] *Let* $t \ge -n$ *and* $\ell \in \mathbf{N}$.
(1) *For each* $f \in \mathbf{C}[z_1, \dots, z_n]$ *satisfying the condition* $(\partial^\alpha f)(0) = 0$ *for* $|\alpha| < \ell$ *and each non-negative integer* k, *we have*

$$\|(N + 1 + n + t)^{-k-1} f\|_t^2 \le \frac{(n + 2k + 2 + t + \ell)^\ell}{(\ell + 1 + n + t)^{2k+2}} \|f\|_{2k+2+t}^2.$$

(2) *For each* $f \in \mathbf{C}[z_1, \dots, z_n]$ *satisfying the condition* $(\partial^\alpha f)(0) = 0$ *for* $|\alpha| < \ell + 1$, *each non-negative integer* k *and each* $1 \le j \le n$, *we have*

$$\|(N+1+n+t)^{-k-1} (M_{z_j}^{(t)*} - M_{z_j}^{(t+2k+2)*}) f\|_t^2 \le (2k+4)^4 \frac{(n + 2k + 4 + t + \ell)^{2\ell}}{(\ell + 1 + n + t)^{2k+2}} \|f\|_{2k+4+t}^2.$$

Let us recall Stirling's asymptotic expansion for $r(r+1) \cdots (r+k)$, $r > 0$, which will be needed for several estimates in this paper. Indeed from the identity

$$\frac{1}{2}\{f(1) + f(0)\} = \int_0^1 f(x) dx - \frac{1}{2} \int_0^1 (x^2 - x) f''(x) dx$$

for C^2-functions one derives the formula

$$\sum_{j=0}^{k} \log(r + j) = \frac{1}{2}\{\log r + \log(r + k)\} + \int_0^k \log(r + x) dx + \frac{1}{2} \sum_{j=0}^{k-1} \int_0^1 \frac{x^2 - x}{(r + j + x)^2} dx,$$

$k \in \mathbf{N}$. Evaluating the integral $\int_0^k$ and then exponentiating both sides, we find that

$$\prod_{j=0}^{k} (r + j) = (r + k)^{r+k+(1/2)} e^{-k} e^{c(r;k)}, \tag{2.2}$$

where $c(r;k)$ has a finite limit (which depends on r) as $k \to \infty$.

Next we turn to a class of Lorentz-like ideals that are naturally involved in the investigation of essential normality. For a bounded operator A, let $s_1(A), \dots, s_k(A), \dots$ denote its s-numbers. Recall that, for each $1 \le p < \infty$, the formula

$$\|A\|_p^+ = \sup_{k \ge 1} \frac{s_1(A) + s_2(A) + \cdots + s_k(A)}{1^{-1/p} + 2^{-1/p} + \cdots + k^{-1/p}} \tag{2.3}$$

defines a symmetric norm for operators [17,Section III.14]. On any Hilbert space $\mathcal{H}$, the set $\mathcal{C}_p^+ = \{A \in \mathcal{B}(\mathcal{H}) : \|A\|_p^+ < \infty\}$ is a norm ideal [17,Section III.2]. It is well known that if $p < p'$, then $\mathcal{C}_p^+$ is contained in the Schatten class $\mathcal{C}_{p'}$.

For a non-increasing sequence of non-negative numbers $\{a_1, \dots, a_k, \dots\}$, if $a_1 + \cdots + a_k \le C(1^{-1/p} + \cdots + k^{-1/p})$, then $ka_k \le C(1^{-1/p} + \cdots + k^{-1/p})$. It follows that if $p > 1$ and if $T \in \mathcal{C}_p^+$, then there is a $0 < C(T) < \infty$ such that $s_k(T) \le C(T)k^{-1/p}$ for every $k \in \mathbf{N}$. Thus if $p > 1$ and if B is a bounded operator such that $B^*B \in \mathcal{C}_p^+$, then $B \in \mathcal{C}_{2p}^+$.

Our next proposition is a slight improvement of Proposition 4.2 in [15]. The improvement lies in the incorporation of an extra "ϵ", which will appear in the proof of Theorem 1.5, and which we believe will be important for future investigations.

Proposition 2.6. *Let $0 \le \epsilon < 1$. Suppose that E is a linear subspace of $\mathbf{C}[z_1, \dots, z_n]$ and that $t \ge -n$. Let $E^{(t)}$ be the closure of E in $\mathcal{H}^{(t)}$, and let $\mathcal{E}^{(t)}$ be the orthogonal projection from $\mathcal{H}^{(t)}$ to $E^{(t)}$. Suppose that $A \in \mathcal{B}(\mathcal{H}^{(t)})$, and suppose that there is a C such that*

$$\|Ag\|_t \le C\|g\|_{t+1-\epsilon} \tag{2.4}$$

for every $g \in E$. Then $A\mathcal{E}^{(t)} \in \mathcal{C}_{2n/(1-\epsilon)}^+$.

Proof. Given $0 \le \epsilon < 1$ and $t \ge -n$, let $J : \mathcal{H}^{(t)} \to \mathcal{H}^{(t+1-\epsilon)}$ be the natural embedding operator. An elementary calculation using (1.2) shows that

$$J^*Je_\alpha^{(t)} = c_\alpha e_\alpha^{(t)}, \quad \alpha \in \mathbf{Z}_+^n, \tag{2.5}$$

where

$$c_\alpha = \frac{\prod_{j=1}^{|\alpha|}(n+t+j)}{\prod_{j=1}^{|\alpha|}(n+t+1-\epsilon+j)}$$

for $\alpha \ne 0$ and $c_0 = 1$. By (2.2), there is a $0 < C_1 < \infty$ that is determined by the values of $n + t$ (≥ 0) and $1 - \epsilon$ (> 0) such that

$$c_\alpha \le \frac{C_1}{(|\alpha| + 1)^{1-\epsilon}} \tag{2.6}$$

for every $\alpha \in \mathbf{Z}_+^n$. Since $\{e_\alpha^{(t)} : \alpha \in \mathbf{Z}_+^n\}$ is an orthonormal basis for $\mathcal{H}^{(t)}$, (2.5) gives us all the s-numbers of J^*J. Thus it follows from (2.6) and (2.3) that $J^*J \in \mathcal{C}_{n/(1-\epsilon)}^+$.

Let E, $\mathcal{E}^{(t)}$ and A be the same as in the statement of the proposition. Then (2.4) implies that the operator inequality

$$(A\mathcal{E}^{(t)})^* A\mathcal{E}^{(t)} \leq C^2 \mathcal{E}^{(t)} J^* J \mathcal{E}^{(t)}$$

holds on the Hilbert space $\mathcal{H}^{(t)}$. Hence $s_k((A\mathcal{E}^{(t)})^* A\mathcal{E}^{(t)}) \leq s_k(C^2\mathcal{E}^{(t)} J^* J\mathcal{E}^{(t)})$ for every $k \in \mathbf{N}$ [17,Lemma II.1.1]. Since $J^*J \in \mathcal{C}^+_{n/(1-\epsilon)}$, it follows that $(A\mathcal{E}^{(t)})^* A\mathcal{E}^{(t)} \in \mathcal{C}^+_{n/(1-\epsilon)}$. Since $n/(1-\epsilon) \geq n \geq 2 > 1$, this implies the membership $A\mathcal{E}^{(t)} \in \mathcal{C}^+_{2n/(1-\epsilon)}$. $\square$

We conclude this section with an elementary inequality that will be needed in the sequel. For each $-n \leq t < \infty$, there exist constants $0 < c(t) \leq C(t) < \infty$ such that

$$c(t)\|f\|_t^2 \leq |f(0)|^2 + \|Rf\|_{t+2}^2 \leq C(t)\|f\|_t^2 \tag{2.7}$$

for every $f \in \mathbf{C}[z_1, \dots, z_n]$. This follows easily from (1.2).

3. Commutation relations

We will see that, given the propositions in Section 2, the proofs of Theorems 1.2 and 1.5 are easily reduced to norm estimates for fRq, $f\partial_j q$ and $fL_{j,k}q$ for the range of weights $-3 \leq t \leq -2$. The purpose of this section is to further reduce these estimates to just one, namely that for norm of fRq alone. In other words, in this section we want to show that the desired norm bound for fRq implies the desired norm bounds for $fL_{j,k}q$ and $f\partial_j q$. For this we must deal with the commutation relations of the differential operators involved, and the work in this section is unfortunately very repetitive and tedious.

We will consider $\partial_1, \dots, \partial_n$ also as operators on the linear space $C^\infty(\mathbf{B})$ in the usual way. That is, for each $j \in \{1, \dots, n\}$,

$$\partial_j = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right),$$

where $x_j + iy_j = z_j$, the j-th complex variable. Similarly, $M_{z_1}, \dots, M_{z_n}$ and $M_{\bar{z}_1}, \dots, M_{\bar{z}_n}$ will be regarded as operators on $C^\infty(\mathbf{B})$. If $j \neq k$, then ∂_j commutes with both M_{z_k} and $M_{\bar{z}_k}$. For each $j \in \{1, \dots, n\}$, we have

$$[\partial_j, M_{z_j}] = 1 \quad \text{while} \quad [\partial_j, M_{\bar{z}_j}] = 0.$$

The radial derivative $R = z_1\partial_1 + \cdots + z_n\partial_n$ also acts on $C^\infty(\mathbf{B})$. Moreover, the above commutation relations lead to

$$[R, M_{z_j}] = M_{z_j} \quad \text{while} \quad [R, M_{\bar{z}_j}] = 0, \tag{3.1}$$

$j \in \{1, \dots, n\}$. Recall that for $j \neq k$ in $\{1, \dots, n\}$, $L_{j,k} = \bar{z}_k\partial_j - \bar{z}_j\partial_k$. Since $[\partial_j, R] = \partial_j$, we have

$$[L_{j,k}, R] = L_{j,k} \tag{3.2}$$

for each pair of $j \neq k$. Also recall that the class $\mathcal{P}_n(t;\epsilon)$ was defined in Definition 1.7.

Lemma 3.1. *Suppose that $t \geq -3$ and $0 \leq \epsilon < 1$. Then for every $q \in \mathcal{P}_n(t;\epsilon)$, there is a constant $0 < C_{3.1}(q) < \infty$ such that*

$$\int |(R(fL_{j,k}q))(z)|^2(1-|z|^2)^{t+4}dv(z) \leq C_{3.1}(q)\|qf\|^2_{t+1-\epsilon}$$

for all $f \in \mathbf{C}[z_1,\dots,z_n]$ and $j \neq k$ in $\{1,\dots,n\}$.

Proof. Since $R(fL_{j,k}q) = Rf \cdot L_{j,k}q + fRL_{j,k}q$, we have

$$\int |(R(fL_{j,k}q))(z)|^2(1-|z|^2)^{t+4}dv(z) \leq 2(X+Y), \tag{3.3}$$

where

$$X = \int |(Rf)(z)(L_{j,k}q)(z)|^2(1-|z|^2)^{t+4}dv(z) \quad \text{and}$$
$$Y = \int |(RL_{j,k}q)(z)f(z)|^2(1-|z|^2)^{t+4}dv(z).$$

We estimate X and Y separately.

For X, since $t+4 \geq -3+4 = 1$, we can apply Proposition 2.1 to q to obtain

$$X \leq C_1 \int |q(z)(Rf)(z)|^2(1-|z|^2)^{t+3}dv(z). \tag{3.4}$$

Note that $qRf = R(qf) - fRq$. Therefore (3.4) implies

$$X \leq 2C_1(X_1 + X_2),$$

where

$$X_1 = \int |(R(qf))(z)|^2(1-|z|^2)^{t+3}dv(z) \quad \text{and}$$
$$X_2 = \int |(Rq)(z)f(z)|^2(1-|z|^2)^{t+3}dv(z).$$

By (2.7) and (2.1) we have

$$X_1 \leq C_2\|qf\|^2_{t+1} \quad \text{and} \quad X_2 = C_3\|fRq\|^2_{t+3},$$

where $C_3 = a^{-1}_{n,t+3}$. By the membership $q \in \mathcal{P}_n(t;\epsilon)$, $\|fRq\|_{t+3} \leq C\|qf\|_{t+1-\epsilon}$. Hence $X_2 \leq C_3C^2\|qf\|^2_{t+1-\epsilon}$. Since $\|qf\|_{t+1} \leq \|qf\|_{t+1-\epsilon}$, combining the above, we obtain

$$X \leq 2C_1(C_2 + C_3C^2)\|qf\|^2_{t+1-\epsilon}. \tag{3.5}$$

To estimate Y, note that (3.2) gives us $RL_{j,k}q = L_{j,k}Rq - L_{j,k}q$. Hence

$$Y \le 2(Y_1 + Y_2),$$

where

$$Y_1 = \int |(L_{j,k}Rq)(z)f(z)|^2(1-|z|^2)^{t+4}dv(z) \quad \text{and}$$
$$Y_2 = \int |(L_{j,k}q)(z)f(z)|^2(1-|z|^2)^{t+4}dv(z).$$

Applying Proposition 2.1 to Rq and then using the membership $q \in \mathcal{P}_n(t;\epsilon)$, we have

$$Y_1 \le C_1 \int |(Rq)(z)f(z)|^2(1-|z|^2)^{t+3}dv(z) = C_1C_3\|fRq\|_{t+3}^2 \le C_1C_3C^2\|qf\|_{t+1-\epsilon}^2.$$

An application of Proposition 2.1 to q then gives us

$$Y_2 \le C_1 \int |q(z)f(z)|^2(1-|z|^2)^{t+3}dv(z) = C_1C_3\|qf\|_{t+3}^2 \le C_1C_3\|qf\|_{t+1-\epsilon}^2.$$

Consequently,

$$Y \le 2(C_1C_3C^2 + C_1C_3)\|qf\|_{t+1-\epsilon}^2.$$

Combining this with (3.5) and (3.3), we find that

$$\int |(R(fL_{j,k}q))(z)|^2(1-|z|^2)^{t+4}dv(z) \le 4\{C_1(C_2 + C_3C^2) + C_1C_3C^2 + C_1C_3\}\|qf\|_{t+1-\epsilon}^2.$$

This proves the lemma. □

Lemma 3.2. *Suppose that $t \ge -3$ and $0 \le \epsilon < 1$. Then for every $q \in \mathcal{P}_n(t;\epsilon)$, there is a constant $0 < C_{3.2}(q) < \infty$ such that*

$$\int |(R\partial_j q)(z)f(z)|^2(1-|z|^2)^{t+5}dv(z) \le C_{3.2}(q)\|qf\|_{t+1-\epsilon}^2$$

for all $f \in \mathbf{C}[z_1, \dots, z_n]$ and $j \in \{1, \dots, n\}$.

Proof. By the commutation relation $R\partial_j = \partial_j R - \partial_j$, we have

$$\int |(R\partial_j q)(z)f(z)|^2(1-|z|^2)^{t+5}dv(z) \le 2(X+Y), \tag{3.6}$$

where

$$X = \int |(\partial_j Rq)(z)f(z)|^2(1-|z|^2)^{t+5}dv(z) \quad \text{and}$$
$$Y = \int |(\partial_j q)(z)f(z)|^2(1-|z|^2)^{t+5}dv(z).$$

Since $t+5 \geq 2$, applying Proposition 2.3 to Rq, we obtain

$$X \leq C_1 \int |(Rq)(z)f(z)|^2(1-|z|^2)^{t+3}dv(z) = C_1C_2\|fRq\|_{t+3}^2,$$

where $C_2 = a_{n,t+3}^{-1}$. Since $q \in \mathcal{P}_n(t;\epsilon)$, we have $\|fRq\|_{t+3} \leq C\|qf\|_{t+1-\epsilon}$. Therefore

$$X \leq C_1C_2C^2\|qf\|_{t+1-\epsilon}^2. \tag{3.7}$$

Also by Proposition 2.3, we have

$$Y \leq C_1 \int |q(z)f(z)|^2(1-|z|^2)^{t+3}dv(z) = C_1C_2\|qf\|_{t+3}^2 \leq C_1C_2\|qf\|_{t+1-\epsilon}^2.$$

Combining this with (3.7) and (3.6), we find that

$$\int |(R\partial_j q)(z)f(z)|^2(1-|z|^2)^{t+5}dv(z) \leq 2(C_1C_2C^2 + C_1C_2)\|qf\|_{t+1-\epsilon}^2,$$

proving the lemma. $\square$

Lemma 3.3. *Suppose that $t \geq -3$ and $0 \leq \epsilon < 1$. Then for every $q \in \mathcal{P}_n(t;\epsilon)$, there is a constant $0 < C_{3.3}(q) < \infty$ such that*

$$\int |(\partial_j q)(z)f(z)|^2(1-|z|^2)^{t+3}dv(z) \leq C_{3.3}(q)\|qf\|_{t+1-\epsilon}^2$$

for all $f \in \mathbf{C}[z_1,\dots,z_n]$ and $j \in \{1,\dots,n\}$.

Proof. By (2.7), there is a constant C_1 such that

$$\int |g(z)|^2(1-|z|^2)^{t+3}dv(z) \leq C_1 \int |(Rg)(z)|^2(1-|z|^2)^{t+5}dv(z)$$

for every $g \in \mathbf{C}[z_1,\dots,z_n]$ satisfying the condition $g(0) = 0$. Since $(\partial_j q)(0)f(0) = 0 \cdot f(0) = 0$, it follows that

$$\int |(\partial_j q)(z)f(z)|^2(1-|z|^2)^{t+3}dv(z) \leq C_1 \int |(R(f\partial_j q))(z)|^2(1-|z|^2)^{t+5}dv(z).$$

Now, $R(f\partial_j q) = Rf \cdot \partial_j q + fR\partial_j q$. Therefore

$$\int |(\partial_j q)(z)f(z)|^2(1-|z|^2)^{t+3}dv(z) \leq 2C_1(X+Y), \tag{3.8}$$

where

$$X = \int |(\partial_j q)(z)(Rf)(z)|^2(1-|z|^2)^{t+5}dv(z) \quad \text{and}$$
$$Y = \int |(R\partial_j q)(z)f(z)|^2(1-|z|^2)^{t+5}dv(z).$$

By Lemma 3.2, we have

$$Y \le C_{3.2}(q)\|qf\|^2_{t+1-\epsilon}. \tag{3.9}$$

On the other hand, by Proposition 2.3,

$$X \le C_2 \int |q(z)(Rf)(z)|^2(1-|z|^2)^{t+3}dv(z).$$

Since $qRf = R(qf) - fRq$, we have $X \le 2C_2(X_1 + X_2)$, where

$$X_1 = \int |(R(qf))(z)|^2(1-|z|^2)^{t+3}dv(z) \quad \text{and}$$
$$X_2 = \int |(Rq)(z)f(z)|^2(1-|z|^2)^{t+3}dv(z).$$

Since $q \in \mathcal{P}_n(t;\epsilon)$, we have $X_2 = C_3\|fRq\|^2_{t+3} \le C_3C^2\|qf\|^2_{t+1-\epsilon}$. By (2.7),

$$X_1 \le C_4\|qf\|^2_{t+1} \le C_4\|qf\|^2_{t+1-\epsilon}.$$

Therefore

$$X \le 2C_2(C_4 + C_3C^2)\|qf\|^2_{t+1-\epsilon}.$$

Combining this with (3.9) and (3.8), we find that

$$\int |(\partial_j q)(z)f(z)|^2(1-|z|^2)^{t+3}dv(z) \le 2C_1\{2C_2(C_4 + C_3C^2) + C_{3.2}(q)\}\|qf\|^2_{t+1-\epsilon}.$$

This completes the proof. □

Lemma 3.4. *Suppose that $t \ge -3$. Then there is a constant $C_{3.4}$ that depends only on t and the complex dimension n such that the following estimate holds: Let $h \in \mathbf{C}[z_1, \dots, z_n]$ satisfy the condition $(\partial^\alpha h)(0) = 0$ for all $\alpha \in \mathbf{Z}^n_+$ such that $|\alpha| \le 1$. Let $g \in \mathbf{C}[z_1, \dots, z_n]$ be such that $g(0) = 0$. Then for every $j \in \{1, \dots, n\}$ we have*

$$\|g - M^{(t+4)*}_{z_j}h\|^2_{t+2} \le C_{3.4}\int \{|(Rg)(z) - \bar{z}_j(Rh)(z)|^2 + |h(z)|^2\}(1-|z|^2)^{t+4}dv(z).$$

Proof. First of all, by (2.7) there is a constant C_1 such that

$$\|f\|^2_{t+2} \le C_1 \int |(Rf)(z)|^2(1-|z|^2)^{t+4}dv(z)$$

for every $f \in \mathbf{C}[z_1, \dots, z_n]$ satisfying the condition $f(0) = 0$. We will show that the constant $C_{3.4} = 2C_1$ works for the lemma. Note that if $h \in \mathbf{C}[z_1, \dots, z_n]$ is such that

$(\partial^\alpha h)(0) = 0$ for all $\alpha \in \mathbf{Z}_+^n$ satisfying the condition $|\alpha| \leq 1$, then $(M_{z_j}^{(t+4)*}h)(0) = 0$. Hence if $g \in \mathbf{C}[z_1, \ldots, z_n]$ satisfies the condition $g(0) = 0$, then

$$\|g - M_{z_j}^{(t+4)*}h\|_{t+2}^2 \leq C_1 \int |(Rg)(z) - (RM_{z_j}^{(t+4)*}h)(z)|^2 (1 - |z|^2)^{t+4} dv(z).$$

Recall from (3.1) that $[R, M_{z_j}] = M_{z_j}$. Taking the adjoint on the Hilbert space $\mathcal{H}^{(t+4)}$, we find that $RM_{z_j}^{(t+4)*} = M_{z_j}^{(t+4)*}R - M_{z_j}^{(t+4)*}$. Thus if we set $C_{3.4} = 2C_1$, then

$$\|g - M_{z_j}^{(t+4)*}h\|_{t+2}^2 \leq C_{3.4}(X + Y), \tag{3.10}$$

where

$$X = \int |(Rg)(z) - (M_{z_j}^{(t+4)*}Rh)(z)|^2 (1 - |z|^2)^{t+4} dv(z) \quad \text{and}$$
$$Y = \int |(M_{z_j}^{(t+4)*}h)(z)|^2 (1 - |z|^2)^{t+4} dv(z).$$

If we write $P^{(t+4)}$ for the orthogonal projection from $L^2(\mathbf{B}, dv_{t+4})$ to the weighted Bergman space $L_a^2(\mathbf{B}, dv_{t+4})$, where $dv_t(z) = a_{n,t+4}(1 - |z|^2)^{t+4} dv(z)$, then

$$Rg - M_{z_j}^{(t+4)*}Rh = P^{(t+4)}(Rg - M_{\bar{z}_j}Rh).$$

Hence

$$X \leq \int |(Rg)(z) - \bar{z}_j(Rh)(z)|^2 (1 - |z|^2)^{t+4} dv(z). \tag{3.11}$$

Similarly, $M_{z_j}^{(t+4)*}h = P^{(t+4)}M_{\bar{z}_j}h$. Therefore

$$Y \leq \int |\bar{z}_j h(z)|^2 (1 - |z|^2)^{t+4} dv(z) \leq \int |h(z)|^2 (1 - |z|^2)^{t+4} dv(z).$$

Combining this with (3.10) and (3.11), the lemma follows. $\square$

We are now ready to tackle the main objective of the section:

Proposition 3.5. *Suppose that $t \geq -3$ and that $0 \leq \epsilon < 1$. Then for every $q \in \mathcal{P}_n(t; \epsilon)$, there is a constant $C_{3.5}(q)$ such that*

$$\|(M_{\partial_j q}^{(t+2)} - M_{z_j}^{(t+4)*}M_{Rq}^{(t+2)})f\|_{t+2}^2 \leq C_{3.5}(q)\|qf\|_{t+1-\epsilon}^2$$

for every $f \in \mathbf{C}[z_1, \ldots, z_n]$ satisfying the condition $f(0) = 0$ and every $j \in \{1, \ldots, n\}$.

Proof. Note that $(Rq)(0) = 0$. Thus for $f \in \mathbf{C}[z_1, \ldots, z_n]$ satisfying the condition $f(0) = 0$, we also have $(\partial_i(fRq))(0) = 0$ for $i = 1, \ldots, n$. Hence, applying Lemma 3.4 to the case where $g = f\partial_j q$ and $h = fRq$, we obtain

$$\|(M_{\partial_j q}^{(t+2)} - M_{z_j}^{(t+4)*}M_{Rq}^{(t+2)})f\|_{t+2}^2 \leq C_{3.4}(X + Y), \tag{3.12}$$

where

$$X = \int |(R(f\partial_j q))(z) - \bar{z}_j(R(fRq))(z)|^2(1-|z|^2)^{t+4}dv(z) \quad \text{and}$$
$$Y = \int |(Rq)(z)f(z)|^2(1-|z|^2)^{t+4}dv(z).$$

Obviously,

$$(3.13) \qquad Y \le \int |(Rq)(z)f(z)|^2(1-|z|^2)^{t+3}dv(z) = C_1\|fRq\|^2_{t+3} \le C_1C^2\|qf\|^2_{t+1-\epsilon},$$

where the second $\le$ is due to the membership $q \in \mathcal{P}_n(t;\epsilon)$.

To estimate X, note that R commutes with $M_{\bar{z}_j}$. Consequently,

$$R(f\partial_j q) - \bar{z}_j R(fRq) = R(f\partial_j q) - R(f\bar{z}_j Rq) = R\{f(\partial_j q - \bar{z}_j Rq)\}.$$

We have $\partial_j - \bar{z}_j R = (1-|z|^2)\partial_j + \sum_{k\neq j} z_k L_{j,k}$. Therefore

$$(3.14) \qquad X \le n(X_1 + \cdots + X_n),$$

where

$$X_k = \int |(R(fz_kL_{j,k}q))(z)|^2(1-|z|^2)^{t+4}dv(z) \quad \text{for } k \neq j \text{ and}$$
$$X_j = \int |(R((1-|z|^2)f\partial_j q))(z)|^2(1-|z|^2)^{t+4}dv(z).$$

By the product rule, $R(fz_kL_{j,k}q) = z_kfL_{k,j}q + z_kR(fL_{j,k}q)$. Hence if $k \neq j$, then

$$X_k \le 2(A_k + B_k),$$

where

$$A_k = \int |(L_{j,k}q)(z)f(z)|^2(1-|z|^2)^{t+4}dv(z) \quad \text{and}$$
$$B_k = \int |(R(fL_{j,k}q))(z)|^2(1-|z|^2)^{t+4}dv(z).$$

Since $t+4 \ge -3+4 = 1$, we can apply Proposition 2.1 to obtain

$$A_k \le C_1 \int |q(z)f(z)|^2(1-|z|^2)^{t+3}dv(z) = C_1C_2\|qf\|^2_{t+3} \le C_1C_2\|qf\|^2_{t+1-\epsilon}.$$

Since $q \in \mathcal{P}_n(t;\epsilon)$, by Lemma 3.1 we have $B_k \le C_{3.1}(q)\|qf\|^2_{t+1-\epsilon}$. Therefore

$$(3.15) \qquad X_k \le 2(C_1C_2 + C_{3.1}(q))\|qf\|^2_{t+1-\epsilon}$$

for every $k \neq j$.

To estimate X_j, again apply the product rule, and note that $R|z|^2 = |z|^2$. We have

$$X_j \le 3(U + V + W),$$

where

$$\begin{aligned} U &= \int |(\partial_j q)(z)(Rf)(z)|^2 (1 - |z|^2)^{t+6} dv(z), \\ V &= \int |(\partial_j q)(z) f(z)|^2 (1 - |z|^2)^{t+4} dv(z) \quad \text{and} \\ W &= \int |(R\partial_j q)(z) f(z)|^2 (1 - |z|^2)^{t+6} dv(z). \end{aligned}$$

Applying Proposition 2.3, we have

$$U \le C_3 \int |q(z)(Rf)(z)|^2 (1 - |z|^2)^{t+4} dv(z) \le 2C_3(U_1 + U_2),$$

where

$$\begin{aligned} U_1 &= \int |(R(qf))(z)|^2 (1 - |z|^2)^{t+4} dv(z) \quad \text{and} \\ U_2 &= \int |(Rq)(z) f(z)|^2 (1 - |z|^2)^{t+4} dv(z). \end{aligned}$$

By (2.7), $U_1 \le C_4 \|qf\|_{t+2}^2 \le C_4 \|qf\|_{t+1-\epsilon}^2$. Also, $U_2 \le C_5 \|fRq\|_{t+3}^2 \le C_5 C^2 \|qf\|_{t+1-\epsilon}^2$. Therefore

$$U \le 2C_3(C_4 + C_5 C^2) \|qf\|_{t+1-\epsilon}^2.$$

Applying Lemma 3.2 to W and Lemma 3.3 to V, we have

$$V + W \le (C_{3.3}(q) + C_{3.2}(q)) \|qf\|_{t+1-\epsilon}^2.$$

Consequently,

$$X_j \le 3\{2C_3(C_4 + C_5 C^2) + C_{3.3}(q) + C_{3.2}(q)\} \|qf\|_{t+1-\epsilon}^2.$$

Recalling (3.14) and (3.15), we see that

$$X \le n\{2(n-1)(C_1 C_2 + C_{3.1}(q)) + 6C_3(C_4 + C_5 C^2) + 3C_{3.3}(q) + 3C_{3.2}(q)\} \|qf\|_{t+1-\epsilon}^2.$$

Finally, combining this inequality with (3.13) and (3.12), the proposition is proved. □

4. Essential normality

Our goal for this section is very clear: we want to show that for weights $t \geq -3$, the membership $q \in \mathcal{P}_n(t;\epsilon)$ implies the desired essential normality for the submodule $[q]^{(t)}$.

Lemma 4.1. *Let $f \in \mathbf{C}[z_1, \dots, z_n]$ be such that $f(0) = 0$. Then for every $t \geq -n$ and every $j \in \{1, \dots, n\}$ we have $\|(M_{z_j}^{(t)*} - M_{z_j}^{(t+4)*})f\|_{t+2}^2 \leq 2^8\|f\|_{t+4}^2$.*

Proof. Let e_j be the element in $\mathbf{Z}_+^n$ whose j-th component is 1 and whose other components are 0. Using (1.2), straightforward calculation gives us

$$M_{z_j}^{(t)*}z^\alpha = \frac{\alpha_j}{n+t+|\alpha|}z^{\alpha-e_j} \tag{4.1}$$

whenever the j-th component α_j of α is greater than 0. Hence

$$(M_{z_j}^{(t)*} - M_{z_j}^{(t+4)*})z^\alpha = \frac{4\alpha_j}{(n+t+|\alpha|)(n+t+4+|\alpha|)}z^{\alpha-e_j} = 4M_{z_j}^{(t+4)*}(N+n+t)^{-1}z^\alpha.$$

Recall that we denote the number operator by N. Since $f(0) = 0$, $(N+n+t)^{-1}f$ is well defined. Thus we can define

$$\tilde{f} = (N+1+n+t+2)(N+n+t)^{-1}f.$$

Obviously, $\|\tilde{f}\|_\tau \leq 4\|f\|_\tau$ for every $\tau \geq -n$ and $\tilde{f}(0) = 0$. From the above we see that

$$(M_{z_j}^{(t)*} - M_{z_j}^{(t+4)*})f = 4M_{z_j}^{(t+4)*}(N+1+n+t+2)^{-1}\tilde{f}.$$

By (4.1), we have $\|M_{z_j}^{(t+4)*}g\|_{t+2} \leq \|M_{z_j}^{(t+2)*}g\|_{t+2} \leq \|g\|_{t+2}$, $g \in \mathbf{C}[z_1, \dots, z_n]$. Hence

$$\|(M_{z_j}^{(t)*} - M_{z_j}^{(t+4)*})f\|_{t+2}^2 \leq 2^4\|(N+1+n+t+2)^{-1}\tilde{f}\|_{t+2}^2. \tag{4.2}$$

Applying Proposition 2.5(1) to the case $\ell = 1$ and $k = 0$, we have

$$\|(N+1+n+t+2)^{-1}\tilde{f}\|_{t+2}^2 \leq \frac{n+2+t+2+1}{(1+1+n+t+2)^2}\|\tilde{f}\|_{t+4}^2 \leq \|\tilde{f}\|_{t+4}^2 \leq 2^4\|f\|_{t+4}^2.$$

Combining this with (4.2), the lemma follows. □

Lemma 4.2. *Suppose that $t \geq -3$ and that $0 \leq \epsilon < 1$. Then for every $q \in \mathcal{P}_n(t;\epsilon)$, there is a constant $C_{4.2}(q)$ such that*

$$\|(N+1+n+t)^{-1}(M_{\partial_j q}^{(t)} - M_{z_j}^{(t)*}M_{Rq}^{(t)})f\|_t^2 \leq C_{4.2}(q)\|qf\|_{t+1-\epsilon}^2$$

for every $f \in \mathbf{C}[z_1, \dots, z_n]$ satisfying the condition $f(0) = 0$ and every $j \in \{1, \dots, n\}$.

Proof. Since $f(0) = 0$ and $(Rq)(0) = 0$, we have $((M_{\partial_j q}^{(t)} - M_{z_j}^{(t)*}M_{Rq}^{(t)})f)(0) = 0$. Thus another application of Proposition 2.5(1) to the case $\ell = 1$ and $k = 0$ gives us

$$\|(N+1+n+t)^{-1}(M_{\partial_j q}^{(t)} - M_{z_j}^{(t)*}M_{Rq}^{(t)})f\|_t^2 \leq \|(M_{\partial_j q}^{(t+2)} - M_{z_j}^{(t)*}M_{Rq}^{(t+2)})f\|_{t+2}^2.$$

Since $M_{z_j}^{(t)*} = M_{z_j}^{(t+4)*} + (M_{z_j}^{(t)*} - M_{z_j}^{(t+4)*})$, we have

$$\|(N+1+n+t)^{-1}(M^{(t)}_{\partial_j q} - M^{(t)*}_{z_j} M^{(t)}_{Rq})f\|_t^2 \le 2(X+Y),$$

where

$$X = \|(M^{(t+2)}_{\partial_j q} - M^{(t+4)*}_{z_j} M^{(t+2)}_{Rq})f\|^2_{t+2} \quad \text{and} \quad Y = \|(M^{(t)*}_{z_j} - M^{(t+4)*}_{z_j})fRq\|^2_{t+2}.$$

By Proposition 3.5, we have $X \le C_{3.5}(q)\|qf\|^2_{t+1-\epsilon}$. Applying Lemma 4.1,

$$Y \le 2^8\|fRq\|^2_{t+4} \le 2^8\|fRq\|^2_{t+3}.$$

Then the membership $q \in \mathcal{P}_n(t;\epsilon)$ gives us $Y \le 2^8C^2\|qf\|^2_{t+1-\epsilon}$. Hence

$$\|(N+1+n+t)^{-1}(M^{(t)}_{\partial_j q} - M^{(t)*}_{z_j} M^{(t)}_{Rq})f\|_t^2 \le 2(C_{3.5}(q) + 2^8C^2)\|qf\|^2_{t+1-\epsilon},$$

proving the lemma. □

Lemma 4.3. *Suppose that* $t \ge -3$ *and that* $0 \le \epsilon < 1$. *For each* $q \in \mathcal{P}_n(t;\epsilon)$, *there is a constant* $C_{4.3} = C_{4.3}(q)$ *such that the following estimate holds*: *Suppose that* $f \in \mathbf{C}[z_1,\dots,z_n]$ *satisfies the condition* $(\partial^\alpha f)(0) = 0$ *for* $|\alpha| \le \ell+1$, *where* $\ell \in \mathbf{N}$. *Then for every natural number* $k \ge 1$ *and every* $j \in \{1,\dots,n\}$,

$$\begin{aligned}\|(N+1+n+t)^{-k-1}&(M^{(t)}_{\partial_j R^k q} - M^{(t)*}_{z_j} M^{(t)}_{R^{k+1}q})f\|_t \\ &\le \frac{(n+2k+4+t+\ell)^{\ell+2}}{(\ell+1+n+t)^{k+1}} C^{k+1}_{4.3}\|qf\|_{t+1-\epsilon}.\end{aligned}$$

Proof. This is similar to the proof of [15,Proposition 5.3], but involves more steps because we now allow $t \ge -3$. Also note that this lemma only considers $k \ge 1$. Since

$$M^{(t)}_{\partial_j R^k q} - M^{(t)*}_{z_j} M^{(t)}_{R^{k+1}q} = (M^{(t)}_{\partial_j R^k q} - M^{(2k+2+t)*}_{z_j} M^{(t)}_{R^{k+1}q}) - (M^{(t)*}_{z_j} - M^{(2k+2+t)*}_{z_j})M^{(t)}_{R^{k+1}q},$$

we have

$$\|(N+1+n+t)^{-k-1}(M^{(t)}_{\partial_j R^k q} - M^{(t)*}_{z_j} M^{(t)}_{R^{k+1}q})f\|_t \le A+B, \tag{4.3}$$

where

$$\begin{aligned}A &= \|(N+1+n+t)^{-k-1}(M^{(t)}_{\partial_j R^k q} - M^{(2k+2+t)*}_{z_j} M^{(t)}_{R^{k+1}q})f\|_t \quad \text{and} \\ B &= \|(N+1+n+t)^{-k-1}(M^{(t)*}_{z_j} - M^{(2k+2+t)*}_{z_j})M^{(t)}_{R^{k+1}q}f\|_t.\end{aligned}$$

For A, we apply Proposition 2.5(1), which gives us

$$A \le \frac{(n+2k+2+t+\ell)^\ell}{(\ell+1+n+t)^{k+1}}\|(M^{(2k+2+t)}_{\partial_j R^k q} - M^{(2k+2+t)*}_{z_j} M^{(2k+2+t)}_{R^{k+1}q})f\|_{2k+2+t}. \tag{4.4}$$

Since $k \geq 1$ and $t \geq -3$, $\mathcal{H}^{(2k+2+t)}$ is a weighted Bergman space, and we have

$$\begin{aligned}
&\|(M^{(2k+2+t)}_{\partial_j R^k q} - M^{(2k+2+t)*}_{z_j} M^{(2k+2+t)}_{R^{k+1}q})f\|_{2k+2+t} \\
&\quad \leq a^{1/2}_{n,2k+2+t}\left(\int |\{(\partial_j R^k q)(z) - \bar{z}_j(R^{k+1}q)(z)\}f(z)|^2(1-|z|^2)^{2k+2+t}dv(z)\right)^{1/2}.
\end{aligned}$$

The identity $\partial_j - \bar{z}_j R = (1-|z|^2)\partial_j + \sum_{i\neq j} z_i L_{j,i}$ then leads to

$$\begin{aligned}
&\|(M^{(2k+2+t)}_{\partial_j R^k q} - M^{(2k+2+t)*}_{z_j} M^{(2k+2+t)}_{R^{k+1}q})f\|_{2k+2+t} \\
&\qquad \leq a^{1/2}_{n,2k+2+t}\left(\int |(\partial_j R^k q)(z)f(z)|^2(1-|z|^2)^{2k+4+t}dv(z)\right)^{1/2} \\
&\qquad + a^{1/2}_{n,2k+2+t}\sum_{i\neq j}\left(\int |(L_{j,i}R^k q)(z)f(z)|^2(1-|z|^2)^{2k+2+t}dv(z)\right)^{1/2}.
\end{aligned} \tag{4.5}$$

Suppose that the degree of q equals $K \geq 1$. Applying Proposition 2.3 to the polynomial $R^k q$ and Proposition 2.2 to Rq, we obtain

$$\begin{aligned}
&\int |(\partial_j R^k q)(z)f(z)|^2(1-|z|^2)^{2k+4+t}dv(z) \\
&\qquad \leq C^{K(2k+4+t)}_{2.3}(K!)^2\int |(R^k q)(z)f(z)|^2(1-|z|^2)^{2k+2+t}dv(z) \\
&\qquad \leq (C_{2.3}C_{2.2})^{K(3k+4+t)}(K!)^4\int |(Rq)(z)f(z)|^2(1-|z|^2)^{4+t}dv(z) \\
&\qquad \leq (C_{2.3}C_{2.2})^{K(3k+4+t)}(K!)^4\|fRq\|^2_{t+3}.
\end{aligned}$$

Since $q \in \mathcal{P}_n(t;\epsilon)$, we now have

$$\int |(\partial_j R^k q)(z)f(z)|^2(1-|z|^2)^{2k+4+t}dv(z) \leq (C_{2.3}C_{2.2})^{K(3k+4+t)}(K!)^4C^2\|qf\|^2_{t+1-\epsilon}. \tag{4.6}$$

Similarly, we apply Propositions 2.1 and 2.2 to obtain

$$\begin{aligned}
&\int |(L_{j,i}R^k q)(z)f(z)|^2(1-|z|^2)^{2k+2+t}dv(z) \\
&\qquad \leq C_{2.1}2^{2k+2+t}(K!)^2\int |(R^k q)(z)f(z)|^2(1-|z|^2)^{2k+1+t}dv(z) \\
&\qquad \leq C_{2.1}(2C_{2.2})^{K(3k+2+t)}(K!)^4\int |(Rq)(z)f(z)|^2(1-|z|^2)^{3+t}dv(z).
\end{aligned}$$

Applying the membership $q \in \mathcal{P}_n(t;\epsilon)$ to bound the last integral, we have

$$\int |(L_{j,i}R^k q)(z)f(z)|^2(1-|z|^2)^{2k+2+t}dv(z) \leq C_{2.1}(2C_{2.2})^{K(3k+2+t)}(K!)^4C^2\|qf\|^2_{t+1-\epsilon}. \tag{4.7}$$

Note that $a_{n,2k+2+t} \le (n!)^{-1}(n+2k+2+t)^n$. Combining (4.5), (4.6) and (4.7), we see that there is a C_1 that depends only on n, t and $q \in \mathcal{P}_n(t;\epsilon)$ such that

$$\|(M^{(2k+2+t)}_{\partial_j R^k q} - M^{(2k+2+t)*}_{z_j} M^{(2k+2+t)}_{R^{k+1}q})f\|_{2k+2+t} \le C_1^{k+1}\|qf\|_{t+1-\epsilon}.$$

Recalling (4.4), this gives us

$$A \le \frac{(n+2k+2+t+\ell)^\ell}{(\ell+1+n+t)^{k+1}} C_1^{k+1}\|qf\|_{t+1-\epsilon}. \tag{4.8}$$

By Proposition 2.5(2), we have

$$B \le \frac{(n+2k+4+t+\ell)^{\ell+2}}{(\ell+1+n+t)^{k+1}} \|M^{(t)}_{R^{k+1}q} f\|_{2k+4+t}.$$

Applying Proposition 2.2 to the polynomial Rq, we obtain

$$\begin{aligned}\|M^{(t)}_{R^{k+1}q} f\|^2_{2k+4+t} &= a_{n,2k+4+t}\int |(R^k Rq)(z)f(z)|^2(1-|z|^2)^{2k+4+t}dv(z)\\ &\le a_{n,2k+4+t} C_{2.2}^{K(3k+4+t)}(K!)^2 \int |(Rq)(z)f(z)|^2(1-|z|^2)^{4+t}dv(z).\end{aligned}$$

Hence

$$\begin{aligned}\|M^{(t)}_{R^{k+1}q} f\|^2_{2k+4+t} &\le a_{n,2k+4+t} C_{2.2}^{K(3k+4+t)}(K!)^2\|fRq\|^2_{t+3}\\ &\le a_{n,2k+4+t} C_{2.2}^{K(3k+4+t)}(K!)^2 C^2\|qf\|^2_{t+1-\epsilon}.\end{aligned}$$

Thus there is a C_2 that depends only on n, t and $q \in \mathcal{P}_n(t;\epsilon)$ such that

$$\|M^{(t)}_{R^{k+1}q} f\|_{2k+4+t} \le C_2^{k+1}\|qf\|_{t+1-\epsilon}.$$

Consequently,

$$B \le \frac{(n+2k+4+t+\ell)^{\ell+2}}{(\ell+1+n+t)^{k+1}} C_2^{k+1}\|qf\|_{t+1-\epsilon}.$$

Combining this with (4.8) and (4.3), the proof of the lemma is complete. □

Proposition 4.4. *Suppose that $t \ge -3$ and that $0 \le \epsilon < 1$. Let $q \in \mathcal{P}_n(t;\epsilon)$. Then the submodule $[q]^{(t)}$ of $\mathcal{H}^{(t)}$ is essentially normal. More precisely, the submodule operators*

$$Z^{(t)}_{q,j} = M_{z_j}|[q]^{(t)}, \quad 1 \le j \le n,$$

have the property $[Z^{(t)*}_{q,j}, Z^{(t)}_{q,i}] \in \mathcal{C}^+_{n/(1-\epsilon)}$ for all $j, i \in \{1, \ldots, n\}$.

Proof. Given $q \in \mathcal{P}_n(t;\epsilon)$, let $C_{4.3} = C_{4.3}(q)$ be the constant provided by Lemma 4.3. We then pick an $\ell \in \mathbf{N}$ satisfying the condition

$$\ell + 1 + n + t > 2C_{4.3}. \tag{4.9}$$

With this ℓ, we define

$$E = \{qf : f \in \mathbf{C}[z_1, \dots, z_n],\ (\partial^\alpha f)(0) = 0 \text{ for } |\alpha| \le \ell + 1\}.$$

Let $Q^{(t)}$ be the orthogonal projection from $\mathcal{H}^{(t)}$ onto $\mathcal{H}^{(t)} \ominus [q]^{(t)}$. Let $j \in \{1, \dots, n\}$, and let $f \in \mathbf{C}[z_1, \dots, z_n]$ be such that $(\partial^\alpha f)(0) = 0$ for $|\alpha| \le \ell + 1$. Then

$$Q^{(t)} M_{z_j}^{(t)*} qf = Q^{(t)} M_{z_j}^{(t)*} M_q^{(t)} f = Q^{(t)} (M_{z_j}^{(t)*} M_q^{(t)} - M_q^{(t)} M_{z_j}^{(t)*}) f.$$

Applying Propositions 2.4, we have

$$\|Q^{(t)} M_{z_j}^{(t)*} qf\|_t \le \sum_{k=0}^{\infty} \|(N + 1 + n + t)^{-k-1} (M^{(t)}_{\partial_j R^k q} - M_{z_j}^{(t)*} M^{(t)}_{R^{k+1} q}) f\|_t.$$

For the sum on the right-hand side, we apply Lemma 4.2 to the term $k = 0$ and Lemma 4.3 to the terms $k \ge 1$. The result of this is

$$\|Q^{(t)} M_{z_j}^{(t)*} qf\|_t \le C_{4.2}^{1/2}(q) \|qf\|_{t+1-\epsilon} + \sum_{k=1}^{\infty} \frac{(n + 2k + 4 + t + \ell)^{\ell+2}}{(\ell + 1 + n + t)^{k+1}} C_{4.3}^{k+1} \|qf\|_{t+1-\epsilon}.$$

That is,

$$\|Q^{(t)} M_{z_j}^{(t)*} g\|_t \le C \|g\|_{t+1-\epsilon} \quad \text{for every} \quad g \in E, \tag{4.10}$$

where

$$C = C_{4.2}^{1/2}(q) + \sum_{k=1}^{\infty} \frac{(n + 2k + 4 + t + \ell)^{\ell+2}}{(\ell + 1 + n + t)^{k+1}} C_{4.3}^{k+1}.$$

Note that (4.9) ensures that $C < \infty$. Let $E^{(t)}$ be the closure of E in $\mathcal{H}^{(t)}$, and let $\mathcal{E}^{(t)} : \mathcal{H}^{(t)} \to E^{(t)}$ be the orthogonal projection. By Proposition 2.6, (4.10) implies that

$$Q^{(t)} M_{z_j}^{(t)*} \mathcal{E}^{(t)} \in \mathcal{C}^+_{2n/(1-\epsilon)}.$$

Let $P^{(t)}$ be the orthogonal projection from $\mathcal{H}^{(t)}$ onto $[q]^{(t)}$. Since $E^{(t)}$ is a subspace of $[q]^{(t)}$ of finite codimension, we have $\operatorname{rank}(P^{(t)} - \mathcal{E}^{(t)}) < \infty$. Therefore

$$Q^{(t)} M_{z_j}^{(t)*} P^{(t)} \in \mathcal{C}^+_{2n/(1-\epsilon)}.$$

On the space $\mathcal{H}^{(t)}$, it is well known that $[M_{z_j}^{(t)*}, M_{z_i}^{(t)}] \in \mathcal{C}^+_n$. Thus it follows from a standard argument that $[Z_{q,j}^{(t)*}, Z_{q,i}^{(t)}] \in \mathcal{C}^+_{n/(1-\epsilon)}$, $i, j \in \{1, \dots, n\}$. This completes the proof. $\square$

5. Estimates of Dirichlet type

Having proved Proposition 4.4, our next goal is to establish the equality $\mathcal{P}_n(t;0) = \mathbf{C}[z_1, \ldots, z_n]$ for the range of weights $-3 < t \le -2$. This together with Proposition 4.4 will obviously imply Theorem 1.2.

Lemma 5.1. *For each real number $s < 1/2$, there exists a constant $0 < C_{5.1}(s) < \infty$ such that the inequality*

$$\sum_{i=1}^{\infty} \left| \frac{1}{i^s} \sum_{j=i}^{\infty} \frac{c_j}{j^{1-s}} \right|^2 \le C_{5.1}(s) \sum_{i=1}^{\infty} |c_i|^2$$

holds for every $\{c_i\} \in \ell^2(\mathbf{N})$.

Proof. This lemma is in fact a discrete variant of a well-known inequality of Hardy. See, e.g., [6,Lemma 3.3.9]. But since its proof is easy enough, let us produce it here anyway.

Given $s < 1/2$, we set

$$C(s) = 2^{2(1-s)} \int_1^{\infty} \frac{1}{y^{2(1-s)}} (2 + \log y)^2 \, dy.$$

The condition $s < 1/2$ ensures that $C(s) < \infty$. If $j \le x \le j+1$ with $j \ge 1$, then $1/j \le 2/x$. Thus for each $i \in \mathbf{N}$ we have

$$(5.1) \qquad \frac{1}{i} \sum_{j=i}^{\infty} \left(\frac{i}{j}\right)^{2(1-s)} \left(2 + \log \frac{j}{i}\right)^2 \le \frac{1}{i} \int_i^{\infty} \left(\frac{2i}{x}\right)^{2(1-s)} \left(2 + \log \frac{x}{i}\right)^2 dx = C(s),$$

where the $=$ is obtained by making the substitution $y = x/i$. By a similar argument, there is a $0 < C_1 < \infty$ such that the inequality

$$(5.2) \qquad \sum_{i=1}^{j} \frac{1}{i(2 + \log(j/i))^2} \le C_1$$

holds for every $j \in \mathbf{N}$. Suppose now that $\{c_i\} \in \ell^2(\mathbf{N})$ is given. Then define

$$b_i = \frac{1}{i^s} \sum_{j=i}^{\infty} \frac{c_j}{j^{1-s}} = \frac{1}{i} \sum_{j=i}^{\infty} \left(\frac{i}{j}\right)^{1-s} c_j$$

for each $i \in \mathbf{N}$. Applying the Cauchy-Schwarz inequality and (5.1), we have

$$|b_i|^2 \le \frac{1}{i} \sum_{j=i}^{\infty} \left(\frac{i}{j}\right)^{2(1-s)} \left(2 + \log \frac{j}{i}\right)^2 \cdot \frac{1}{i} \sum_{j=i}^{\infty} \frac{|c_j|^2}{(2 + \log(j/i))^2} \le \frac{C(s)}{i} \sum_{j=i}^{\infty} \frac{|c_j|^2}{(2 + \log(j/i))^2}.$$

Therefore

$$\sum_{i=1}^{\infty}|b_i|^2 \le \sum_{i=1}^{\infty}\frac{C(s)}{i}\sum_{j=i}^{\infty}\frac{|c_j|^2}{(2+\log(j/i))^2} = C(s)\sum_{j=1}^{\infty}|c_j|^2\sum_{i=1}^{j}\frac{1}{i(2+\log(j/i))^2}$$
$$\le C(s)C_1\sum_{j=1}^{\infty}|c_j|^2,$$

where the second $\le$ follows from (5.2). This completes the proof. $\square$

Denote $D = \{z \in \mathbf{C} : |z| < 1\}$, the open unit disc in the complex plane. Let dA be the area measure on D with the normalization $A(D) = 1$. Let $\mathbf{T}$ denote the unit circle $\{\tau \in \mathbf{C} : |\tau| = 1\}$. For this section, we will write ∂ for the one-variable differentiation d/dz on $\mathbf{C}$. But we always write $\mathcal{R}$ for the one-variable radial derivative. That is, $\mathcal{R} = z\partial$.

For each $-1 < t \le 1$ and each one-variable polynomial f, define

$$\mathcal{N}_t(f) = |f(0)|^2 + \int |(\mathcal{R}f)(z)|^2(1-|z|^2)^t dA(z). \tag{5.3}$$

Keep in mind that $\mathcal{N}_t$ has the following rotation invariance: For any polynomial f and any $\tau \in \mathbf{T}$, if we set $f_\tau(z) = f(\tau z)$, then $\mathcal{N}_t(f_\tau) = \mathcal{N}_t(f)$.

Lemma 5.2. *For each $0 < t \le 1$, there is a $0 < C_{5.2}(t) < \infty$ such that the inequality*

$$\int |f(z)|^2\frac{(1-|z|^2)^t}{|1-z|^2}dA(z) \le C_{5.2}(t)\mathcal{N}_t(f)$$

holds for every one-variable polynomial f.

Proof. Let $0 < t \le 1$ be given. Then an easy integration shows that

$$\int |z^k|^2(1-|z|^2)^t dA(z) = \frac{k!}{\prod_{j=0}^{k}(t+1+j)} \tag{5.4}$$

for each $k \in \mathbf{Z}_+$. It follows from the asymptotic formula (2.2) that there are positive numbers $\alpha = \alpha(t)$ and $\beta = \beta(t)$ such that

$$\frac{\alpha}{(k+1)^{1+t}} \le \int |z^k|^2(1-|z|^2)^t dA(z) \le \frac{\beta}{(k+1)^{1+t}} \tag{5.5}$$

for every $k \in \mathbf{Z}_+$.

Given a one-variable polynomial f, we write it in the form $f(z) = \sum_{k=0}^{\infty} u_k z^k$, where $u_k \in \mathbf{C}$, and $u_k = 0$ for all but a finite number of k's. Then

$$\frac{f(z)}{1-z} = \sum_{k=0}^{\infty}\sum_{i=0}^{k} u_i z^k,$$

$z \in D$. Thus it follows from (5.5) that

$$\int |f(z)|^2 \frac{(1-|z|^2)^t}{|1-z|^2} dA(z) \le \sum_{k=0}^{\infty} \frac{\beta}{(k+1)^{1+t}} \left| \sum_{i=0}^{k} u_i \right|^2 \le \sum_{k=0}^{\infty} \frac{2\beta}{(k+1)^{1+t}} \sum_{0 \le i \le j \le k} |u_i||u_j|.$$

A change of the order of summation in the above gives us

$$\int |f(z)|^2 \frac{(1-|z|^2)^t}{|1-z|^2} dA(z) \le \sum_{i=0}^{\infty} |u_i| \sum_{j=i}^{\infty} |u_j| \sum_{k=j}^{\infty} \frac{2\beta}{(k+1)^{1+t}} \le \frac{8\beta}{t} \sum_{i=0}^{\infty} |u_i| \sum_{j=i}^{\infty} \frac{|u_j|}{(j+1)^t}.$$

Now set $s = (1-t)/2$. Since $t > 0$, we have $s < 1/2$. For each $i \in \mathbf{N}$, define $c_i = |u_{i-1}| i^s = |u_{i-1}| i^{(1-t)/2}$. Since $t + s = (1+t)/2 = 1 - s$, the above inequality can be rewritten as

$$\int |f(z)|^2 \frac{(1-|z|^2)^t}{|1-z|^2} dA(z) \le \frac{8\beta}{t} \sum_{i=1}^{\infty} \frac{c_i}{i^s} \sum_{j=i}^{\infty} \frac{c_j}{j^{1-s}}.$$

By the Cauchy-Schwarz inequality, we now have

$$\int |f(z)|^2 \frac{(1-|z|^2)^t}{|1-z|^2} dA(z) \le \frac{8\beta}{t} \left(\sum_{i=1}^{\infty} c_i^2 \right)^{1/2} \left(\sum_{i=1}^{\infty} \left\{ \frac{1}{i^s} \sum_{j=i}^{\infty} \frac{c_j}{j^{1-s}} \right\}^2 \right)^{1/2}.$$

An application of Lemma 5.1 then gives us

$$\int |f(z)|^2 \frac{(1-|z|^2)^t}{|1-z|^2} dA(z) \le \frac{8\beta}{t} C_{5.1}^{1/2}(s) \sum_{i=1}^{\infty} c_i^2 = \frac{8\beta}{t} C_{5.1}^{1/2}(s) \sum_{i=0}^{\infty} |u_i|^2 (i+1)^{1-t}. \tag{5.6}$$

On the other hand,

$$(\mathcal{R}f)(z) = \sum_{k=1}^{\infty} k u_k z^k.$$

Applying (5.5), we have

$$\begin{aligned} |f(0)|^2 + \int |(\mathcal{R}f)(z)|^2 (1-|z|^2)^t dA(z) &\ge |u_0|^2 + \sum_{k=1}^{\infty} \frac{\alpha k^2 |u_k|^2}{(k+1)^{1+t}} \\ &\ge \frac{1}{4} \min\{\alpha, 1\} \sum_{k=0}^{\infty} (k+1)^{1-t} |u_k|^2. \end{aligned}$$

Combining this inequality with (5.6), the lemma follows. □

Lemma 5.3. *For each $0 < t \le 1$, there is a $0 < C_{5.3}(t) < \infty$ such that the inequality*

$$|f(w)|^2 (1-|w|^2)^t \le C_{5.3}(t) \mathcal{N}_t(f)$$

holds for every $w \in D$ and every one-variable polynomial f.

Proof. Let $0 < t \le 1$ be given. Set $c_0^{(t)} = 1$. For each $k \in \mathbf{N}$, set

$$c_k^{(t)} = \left(k^2 \int |z^k|^2(1-|z|^2)^t dA(z)\right)^{-1}.$$

For each $w \in D$, define

$$K_w^{(t)}(z) = \sum_{k=0}^{\infty} c_k^{(t)} \bar{w}^k z^k.$$

We first show that

$$\int |(\mathcal{R}K_w^{(t)})(z)|^2(1-|z|^2)^t dA(z) \le \frac{(6/t)}{(1-|w|^2)^t} \tag{5.7}$$

for every $w \in D$. Indeed since $(\mathcal{R}K_w^{(t)})(z) = \sum_{k=1}^{\infty} kc_k^{(t)}\bar{w}^k z^k$, we have

$$\int |(\mathcal{R}K_w^{(t)})(z)|^2(1-|z|^2)^t dA(z) = \sum_{k=1}^{\infty} |kc_k^{(t)}\bar{w}^k|^2 \int |z^k|^2(1-|z|^2)^t dA(z) = \sum_{k=1}^{\infty} c_k^{(t)}|w|^{2k}.$$

By (5.4), for each $k \in \mathbf{N}$ we have

$$c_k^{(t)} = \frac{(t+k)(t+k+1)}{k^2 t} \cdot \frac{1}{k!}\prod_{j=0}^{k-1}(t+j) \le \left(\frac{6}{t}\right)\frac{1}{k!}\prod_{j=0}^{k-1}(t+j).$$

Therefore

$$\int |(\mathcal{R}K_w^{(t)})(z)|^2(1-|z|^2)^t dA(z) \le \frac{6}{t}\sum_{k=1}^{\infty}\frac{(|w|^2)^k}{k!}\prod_{j=0}^{k-1}(t+j).$$

Comparing this with the power series expansion

$$\frac{1}{(1-u)^t} = 1 + \sum_{k=1}^{\infty}\frac{u^k}{k!}\prod_{j=0}^{k-1}(t+j), \quad u \in D,$$

(5.7) is proved.

Let a polynomial f be given, and again write it in the form $f(z) = \sum_{k=0}^{\infty} u_k z^k$. Then for each $w \in D$, it follows from the definition of $c_k^{(t)}$ that

$$f(w) = u_0 + \sum_{k=1}^{\infty} u_k w^k = f(0)\overline{K_w^{(t)}(0)} + \int (\mathcal{R}f)(z)\overline{(\mathcal{R}K_w^{(t)})(z)}(1-|z|^2)^t dA(z).$$

Applying the Cauchy-Schwarz inequality and (5.7), we now have

$$
\begin{aligned}
|f(w)|^2 &\le 2|f(0)|^2 + 2\int |(\mathcal{R}f)(z)|^2(1-|z|^2)^t dA(z) \int |(\mathcal{R}K_w^{(t)})(z)|^2(1-|z|^2)^t dA(z) \\
&\le 2|f(0)|^2 + \frac{(12/t)}{(1-|w|^2)^t}\int |(\mathcal{R}f)(z)|^2(1-|z|^2)^t dA(z) \le \frac{(12/t)}{(1-|w|^2)^t}\mathcal{N}_t(f).
\end{aligned}
$$

This proves the lemma. □

For each $a \in D$, we define the disc $D(a) = \{w \in D : |w-a| < (1/2)(1-|a|)\}$.

Lemma 5.4. *Let $0 < t \le 1$ be given. Then there is a constant $0 < C_{5.4}(t) < \infty$ such that for every one-variable polynomial f and every $a \in D$, we have*

$$\int_{D(a)} |f(z)|^2(1-|z|^2)^t dA(z) \le C_{5.4}(t)\mathcal{N}_t(\Phi_{f,a}),$$

where $\Phi_{f,a}(z) = (z-a)f(z)$.

Proof. For each $a \in D$, define $W(a) = \{w \in D : (3/4)(1-|a|) < |w-a| < (7/8)(1-|a|)\}$. It is elementary that there is a C_1 such that the inequality

$$\sup_{z \in D(a)} |f(z)|^2 \le \frac{C_1}{(1-|a|)^2}\int_{W(a)} |f|^2 dA$$

holds for every $a \in D$ and every polynomial f. If $w \in W(a)$, then $(4/3)(1-|a|)^{-1}|w-a| > 1$ by definition. Therefore

$$\sup_{z \in D(a)} |f(z)|^2 \le \frac{2C_1}{(1-|a|)^2}\int_{W(a)} \left|\frac{w-a}{1-|a|}f(w)\right|^2 dA(w) = \frac{2C_1}{(1-|a|)^4}\int_{W(a)} |\Phi_{f,a}|^2 dA.$$

If $z \in D(a)$, then $1-|z| \le 1-|a|+|a-z| < (3/2)(1-|a|)$. If $w \in W(a)$, then $1-|w| \ge 1-|a|-|a-w| \ge (1/8)(1-|a|)$. Thus the inequality

$$1-|z| \le 12(1-|w|)$$

holds for every pair of $z \in D(a)$ and $w \in W(a)$. Let $0 < t \le 1$. Then the above yields

$$\sup_{z \in D(a)} |f(z)|^2(1-|z|^2)^t \le \frac{48C_1}{(1-|a|)^4}\int_{W(a)} |\Phi_{f,a}(w)|^2(1-|w|^2)^t dA(w).$$

Lemma 5.3 tells us that

$$|\Phi_{f,a}(w)|^2(1-|w|^2)^t \le C_{5.3}(t)\mathcal{N}_t(\Phi_{f,a})$$

for every $w \in D$. Therefore

$$\sup_{z \in D(a)} |f(z)|^2(1-|z|^2)^t \le \frac{48C_1 A(W(a))}{(1-|a|)^4}C_{5.3}(t)\mathcal{N}_t(\Phi_{f,a}) \le \frac{48C_1}{(1-|a|)^2}C_{5.3}(t)\mathcal{N}_t(\Phi_{f,a}).$$

Consequently

$$\int_{D(a)} |f(z)|^2(1-|z|^2)^t dA(z) \le \frac{48C_1A(D(a))}{(1-|a|)^2} C_{5.3}(t)\mathcal{N}_t(\Phi_{f,a}) = 12C_1C_{5.3}(t)\mathcal{N}_t(\Phi_{f,a}).$$

This completes the proof. □

Lemma 5.5. *Let $0 < t \le 1$ be given. Then there is a constant $0 < C_{5.5}(t) < \infty$ such that for every one-variable polynomial f and every $a \in \mathbf{C}$, we have*

$$\int |f(z)|^2(1-|z|^2)^t dA(z) \le C_{5.5}(t)\mathcal{N}_t(\Phi_{f,a}),$$

where $\Phi_{f,a}(z) = (z-a)f(z)$ and $\mathcal{N}_t$ was defined by (5.3).

Proof. Given an $a \in \mathbf{C}$, write it in the form $a = \rho\tau$, where $\rho \in [0,\infty)$ and $\tau \in \mathbf{T}$. (1) First suppose that $a \notin D$, i.e., $\rho \ge 1$. This leads to the inequality

$$|z-\rho| \ge |z-1|$$

for every $z \in D$. Given a polynomial f, define $h(z) = (\tau z - a)f(\tau z) = \tau(z-\rho)f(\tau z)$. Using the rotation invariance of dA and applying Lemma 5.2 to h, we have

$$\int |f(z)|^2(1-|z|^2)^t dA(z) = \int |f(\tau z)|^2(1-|z|^2)^t dA(z) = \int |h(z)|^2\frac{(1-|z|^2)^t}{|z-\rho|^2}dA(z)$$
$$\le \int |h(z)|^2\frac{(1-|z|^2)^t}{|z-1|^2}dA(z) \le C_{5.2}(t)\mathcal{N}_t(h) = C_{5.2}(t)\mathcal{N}_t(\Phi_{f,a}),$$

where the last = follows from the relation $h(z) = \Phi_{f,a}(\tau z)$ and the rotation invariance of $\mathcal{N}_t$. This proves the lemma in the case $a \notin D$.

(2) Now let us suppose that $a \in D$, i.e., $\rho \in [0,1)$. Given a polynomial f, we again define $h(z) = (\tau z - a)f(\tau z)$. We have

$$\int |f(z)|^2(1-|z|^2)^t dA(z) = X + Y, \tag{5.8}$$

where

$$X = \int_{D(a)} |f(z)|^2(1-|z|^2)^t dA(z) \quad \text{and} \quad Y = \int_{D\backslash D(a)} |f(z)|^2(1-|z|^2)^t dA(z).$$

Since Lemma 5.4 tells us that

$$X \le C_{5.4}(t)\mathcal{N}_t(\Phi_{f,a}), \tag{5.9}$$

we only need to estimate Y. For this, note that $|\tau - a| = 1-\rho = 1-|a|$. Thus, by the definition of $D(a)$, if $z \notin D(a)$, then $|z-a| \ge (1/2)|\tau - a|$. Hence for each $z \in D\backslash D(a)$,

$$|z-a| = \frac{1}{3}|z-a| + \frac{2}{3}|z-a| \ge \frac{1}{3}\{|z-a| + |\tau - a|\} \ge \frac{1}{3}|z-\tau|.$$

Consequently,

$$\begin{aligned} Y &= \int_{D\setminus D(a)} |(z-a)f(z)|^2 \frac{(1-|z|^2)^t}{|z-a|^2} dA(z) \le 9 \int |(z-a)f(z)|^2 \frac{(1-|z|^2)^t}{|z-\tau|^2} dA(z) \\ &= 9 \int |h(z)|^2 \frac{(1-|z|^2)^t}{|\tau z-\tau|^2} dA(z) = 9 \int |h(z)|^2 \frac{(1-|z|^2)^t}{|z-1|^2} dA(z). \end{aligned}$$

Applying Lemma 5.2 to h, we have

$$Y \le 9C_{5.2}(t)\mathcal{N}_t(h) = 9C_{5.2}(t)\mathcal{N}_t(\Phi_{f,a}),$$

where the $=$ is again due to the relation $h(z) = \Phi_{f,a}(\tau z)$ and the rotation invariance of $\mathcal{N}_t$. Combining this with (5.8) and (5.9), the proof is now complete. □

Proposition 5.6. *Suppose that* $0 < t \le 1$. *Let* g *and* f *be one-variable polynomials. If the degree of* g *equals* $K \ge 1$, *then*

$$\int |(\partial g)(z)f(z)|^2(1-|z|^2)^t dA(z) \le C_{5.5}(t)K^2\mathcal{N}_t(gf),$$

where $C_{5.5}(t)$ *is the constant given in Lemma* 5.5.

Proof. If the degree of g equals K, then there are $c, a_1, \dots, a_K \in \mathbf{C}$ such that

$$g(z) = c(z-a_1)\cdots(z-a_K).$$

By the product rule for differentiation, $\partial g = g_1 + \cdots + g_K$, where

$$g_j(z) = c\prod_{i\ne j}(z-a_i).$$

By the Cauchy-Schwarz inequality, we have

$$\int |(\partial g)(z)f(z)|^2(1-|z|^2)^t dA(z) \le K\sum_{j=1}^{K} \int |g_j(z)f(z)|^2(1-|z|^2)^t dA(z). \tag{5.10}$$

For each $1 \le j \le K$, note that if we define $\Phi_{g_jf,a_j}(z) = (z-a_j)(g_jf)(z)$ as before, then $\Phi_{g_jf,a_j} = gf$. Thus, applying Lemma 5.5 to g_jf and a_j, we obtain

$$\int |g_j(z)f(z)|^2(1-|z|^2)^t dA(z) \le C_{5.5}(t)\mathcal{N}_t(\Phi_{g_jf,a_j}) = C_{5.5}(t)\mathcal{N}_t(gf).$$

Combining this with (5.10), the proposition follows. □

Let $d\sigma$ be the positive, regular Borel measure on S that is invariant under the orthogonal group $O(2n)$, i.e., the group of isometries on $\mathbf{C}^n \cong \mathbf{R}^{2n}$ that fix 0. We take the usual normalization $\sigma(S) = 1$. If h is a function on $\mathbf{B}$, for each $\xi \in S$ we define the function

$$h_\xi(z) = h(z\xi), \quad z \in D,$$

on the unit disc. For each $\xi \in S$, h_ξ is often called a "slice" of h. Since $dv = 2nr^{2n-1}drd\sigma$, $dA = 2rdrdm$, and since $d\sigma$ is invariant under rotation, we have

$$\int G dv = n \int \left(\int G_\xi(z)|z|^{2n-2} dA(z) \right) d\sigma(\xi) \tag{5.11}$$

for every G that is continuous on the closure of $\mathbf{B}$.

Lemma 5.7. *There is a constant* $0 < C_{5.7} < \infty$ *such that the inequality*

$$\int \left(\int |f_\xi(z)|^2 (1 - |z|^2)^t dA(z) \right) d\sigma(\xi) \le C_{5.7} \int |f(\zeta)|^2 (1 - |\zeta|^2)^t dv(\zeta)$$

holds for every $f \in \mathbf{C}[z_1, \ldots, z_n]$ *and every* $0 < t \le 1$.

Proof. First of all, it is an easy exercise to show that there is a constant $0 < C < \infty$ such that the inequality

$$\int_{|z|<1/2} |\varphi(z)|^2 dA(z) \le C \int_{1/2 \le |z| < 1} |\varphi(z)|^2 (1 - |z|^2) dA(z)$$

holds for every one-variable polynomial φ. Thus for each $f \in \mathbf{C}[z_1, \ldots, z_n]$ and each $0 < t \le 1$, we have

$$\begin{aligned} \int \int |f_\xi(z)|^2 (1 - |z|^2)^t dA(z) d\sigma(\xi) &\le (1 + C) \int \int_{1/2 \le |z| < 1} |f_\xi(z)|^2 (1 - |z|^2)^t dA(z) d\sigma(\xi) \\ &\le (1 + C) 2^{2n-2} \int \int_{1/2 \le |z| < 1} |f_\xi(z)|^2 (1 - |z\xi|^2)^t |z|^{2n-2} dA(z) d\sigma(\xi). \end{aligned}$$

Combining this with (5.11), the lemma follows. □

Recall that for each $g \in \mathbf{C}[z_1, \ldots, z_n]$ and each $\xi \in S$, we have the relation $(\mathcal{R} g_\xi)(z) = (Rg)_\xi(z)$, $z \in \mathbf{C}$.

Lemma 5.8. *For each* $0 < t \le 1$, *there is a constant* $0 < C_{5.8}(t) < \infty$ *such that*

$$\int \mathcal{N}_t(f_\xi) d\sigma(\xi) \le C_{5.8}(t) \|f\|_{t-2}^2 \tag{5.12}$$

for every $f \in \mathbf{C}[z_1, \ldots, z_n]$.

Proof. For $f \in \mathbf{C}[z_1, \ldots, z_n]$ and $0 < t \le 1$, Lemma 5.7 gives us

$$\begin{aligned} \int \int |(\mathcal{R} f_\xi)(z)|^2 (1 - |z|^2)^t dA(z) d\sigma(\xi) &= \int \int |(Rf)_\xi(z)|^2 (1 - |z|^2)^t dA(z) d\sigma(\xi) \\ &\le C_{5.7} \int |(Rf)(\zeta)|^2 (1 - |\zeta|^2)^t dv(\zeta). \end{aligned}$$

Combining this with (5.3) and (2.7), we obtain (5.12). □

Proposition 5.9. *For each $0 < t \le 1$, there is a constant $0 < C_{5.9}(t) < \infty$ such that the following estimate holds: Let $q, f \in \mathbf{C}[z_1, \dots, z_n]$. If the degree of q equals $K \ge 1$, then*

$$\int |(Rq)(\zeta)f(\zeta)|^2(1-|\zeta|^2)^t dv(\zeta) \le C_{5.9}(t)K^2\|qf\|_{t-2}^2.$$

Proof. Let $0 < t \le 1$ and $q, f \in \mathbf{C}[z_1, \dots, z_n]$. If $\deg(q) = K$, then Proposition 5.6 gives us

$$\begin{aligned}\int |(Rq)_\xi(z)f_\xi(z)|^2(1-|z\xi|^2)^t dA(z) &= \int |(\mathcal{R}q_\xi)(z)f_\xi(z)|^2(1-|z|^2)^t dA(z)\\ &\le C_{5.5}(t)K^2\mathcal{N}_t(q_\xi f_\xi) = C_{5.5}(t)K^2\mathcal{N}_t((qf)_\xi)\end{aligned}$$

for every $\xi \in S$. Integrating both side with respect to $d\sigma$ and applying (5.11) and Lemma 5.8, we have

$$\begin{aligned}\int |(Rq)(\zeta)f(\zeta)|^2(1-|\zeta|^2)^t dv(\zeta) &\le n\int\int |(Rq)_\xi(z)f_\xi(z)|^2(1-|z\xi|^2)^t dA(z)d\sigma(\xi)\\ &\le nC_{5.5}(t)K^2\int \mathcal{N}_t((qf)_\xi)d\sigma(\xi) \le nC_{5.5}(t)K^2C_{5.8}(t)\|qf\|_{t-2}^2\end{aligned}$$

as promised. □

Proposition 5.10. *For each $-3 < t \le -2$ we have $\mathcal{P}_n(t;0) = \mathbf{C}[z_1, \dots, z_n]$.*

Proof. Let $q \in \mathbf{C}[z_1, \dots, z_n]$ and suppose that the degree of q equals K. If $-3 < t \le -2$, then $0 < t+3 \le 1$. Thus we can apply Proposition 5.9 to obtain

$$\int |(Rq)(\zeta)f(\zeta)|^2(1-|\zeta|^2)^{3+t}dv(\zeta) \le C\|qf\|_{3+t-2}^2 = C\|qf\|_{t+1}^2$$

for every $f \in \mathbf{C}[z_1, \dots, z_n]$, where $C = C_{5.9}(t+3)K^2$. Since $t+3 > 0$, we have

$$\|fRq\|_{t+3}^2 = a_{n,t+3}\int |(Rq)(\zeta)f(\zeta)|^2(1-|\zeta|^2)^{3+t}dv(\zeta).$$

Therefore $\|fRq\|_{t+3}^2 \le a_{n,t+3}C\|qf\|_{t+1}^2$ for every $f \in \mathbf{C}[z_1, \dots, z_n]$. By Definition 1.7, this means $q \in \mathcal{P}_n(t;0)$, completing the proof. □

Proof of Theorem 1.2. This follows immediately from Propositions 5.10 and 4.4. □

6. The weight $t = -3$

In view of the proof of Theorem 1.2 above, for the case $t = -3$ we obviously would like to establish the equality $\mathcal{P}_n(-3;0) = \mathbf{C}[z_1, \dots, z_n]$, or at least $\mathcal{P}_n(-3;\epsilon) = \mathbf{C}[z_1, \dots, z_n]$ for $0 < \epsilon < 1$. But unfortunately we are not able to do that at the moment. This is because the approach in Section 5 breaks down for the weight $t = -3$. In fact, the breakdown occurs in one-variable estimates. Namely, in order to prove the analogue of Proposition 5.10 for $t = -3$ using the approach in Section 5, one would have to prove Proposition 5.6 for the case $t = 0$. That is, one would have to prove an inequality of the form

$$\int |(\partial g)(z)f(z)|^2 dA(z) \le C\mathcal{N}_0(gf), \tag{6.1}$$

where C is independent of f. But if one simply tries $g(z) = 1 - z$ and $f(z) = 1 + rz + \cdots + (rz)^k$ for $0 < r < 1$ and $k \in \mathbf{N}$, one sees that (6.1) in general fails.

At first, this failure might suggest that the case $t = -3$ is hopeless. But if one carefully analyzes *how* (6.1) fails, one sees that it is still possible to show that $\mathcal{P}_n(-3;\epsilon)$ contains a substantial subset of $\mathbf{C}[z_1, \dots, z_n]$.

Indeed a careful analysis shows that the example we gave above is already the worst case scenario for (6.1), namely g has a zero on the unit circle $\mathbf{T}$. Recall from Section 5 that g represents the slices q_ξ, $\xi \in S$, of the n-variable polynomial q under consideration. Thus our analysis tells us that if the circle

$$\{\tau\xi : \tau \in \mathbf{T}\}$$

runs through the zero locus $\mathcal{Z}(q)$, then q_ξ is a bad slice for q. But fortunately, there are not too many such bad slices for each $q \in \mathcal{G}_n$, and the other slices of such a q are all "salvageable". This is the idea behind the proof of Theorem 1.5. But it takes quite a bit of work to bring this idea to fruition.

Lemma 6.1. *Given any $0 < \epsilon < 1$, there is a $0 < C_{6.1}(\epsilon) < \infty$ such that the inequality*

$$\int \frac{|f(z)|^2}{|z-a|^2} dA(z) \le \frac{C_{6.1}(\epsilon)}{(|a|-1)^\epsilon}\mathcal{N}_0(f)$$

holds for every $a \in \mathbf{C}$ with $|a| > 1$ and every one-variable polynomial f.

Proof. Let $0 < \epsilon < 1$ be given. Since $\mathcal{N}_0(f) = \mathcal{N}_0(f_\tau)$ for $\tau \in \mathbf{T}$, where $f_\tau(z) = f(\tau z)$, it suffices to consider the case where a is real and $a > 1$. For such an a, we have

$$\frac{1}{|z-a|^2} \le \frac{1}{(a-1)^\epsilon} \cdot \frac{1}{|z-a|^{2-\epsilon}} \le \frac{1}{(a-1)^\epsilon} \cdot \frac{1}{|1-z|^{2-\epsilon}}$$

for every $z \in D$. Therefore

$$\int \frac{|f(z)|^2}{|z-a|^2} dA(z) \le \frac{1}{(a-1)^\epsilon}\int \frac{|f(z)|^2}{|1-z|^{2-\epsilon}} dA(z). \tag{6.2}$$

On the unit disc D, we have the power series expansion

$$\frac{1}{(1-z)^{1-(\epsilon/2)}} = \sum_{j=0}^{\infty} b_j z^j, \quad \text{where} \quad b_j = \frac{1}{j!}\prod_{i=0}^{j-1}(1-(\epsilon/2)+i) \quad \text{for} \quad j \ge 1.$$

By the asymptotic expansion (2.2), there is a constant C such that

$$b_j \le \frac{C}{(j+1)^{\epsilon/2}} \quad \text{for every} \quad j \ge 0.$$

Given a one-variable polynomial f, we write it in the form $f(z) = \sum_{k=0}^{\infty} u_k z^k$, where $u_k \in \mathbf{C}$, and $u_k = 0$ for all but a finite number of k's. Then

$$\frac{f(z)}{(1-z)^{1-(\epsilon/2)}} = \sum_{k=0}^{\infty} \sum_{i=0}^{k} u_i b_{k-i} z^k,$$

$z \in D$. Consequently,

$$\begin{aligned}\int \frac{|f(z)|^2}{|1-z|^{2-\epsilon}} dA(z) &= \sum_{k=0}^{\infty} \frac{1}{k+1} \left| \sum_{i=0}^{k} u_i b_{k-i} \right|^2 \\ &\le \sum_{k=0}^{\infty} \frac{2C^2}{k+1} \sum_{0 \le i \le j \le k} \frac{|u_i||u_j|}{(k-i+1)^{\epsilon/2}(k-j+1)^{\epsilon/2}}.\end{aligned}$$

A change of the order of summation in the above gives us

$$\int \frac{|f(z)|^2}{|1-z|^{2-\epsilon}} dA(z) \le \sum_{i=0}^{\infty} |u_i| \sum_{j=i}^{\infty} |u_j| \sum_{k=j}^{\infty} \frac{2C^2}{(k+1)(k-i+1)^{\epsilon/2}(k-j+1)^{\epsilon/2}}.$$

Obviously, for all integers $k \ge j \ge i \ge 0$ we have

$$\sum_{k=j}^{\infty} \frac{1}{(k+1)^{1-(\epsilon/2)}(k-i+1)^{\epsilon/2}(k-j+1)^{\epsilon/2}} \le \sum_{k=0}^{\infty} \frac{1}{(k+1)^{1+(\epsilon/2)}} = C_1 < \infty.$$

Therefore

$$\int \frac{|f(z)|^2}{|1-z|^{2-\epsilon}} dA(z) \le C_2 \sum_{i=0}^{\infty} |u_i| \sum_{j=i}^{\infty} \frac{|u_j|}{(j+1)^{\epsilon/2}},$$

where $C_2 = 2C^2 C_1$. Set $s = (1-(\epsilon/2))/2$. Since $\epsilon > 0$, we have $s < 1/2$. For each $i \in \mathbf{N}$, define $c_i = |u_{i-1}| i^s = |u_{i-1}| i^{(1-(\epsilon/2))/2}$. Since $(\epsilon/2) + s = (1+(\epsilon/2))/2 = 1-s$, the above inequality can be rewritten as

$$\int \frac{|f(z)|^2}{|1-z|^{2-\epsilon}} dA(z) \le C_2 \sum_{i=1}^{\infty} \frac{c_i}{i^s} \sum_{j=i}^{\infty} \frac{c_j}{j^{1-s}}.$$

As in the proof of Lemma 5.2, an application of Lemma 5.1 now gives us

$$\int \frac{|f(z)|^2}{|1-z|^{2-\epsilon}} dA(z) \le C_2 C_{5.1}^{1/2}(s) \sum_{i=1}^{\infty} c_i^2 = C_2 C_{5.1}^{1/2}(s) \sum_{i=0}^{\infty} |u_i|^2 (i+1)^{1-(\epsilon/2)}.$$

On the other hand,

$$\mathcal{N}_0(f) = |f(0)|^2 + \int |(\mathcal{R}f)(z)|^2 dA(z) = |u_0|^2 + \sum_{k=1}^{\infty} \frac{k^2|u_k|^2}{k+1} \ge \frac{1}{4}\sum_{k=0}^{\infty}(k+1)|u_k|^2.$$

Thus

$$\int \frac{|f(z)|^2}{|1-z|^{2-\epsilon}} dA(z) \le 4C_2 C_{5.1}^{1/2}(s)\mathcal{N}_0(f).$$

Combining this inequality with (6.2), the lemma follows. □

Lemma 6.2. *Given any* $0 < \epsilon < 1$, *there is a* $0 < C_{6.2}(\epsilon) < \infty$ *such that*

$$\int |f(z)|^2 dA(z) \le \frac{C_{6.2}(\epsilon)}{||a|-1|^{\epsilon}} \mathcal{N}_0(\Phi_{f,a}) \tag{6.3}$$

for every $a \in \mathbf{C}\backslash\mathbf{T}$ *and every one-variable polynomial* f, *where* $\Phi_{f,a}(z) = (z-a)f(z)$.

Proof. In the case $|a| > 1$, since $f(z) = \Phi_{f,a}(z)/(z-a)$, (6.3) is obtained by applying Lemma 6.1 to $\Phi_{f,a}$. Thus let us suppose that $|a| < 1$. That is, $a = re^{i\theta}$ for some $r \in [0,1)$ and $\theta \in \mathbf{R}$. We define

$$b = (1 + (1-r))e^{i\theta} = (2-r)e^{i\theta}.$$

Recall that $D(a) = \{w \in D : |w-a| < (1/2)(1-|a|)\}$. Now, if $z \in D\backslash D(a)$, then

$$|z-b| \le |z-a| + |b-a| = |z-a| + 2(1-|a|) \le |z-a| + 4|z-a| = 5|z-a|.$$

That is, $5|z-a|/|z-b| \ge 1$ for $z \in D\backslash D(a)$. By this inequality and Lemma 6.1, we have

$$\begin{aligned} \int_{D\backslash D(a)} |f(z)|^2 dA(z) &\le 25 \int_{D\backslash D(a)} \frac{|\Phi_{f,a}(z)|^2}{|z-b|^2} dA(z) \\ &\le \frac{25C_{6.1}(\epsilon)}{(|b|-1)^{\epsilon}} \mathcal{N}_0(\Phi_{f,a}) = \frac{25C_{6.1}(\epsilon)}{(1-|a|)^{\epsilon}} \mathcal{N}_0(\Phi_{f,a}). \end{aligned} \tag{6.4}$$

On the other hand, by Lemma 5.4, we have

$$\int_{D(a)} |f(z)|^2 (1-|z|^2)^{\epsilon} dA(z) \le C_{5.4}(\epsilon)\mathcal{N}_{\epsilon}(\Phi_{f,a}) \le C_{5.4}(\epsilon)\mathcal{N}_0(\Phi_{f,a}).$$

But if $z \in D(a)$, then $1-|z| \ge 1-|a|-|z-a| \ge (1/2)(1-|a|)$. Hence the above implies

$$\int_{D(a)} |f(z)|^2 dA(z) \le \frac{2C_{5.4}(\epsilon)}{(1-|a|)^{\epsilon}} \mathcal{N}_0(\Phi_{f,a}).$$

Combining this with (6.4), the lemma follows. □

Definition 6.3. For a one-variable polynomial g of degree at least 1, we write

$$\Delta(g) = \inf\{|a-\tau| : g(a) = 0,\ \tau \in \mathbf{T}\}.$$

Proposition 6.4. *Suppose that $0 < \epsilon < 1$. Let g and f be one-variable polynomials. If the degree of g equals $K \geq 1$ and if g has no zeros on the unit circle $\mathbf{T}$, then*

$$\int |(\partial g)(z)f(z)|^2 dA(z) \leq \frac{C_{6.2}(\epsilon)}{(\Delta(g))^\epsilon} K^2 \mathcal{N}_0(gf),$$

where $C_{6.2}(\epsilon)$ is the constant given in Lemma 6.2.

Proof. Since $\Delta(g) = \min\{||a| - 1| : g(a) = 0\}$, this proposition is derived from Lemma 6.2 in exactly the same way that Proposition 5.6 was derived from Lemma 5.5. □

The estimate in Proposition 6.4 tells us how to proceed. First, let us introduce

Definition 6.5. Let $q \in \mathbf{C}[z_1, \dots, z_n]$.
(1) Set $\mathcal{B}(q) = \{\tau\zeta : \zeta \in S \cap \mathcal{Z}(q), \tau \in \mathbf{T}\}$.
(2) For each $\xi \in S$, let $\Delta(q;\xi) = \inf\{|\tau\xi - \zeta| : \tau \in \mathbf{T}, \zeta \in \mathcal{Z}(q)\}$.

In other words, $\Delta(q;\xi)$ is the distance between the circular slice $\{\tau\xi : \tau \in \mathbf{T}\}$ and the zero locus $\mathcal{Z}(q)$ of q. In the case where $\mathcal{Z}(q) = \emptyset$, i.e., when q is a nonzero constant, we interpret $\Delta(q;\xi)$ as ∞ and $1/\Delta(q;\xi)$ as 0.

For our purpose, $\mathcal{B}(q)$ is the bad set for q, hence the notation. This set is bad because if $\xi \in \mathcal{B}(q)$, then Proposition 6.4 is useless for the slice $\{z\xi : z \in D\}$.

Recalling Definition 1.7, in the case $t = -3$ we need to estimate $\|fRq\|_{-3+3} = \|fRq\|_0$. By (5.11) and the relation $(Rq)_\xi(z) = (\mathcal{R}q_\xi)(z)$, $z \in \mathbf{C}$, we have

$$\|fRq\|_0^2 = \int |(Rq)(\zeta)f(\zeta)|^2 dv(\zeta) \leq n \int\int |(\mathcal{R}q_\xi)(z)f_\xi(z)|^2 dA(z)d\sigma(\xi).$$

If the polynomial q somehow has the property $\sigma(\mathcal{B}(q)) = 0$, then

$$\|fRq\|_0^2 \leq n \int_{S\backslash\mathcal{B}(q)} \int |(\mathcal{R}q_\xi)(z)f_\xi(z)|^2 dA(z)d\sigma(\xi).$$

Let $0 < \epsilon < 1$. For each $\xi \in S\backslash\mathcal{B}(q)$, since q_ξ does not vanish on $\mathbf{T}$, Proposition 6.4 is applicable. Since $\Delta(q;\xi) \leq \Delta(q_\xi)$, $\xi \in S$, Proposition 6.4 gives us

$$\begin{aligned}
\|fRq\|_0^2 &\leq nC_{6.2}(\epsilon)K^2 \int_{S\backslash\mathcal{B}(q)} \frac{\mathcal{N}_0(q_\xi f_\xi)}{(\Delta(q;\xi))^\epsilon} d\sigma(\xi) = nC_{6.2}(\epsilon)K^2 \int_{S\backslash\mathcal{B}(q)} \frac{\mathcal{N}_0((qf)_\xi)}{(\Delta(q;\xi))^\epsilon} d\sigma(\xi) \\
&= nC_{6.2}(\epsilon)K^2 \int_{S\backslash\mathcal{B}(q)} \left(|q(0)f(0)|^2 + \int |(\mathcal{R}(qf)_\xi)(z)|^2 dA(z)\right) \frac{1}{(\Delta(q;\xi))^\epsilon} d\sigma(\xi) \\
&= nC_{6.2}(\epsilon)K^2 \int_{S\backslash\mathcal{B}(q)} \left(|q(0)f(0)|^2 + \int |(R(qf))(z\xi)|^2 dA(z)\right) \frac{1}{(\Delta(q;\xi))^\epsilon} d\sigma(\xi), \qquad (6.5)
\end{aligned}$$

where K is the degree of q. This tells us to further introduce

Definition 6.6. Given any $q \in \mathbf{C}[z_1, \dots, z_n]$ and $0 < \epsilon < 1$, we let $\mu_{q;\epsilon}$ be the measure on S given by the formula

$$\mu_{q;\epsilon}(A) = \int_{A \setminus \mathcal{B}(q)} \frac{1}{(\Delta(q;\xi))^\epsilon} d\sigma(\xi)$$

for Borel sets $A \subset S$.

Proposition 6.7. *For $q \in \mathbf{C}[z_1, \dots, z_n]$ and $0 < \epsilon < 1/2$, we have $q \in \mathcal{P}_n(-3;\epsilon)$ whenever the following two conditions are satisfied*:
(1) $\sigma(\mathcal{B}(q)) = 0$.
(2) *There is a constant C such that*

$$\int \int |h(z\xi)|^2 dA(z) d\mu_{q;2\epsilon}(\xi) \le C \int |h(\zeta)|^2 (1 - |\zeta|^2)^{-\epsilon} dv(\zeta)$$

for every $h \in \mathbf{C}[z_1, \dots, z_n]$.

Proof. First of all, condition (2) implies $\mu_{q;2\epsilon}(S) < \infty$. Continuing with (6.5), if we apply condition (2) to the polynomial $h = R(qf)$, $f \in \mathbf{C}[z_1, \dots, z_n]$, then

$$\|fRq\|_0^2 \le nC_{6.2}(2\epsilon)K^2 \left(\mu_{q;2\epsilon}(S)|q(0)f(0)|^2 + C \int |(R(qf))(\zeta)|^2 (1 - |\zeta|^2)^{-\epsilon} dv(\zeta) \right),$$

where K is the degree of q. On the other hand, (2.7) tells us that

$$|q(0)f(0)|^2 + \int |(R(qf))(\zeta)|^2 (1 - |\zeta|^2)^{-\epsilon} dv(\zeta) \le C_1 \|qf\|_{-\epsilon-2}^2 = C_1 \|qf\|_{-3+1-\epsilon}^2.$$

Thus the membership $q \in \mathcal{P}_n(-3;\epsilon)$ follows from these two inequalities. □

7. A growth condition for $\mu_{q;\epsilon}$

From Propositions 4.4 and 6.7 we see the roadmap for the proof of Theorem 1.5: it suffices to show that if $q \in \mathcal{G}_n$, $n \ge 3$, then q satisfies conditions (1) and (2) in Proposition 6.7 for $0 < \epsilon < 1/2$. But this will take a few steps. Our goal for this section is to show that the membership $q \in \mathcal{G}_n$ implies a certain growth condition for the measure $\mu_{q;\epsilon}$.

For $\zeta \in \mathbf{C}^n$ and $r > 0$, let $E(\zeta, r)$ be the corresponding Euclidian ball in $\mathbf{C}^n$. That is,

$$E(\zeta, r) = \{w \in \mathbf{C}^n : |w - \zeta| < r\}.$$

We begin this section with a consequence of condition (a) in Definition 1.4.

Lemma 7.1. *Let $q \in \mathbf{C}[z_1, \dots, z_n]$ and suppose that q satisfies condition* (a) *in Definition* 1.4. *That is, suppose that Rq does not vanish on the set $\mathcal{Z}(q) \cap S$. Then for every given $1 \le C < \infty$, there exist $L, M \in \mathbf{N}$ such that for every pair of $\ell \ge L$ and $\xi \in S$, we have*

$$\text{card}\{k \in \mathbf{Z}_+ : 0 \le k \le \ell - 1, \quad E(e^{2k\pi i/\ell}\xi, C/\ell) \cap \mathcal{Z}(q) \ne \emptyset\} \le M.$$

Proof. Let q be given as in the statement. We divide the proof into five steps.

(1) Since Rq does not vanish on $\mathcal{Z}(q) \cap S$ and since $(Rq)_\xi(z) = (\mathcal{R}q_\xi)(z)$, $z \in \mathbf{C}$, there is no $\xi \in S$ for which the one-variable polynomial q_ξ is identically zero. Thus, by the compactness of S, there is a $c > 0$ such that $\max_{\tau\in\mathbf{T}} |q_\xi(\tau)| \geq c$ for every $\xi \in S$.

(2) Suppose that the $\deg(q) = K \geq 1$. For each $\xi \in S$, we factor q_ξ in the form

$$q_\xi(z) = b(\xi) \prod_{j=1}^{K(\xi)} (z - a_j(\xi)), \quad z \in \mathbf{C}, \tag{7.1}$$

where $K(\xi) \leq K$ and $b(\xi), a_1(\xi), \ldots, a_{K(\xi)}(\xi) \in \mathbf{C}$. Accordingly, we have the partition

$$\{1, \ldots, K(\xi)\} = X(\xi) \cup Y(\xi),$$

where

$$X(\xi) = \{j \in \{1, \ldots, K(\xi)\} : |a_j(\xi)| < 2\} \quad \text{and}$$
$$Y(\xi) = \{j \in \{1, \ldots, K(\xi)\} : |a_j(\xi)| \geq 2\}.$$

Set $c_1 = 3^{-K}c$. We claim that

$$\min_{\tau\in\mathbf{T}} |b(\xi)| \prod_{j\in Y(\xi)} |\tau - a_j(\xi)| \geq c_1 \quad \text{for every} \quad \xi \in S. \tag{7.2}$$

(For any $\xi \in S$ for which $Y(\xi) = \emptyset$, the above simply means $|b(\xi)| \geq c_1$, which is in keeping with the usual convention that $\prod_{j\in\emptyset} \cdots$ means 1. In fact, the same convention also applies to the product in (7.1) and to the products below.) To prove this, note that if $a \in \mathbf{C}$ and $|a| \geq 2$, then for any pair of $\tau, \omega \in \mathbf{T}$ we have

$$|\tau - a| \geq |a| - 1 \geq \frac{1}{2}|a| = \frac{1}{2}(|a| + 1 - 1) \geq \frac{1}{2} \cdot \frac{2}{3}(|a| + 1) \geq \frac{1}{3}|\omega - a|. \tag{7.3}$$

For each $\xi \in S$, by (1) there is an $\omega = \omega(\xi) \in \mathbf{T}$ such that $|q_\xi(\omega)| \geq c$. For each $j \in X(\xi)$, we have $|\omega - a_j(\xi)| \leq 3$. Therefore

$$|b(\xi)| \prod_{j\in Y(\xi)} |\omega - a_j(\xi)| \geq 3^{-\mathrm{card}(X(\xi))}|q_\xi(\omega)| \geq 3^{-\mathrm{card}(X(\xi))}c.$$

Clearly, (7.2) follows from this inequality and (7.3).

(3) We claim that there is a $0 < d < 1/2$ such that for each $\xi \in S$, if j, k are distinct elements in $X(\xi)$ and if both $a_j(\xi)$ and $a_k(\xi)$ belong to the annulus

$$\{z \in \mathbf{C} : 1 - d \leq |z| \leq 1 + d\},$$

then

$$\left| \frac{a_j(\xi)}{|a_j(\xi)|} - \frac{a_k(\xi)}{|a_k(\xi)|} \right| \geq d.$$

Indeed if such a d did not exist, then for each integer $\nu \geq 3$, there would be a $\xi_\nu \in S$ and a pair of $j_\nu \neq k_\nu$ in $X(\xi_\nu)$ such that

$$|1 - |a_{j_\nu}(\xi_\nu)|| \leq 1/\nu, \quad |1 - |a_{k_\nu}(\xi_\nu)|| \leq 1/\nu, \quad \text{and} \quad \left| \frac{a_{j_\nu}(\xi_\nu)}{|a_{j_\nu}(\xi_\nu)|} - \frac{a_{k_\nu}(\xi_\nu)}{|a_{k_\nu}(\xi_\nu)|} \right| \leq 1/\nu.$$

Thus there exist a subsequence $\{\nu_1, \dots, \nu_i, \dots\}$ of $\{3, 4, 5, \dots\}$ and $\xi_0 \in S$ and $a_0 \in \mathbf{T}$ such that $|\xi_{\nu_i} - \xi_0| \to 0$ and $|a_{j_{\nu_i}}(\xi_{\nu_i}) - a_0| \to 0$ as $i \to \infty$. By the above inequalities, we also have $|a_{k_{\nu_i}}(\xi_{\nu_i}) - a_0| \to 0$ as $i \to \infty$. Since $j_\nu \neq k_\nu$, this means that for any open neighborhood U of a_0, if i is sufficiently large, then the one-variable polynomial $q_{\xi_{\nu_i}}$ has at least two zeros (counting multiplicity) in U. Since

$$\lim_{i \to \infty} \sup_{|z| \leq 2} |q_{\xi_{\nu_i}}(z) - q_{\xi_0}(z)| = 0,$$

by the argument principle, a_0 is a zero of multiplicity at least 2 for q_{ξ_0}. Thus we have both $q_{\xi_0}(a_0) = 0$ and $(\mathcal{R}q_{\xi_0})(a_0) = 0$. Since $q_{\xi_0}(a_0) = q(a_0\xi_0)$ and $(\mathcal{R}q_{\xi_0})(a_0) = (Rq)_{\xi_0}(a_0) = (Rq)(a_0\xi_0)$, this means the point $a_0\xi_0 \in S$ is a zero for both q and Rq. This contradicts the assumption that Rq does not vanish on $\mathcal{Z}(q) \cap S$.

(4) With the d from (3), for each $\xi \in S$, we have the partition

$$X(\xi) = A(\xi) \cup B(\xi),$$

where

$$\begin{aligned} A(\xi) &= \{j \in X(\xi) : 1 - d \leq |a_j(\xi)| \leq 1 + d\} \quad \text{and} \\ B(\xi) &= \{j \in X(\xi) : \text{either } |a_j(\xi)| < 1 - d \text{ or } |a_j(\xi)| > 1 + d\}. \end{aligned}$$

Obviously, we have

$$\prod_{j \in B(\xi)} |\tau - a_j(\xi)| \geq d^{\operatorname{card}(B(\xi))} \tag{7.4}$$

for all $\xi \in S$ and $\tau \in \mathbf{T}$. For each $j \in A(\xi)$, we factor $a_j(\xi)$ in the form

$$a_j(\xi) = |a_j(\xi)|\tau_j(\xi), \quad \text{where } \tau_j(\xi) \in \mathbf{T}.$$

Then $|\tau_j(\xi) - \tau_k(\xi)| \geq d$ for all $j \neq k$ in $A(\xi)$. Moreover, for each $j \in A(\xi)$ we define

$$I_j(\xi) = \{\tau \in \mathbf{T} : |\tau - \tau_j(\xi)| \leq d/2\}.$$

It is elementary that if $u \in \mathbf{C}$, $|u| \le 1$, and $0 \le r \le 1$, then $2|1 - ru| \ge |1 - u|$. Therefore for every pair of $k \in A(\xi)$ and $\tau \in \mathbf{T} \backslash \cup_{j \in A(\xi)} I_j(\xi)$ we have

$$|\tau - a_k(\xi)| \ge (1/2)|\tau - \tau_k(\xi)| \ge d/4.$$

Combining this with (7.2) and (7.4), we conclude that

$$|q_\xi(\tau)| \ge c_2 \quad \text{whenever} \quad \tau \in \mathbf{T} \backslash \cup_{j \in A(\xi)} I_j(\xi) \ , \tag{7.5}$$

where $c_2 = (d/4)^K c_1$. Let $j, k \in A(\xi)$ and suppose that $j \ne k$. If $\tau \in I_j(\xi)$, then

$$|\tau - a_k(\xi)| \ge \frac{1}{2}|\tau - \tau_k(\xi)| \ge \frac{1}{2}(|\tau_j(\xi) - \tau_k(\xi)| - |\tau - \tau_j(\xi)|) \ge \frac{1}{2}(d - (1/2)d) = \frac{1}{4}d.$$

Combining this again with (7.2) and (7.4), we obtain

$$|q_\xi(\tau)| \ge c_2|\tau - \tau_j(\xi)| \quad \text{if} \quad \tau \in I_j(\xi), \quad j \in A(\xi).$$

Thus, by (7.5), for each $0 < r < c_2$ we have

$$\{\tau \in \mathbf{T} : |q_\xi(\tau)| \le r\} \subset \bigcup_{j \in A(\xi)} \{\tau \in I_j(\xi) : |\tau - \tau_j(\xi)| \le (r/c_2)\}.$$

Let m be the Lebesgue measure on $\mathbf{T}$ with the normalization $m(\mathbf{T}) = 1$. Since $\mathrm{card}(A(\xi)) \le K(\xi) \le K$, from the above inclusion we deduce that there is a constant C_1 such that

$$m(\{\tau \in \mathbf{T} : |q_\xi(\tau)| \le r\}) \le C_1 r \quad \text{for all} \quad \xi \in S \quad \text{and} \quad 0 < r < c_2. \tag{7.6}$$

(5) Let $1 \le C < \infty$ be given. Obviously, q satisfies a Lipschitz condition on the ball $\{\zeta \in \mathbf{C}^n : |\zeta| \le 1 + C\}$. To be more precise, there is a C_2 such that

$$|q(\zeta) - q(w)| \le C_2|\zeta - w| \quad \text{if} \quad |\zeta| \le 1 + C \quad \text{and} \quad |w| \le 1 + C. \tag{7.7}$$

For $\ell \in \mathbf{N}$ and $0 \le j \le \ell - 1$, define the arc

$$I_{\ell,j} = \{e^{i\theta} : 2j\pi/\ell \le \theta < 2(j+1)\pi/\ell\}.$$

If $\ell \in \mathbf{N}$ and $0 \le j \le \ell - 1$, then for all $\xi \in S$ and $\tau \in I_{\ell,j}$ we have $|\tau\xi - e^{2j\pi i/\ell}\xi| \le 2\pi/\ell$. Thus if $0 \le j \le \ell - 1$ is such that $E(e^{2j\pi i/\ell}\xi, C/\ell) \cap \mathcal{Z}(q) \ne \emptyset$, then (7.7) implies that

$$|q_\xi(\tau)| = |q(\tau\xi)| \le C_2(C + 2\pi)/\ell \quad \text{for every} \quad \tau \in I_{\ell,j}.$$

For each $\xi \in S$, define $J_\ell(\xi) = \{j : 0 \le j \le \ell - 1, E(e^{2j\pi i/\ell}\xi, C/\ell) \cap \mathcal{Z}(q) \ne \emptyset\}$. Let $L \in \mathbf{N}$ be such that $C_2(C + 2\pi)/L < c_2$. For each $\ell \ge L$, we have

$$\begin{aligned} \mathrm{crad}(J_\ell(\xi))(1/\ell) &= m\left(\cup_{j \in J_\ell(\xi)} I_{\ell,j}\right) \\ &\le m(\{\tau \in \mathbf{T} : |q_\xi(\tau)| \le C_2(C + 2\pi)/\ell\}) \le C_1C_2(C + 2\pi)/\ell, \end{aligned}$$

where the second $\leq$ follows from (7.6). Cancelling out $1/\ell$ from both sides, we find that $\mathrm{card}(J_\ell(\xi)) \leq C_1C_2(C+2\pi)$. That is, for the above choice of L, any integer $M \geq C_1C_2(C+2\pi)$ works for the lemma. $\square$

To proceed further, we need a better understanding of the two conditions in Definition 1.4. Given a $q \in \mathbf{C}[z_1, \ldots, z_n]$, consider the real-valued functions

$$\begin{cases} u(x_1, y_1, \ldots, x_n, y_n) = \mathrm{Re}\{q(x_1 + iy_1, \ldots, x_n + iy_n)\}, \\ \\ v(x_1, y_1, \ldots, x_n, y_n) = \mathrm{Im}\{q(x_1 + iy_1, \ldots, x_n + iy_n)\} \end{cases} \tag{7.8}$$

on $\mathbf{R}^{2n}$. Condition (b) in Definition 1.4 is equivalent to the condition that for every $\xi = (x_1 + iy_1, \ldots, x_n + iy_n)$ in $S \cap \mathcal{Z}(q)$, the real normal vector $(x_1, y_1, \ldots, x_n, y_n)$ to S is not contained in the real linear span of the real gradient vectors

$$(\nabla u)(x_1, y_1, \ldots, x_n, y_n) \qquad \text{and} \qquad (\nabla v)(x_1, y_1, \ldots, x_n, y_n),$$

which are nonzero because of (a). Note that by the Cauchy-Riemann equations, we have

$$(\nabla u)(x) \perp (\nabla v)(x) \qquad \text{and} \qquad |(\nabla u)(x)| = |(\nabla v)(x)| \tag{7.9}$$

for every $x \in \mathbf{R}^{2n}$. Hence if $x \in \mathbf{R}^{2n}$ is such that $(\nabla u)(x) \neq 0$, then the condition $x \in \mathrm{span}\{(\nabla u)(x), (\nabla v)(x)\}$ is equivalent to Parseval's identity

$$\langle x, (\nabla u)(x)\rangle^2 + \langle x, (\nabla v)(x)\rangle^2 = |x|^2|(\nabla u)(x)|^2.$$

Combining this fact with the Cauchy-Riemann equations, we see that the two conditions in Definition1.4 together are simply equivalent to the strict inequality

$$0 < |(Rq)(\xi)| < |(\partial q)(\xi)| \quad \text{for every} \ \ \xi \in S \cap \mathcal{Z}(q), \tag{7.10}$$

where, as we recall, ∂q denotes the analytic gradient $(\partial_1 q, \ldots, \partial_n q)$.

For $\xi \in S$ and $r > 0$, let us denote

$$S(\xi, r) = E(\xi, r) \cap S.$$

Obviously, there is a constant C_n determined by the complex dimension n such that $\sigma(S(\xi, r)) \leq C_n r^{2n-1}$ for all $\xi \in S$ and $r > 0$. With this in mind, we have

Lemma 7.2. *For each $q \in \mathcal{G}_n$, there exist $r_0 > 0$ and $0 < C < \infty$ such that the inequality*

$$\sigma(\{w \in S(\xi, r) : \Delta(q; w) < \rho\}) \leq Cr^{2n-2}\rho$$

holds for all $\xi \in S$ and $0 < \rho \leq r < r_0$.

Proof. The idea for the proof is actually quite simple. Namely, for sufficiently small $0 < \rho \leq r$, we can cover the set $\{w \in S(\xi, r) : \Delta(q; w) < \rho\}$ with a family of balls $\{E_\nu$

$: \nu \in \mathcal{N}\}$ in the Euclidian metric, where the radius of each E_ν is on the order of ρ and where the cardinality of $\mathcal{N}$ is on the order of $(r/\rho)^{2n-2}$. Then, since $\sigma(S \cap E_\nu)$ is on the order of ρ^{2n-1}, the desired estimate follows. The key to the proof is a counting argument, which is unfortunately quite complicated in details as shown below. We alert the reader that in the proof, we will freely switch between $\mathbf{C}^n$ and its real version, $\mathbf{R}^{2n}$.

Let $q \in \mathcal{G}_n$, and let u and v be the same as in (7.8). Consider an arbitrary $x \in \mathcal{Z}(q) \cap S$. By (7.9) and (7.10), we have $|(\nabla u)(x)| = |(\nabla v)(x)| > 0$. Set

$$e(x) = \frac{1}{|(\nabla u)(x)|}(\nabla u)(x) \quad \text{and} \quad f(x) = \frac{1}{|(\nabla v)(x)|}(\nabla v)(x).$$

Then $\{e(x), f(x)\}$ is an orthonormal basis for $\text{span}\{(\nabla u)(x), (\nabla v)(x)\}$. Define the subspace

$$T_x = \mathbf{R}^{2n} \ominus \text{span}\{(\nabla u)(x), (\nabla v)(x)\}.$$

Let $p(x)$ be the orthogonal projection of the real vector x on the subspace T_x. Condition (b) in Definition 1.4 implies that $p(x)$ is not the zero vector. This enables us to define

$$g(x) = \frac{1}{|p(x)|}p(x). \tag{7.11}$$

We then complete the single unit vector $g(x)$ to an orthonormal basis

$$\{g(x), g_2, \ldots, g_{2n-2}\}$$

for the linear subspace T_x. Define the isometry $A : \mathbf{R}^{2n-2} \to T_x$ by the formula

$$A(s_1, s_2, \ldots, s_{2n-2}) = s_1 g(x) + s_2 g_2 + \cdots + s_{2n-2} g_{2n-2}.$$

Also, define the isometry $B : \mathbf{R}^2 \to \text{span}\{(\nabla u)(x), (\nabla v)(x)\}$ by the formula

$$B(\eta_1, \eta_2) = \eta_1 e(x) + \eta_2 f(x).$$

The vector x has the orthogonal decomposition

$$x = a_1 g(x) + b_1 e(x) + b_2 f(x). \tag{7.12}$$

Thus if we define $a = (a_1, 0, \ldots, 0)$ in $\mathbf{R}^{2n-2}$ and $b = (b_1, b_2)$ in $\mathbf{R}^2$, then

$$x = Aa + Bb.$$

Finally, define the map Q from $\mathbf{R}^{2n-2} \times \mathbf{R}^2 = \mathbf{R}^{2n}$ to $\mathbf{R}^2$ by the formula

$$Q(s, \eta) = (u(As + B\eta), v(As + B\eta)),$$

$s \in \mathbf{R}^{2n-2}$ and $\eta \in \mathbf{R}^2$. We have $Q(a, b) = (u(x), v(x)) = 0$.

By the standard inverse function theorem and implicit function theorem (see, e.g., [21,Sections 8 and 9]), there is a $d = d(x) > 0$ such that the following hold true:

(i) There is a C^∞-map h from $\{s \in \mathbf{R}^{2n-2} : |s - a| < d\}$ into $\mathbf{R}^2$ such that $h(a) = b$ and $Q(s, h(s)) = 0$ for every $s \in \mathbf{R}^{2n-2}$ satisfying the condition $|a - s| < d$.

(ii) If $s \in \mathbf{R}^{2n-2}$ and $\eta \in \mathbf{R}^2$ satisfy the conditions $|a - s| < d$ and $|b - \eta| < d$, and if $Q(s, \eta) = 0$, then $\eta = h(s)$.

Furthermore, by (i), for every vector $w \in \mathbf{R}^{2n-2}$ we have the equation

$$\frac{d}{dt}Q(a + wt, h(a + wt))\bigg|_{t=0} = \frac{d0}{dt}\bigg|_{t=0} = 0.$$

Now solve this equation using the chain rule. Since

$$\langle(\nabla u)(x), Aw\rangle = 0 = \langle(\nabla v)(x), Aw\rangle$$

for every $w \in \mathbf{R}^{2n-2}$, we obtain $(Dh)(a) = 0$. (As usual, we write $(Dh)(s)$ for the derivative of h at the point s, which is a linear transformation from $\mathbf{R}^{2n-2}$ to $\mathbf{R}^2$.) Combining this fact with basic calculus, there is a $0 < d_1 < d$ such that

(iii) $||h(s)|^2 - |h(s')|^2| \le 4^{-1}|p(x)||s - s'|$ for all $s, s' \in \mathbf{R}^{2n-2}$ satisfying the conditions $|a - s| < d_1$ and $|a - s'| < d_1$. Also, $|(Dh)(s)| \le 1/2$ if $s \in \mathbf{R}^{2n-2}$ and $|a - s| < d_1$.

By (7.11) and (7.12) we have

$$a_1 = \langle x, g(x)\rangle = \frac{\langle x, p(x)\rangle}{|p(x)|} = |p(x)|.$$

Set $d_2 = \min\{|p(x)|/8, d_1\}$

Suppose that $0 < \rho < d_2$. Suppose that $s, s' \in \mathbf{R}^{2n-2}$ satisfy the conditions $|a - s| < d_2$, $|a - s'| < d_2$ and that these vectors have the representation

$$\text{(7.13)} \qquad \begin{cases} s = (a_1 + k\rho, s_2, \ldots, s_{2n-2}), \\ s' = (a_1 + m\rho, s_2, \ldots, s_{2n-2}), \end{cases} \quad \text{where} \quad k, m \in \mathbf{Z}.$$

We claim that there is a $c(x) > 0$ such that

$$\text{(7.14)} \qquad ||(s, h(s))| - |(s', h(s'))|| \ge c(x)|k - m|\rho$$

To prove this, note that the condition $|a - s| < d_2$ implies $|k|\rho < |p(x)|/8$. Therefore $a_1 + k\rho \ge |p(x)|/2$. Similarly, we also have $a_1 + m\rho \ge |p(x)|/2$. Hence for such a pair of s, s' we have

$$\begin{aligned}|(s, h(s))| - |(s', h(s'))| &= \frac{|(s, h(s))|^2 - |(s', h(s'))|^2}{|(s, h(s))| + |(s', h(s'))|} = \frac{|s|^2 + |h(s)|^2 - |s'|^2 - |h(s')|^2}{|(s, h(s))| + |(s', h(s'))|} \\ &= \frac{(a_1 + k\rho)^2 - (a_1 + m\rho)^2 + |h(s)|^2 - |h(s')|^2}{|(s, h(s))| + |(s', h(s'))|}.\end{aligned}$$

On the other hand,

$$|(a_1+k\rho)^2-(a_1+m\rho)^2| = (a_1+k\rho+a_1+m\rho)|k-m|\rho \geq |p(x)||k-m|\rho$$

and, by (iii),

$$||h(s)|^2-|h(s')|^2| \leq 4^{-1}|p(x)||s-s'| = 4^{-1}|p(x)||k-m|\rho.$$

Thus for such a pair of s, s' we have

$$\begin{aligned} ||(s,h(s))|-|(s',h(s'))|| &\geq \frac{(3/4)|p(x)|}{|(s,h(s))|+|(s',h(s'))|}|k-m|\rho \\ &\geq \frac{(3/4)|p(x)|}{|s|+|h(s)|+|s'|+|h(s')|}|k-m|\rho \geq \frac{(3/4)|p(x)|}{2(|a|+d_1+|b|+d_1)}|k-m|\rho, \end{aligned}$$

where the last $\leq$ uses the fact $h(a)=b$ and the bound on Dh in (iii). This proves (7.14).

For each $0<\rho<d_2$, let Γ_ρ be the collection of vectors $a+\rho\beta$, $\beta \in \mathbf{Z}^{2n-2}$, satisfying the condition $\rho|\beta|<d_2$, where $|\beta|$ is the Euclidian length of β. Next we set $d_3=(1+\sqrt{2n})^{-1}d_2$. Suppose that ζ is a point in $E(x,d_3)\cap\mathcal{Z}(q)$. Then $\zeta = As+B\eta$, where s and η satisfy the conditions $|a-s|\leq|x-\zeta|<d_3$ and $|b-\eta|\leq|x-\zeta|<d_3$. Since $d_3<d$, by (ii) we have $\zeta = As+Bh(s)$. Now, if we further require that $0<\rho<d_3$, then there is a $\gamma\in\Gamma_\rho$ such that $|\gamma-s|\leq\sqrt{2n-2}\rho$. Thus

$$|A\gamma+Bh(\gamma)-\zeta| \leq |\gamma-s|+|h(\gamma)-h(s)| \leq 2\sqrt{2n-2}\rho,$$

where for the second $\leq$ we again use the bound on Dh in (iii). For $z\in E(x,d_3/2)$, if $|z-\zeta|<d_3/2$, then $\zeta\in E(x,d_3)$. Set $d_4=d_3/2$ and $C_1=1+2\sqrt{2n-2}$. Combining these facts, we see that for every $0<\rho<d_4$ we have

$$\left\{z\in E(x,d_4) : \inf_{\zeta\in\mathcal{Z}(q)}|z-\zeta|<\rho\right\} \subset \bigcup_{\gamma\in\Gamma_\rho} E(A\gamma+Bh(\gamma),C_1\rho). \tag{7.15}$$

This positive number d_4, of course, depends not only on q but also on the point x in $\mathcal{Z}(q)\cap S$. For this reason we write $r(x)=d_4$. Thus, repeating the above construction, we obtain an $r(x)>0$ for every $x\in\mathcal{Z}(q)\cap S$.

Obviously, the family of balls $E(x,r(x)/26)$, $x\in\mathcal{Z}(q)\cap S$, is an open cover of the compact set $\mathcal{Z}(q)\cap S$. Therefore there is a finite subset F of $\mathcal{Z}(q)\cap S$ such that if we let

$$U=\bigcup_{x\in F}E(x,r(x)/26),$$

then $U\supset\mathcal{Z}(q)\cap S$. Since U is an open set, there is an $r_1>0$ such that

$$|1-|\zeta||\geq r_1 \quad \text{whenever} \quad \zeta\in\mathcal{Z}(q)\backslash U. \tag{7.16}$$

We now apply Lemma 7.1 to this q and the constant $C = 2\pi + 4$, and accordingly we obtain constants $L, M \in \mathbf{N}$ from that lemma. We set

$$r_0 = \min \left\{ \frac{1}{L}, \frac{1}{12} r_1, \min_{x \in F} \frac{1}{26} r(x) \right\}.$$

Obviously, $r_0 > 0$. What remains is to show that this r_0 has the property promised in the statement of the lemma.

As the first step, let us show that for any $\xi \in S$ and $0 < r < r_0$, if $E(\xi, 12r) \cap \mathcal{Z}(q) \neq \emptyset$, then there is an $x \in F$ such that

$$E(\xi, 12r) \subset E(x, r(x)). \tag{7.17}$$

Indeed if there is a $\zeta \in E(\xi, 12r) \cap \mathcal{Z}(q)$, then $|\xi - \zeta| < 12r$. Since $\xi \in S$, this means $|1 - |\zeta|| < 12r < 12r_0 \leq r_1$. Since $\zeta \in \mathcal{Z}(q)$, by (7.16) we have $\zeta \in U$. That is, there is some $x \in F$ such that $\zeta \in E(x, r(x)/26)$. Since $r < r(x)/26$, we have $|\xi - x| < 13r(x)/26$. Therefore, if $z \in E(\xi, 12r)$, then $|z - x| \leq 12r + (13/26)r(x) < r(x)$, proving (7.17).

Again, assume that $0 < r < r_0$. Since $r < 1/L$, there is a natural number $\ell \geq L$ such that $1/(\ell + 1) \leq r < 1/\ell$. For each integer $0 \leq k \leq \ell$, define the interval

$$I_{\ell,k} = \left[\frac{2k\pi}{\ell + 1}, \frac{2(k+1)\pi}{\ell + 1} \right).$$

Let $\xi \in S$ be given. Then it is elementary that

$$E\left(e^{\frac{2k\pi i}{\ell+1}} \xi, 12r\right) \supset E\left(e^{\frac{2k\pi i}{\ell+1}} \xi, \frac{2\pi + 4}{\ell + 1}\right) \supset E(e^{i\theta}\xi, 2r) \quad \text{if} \quad \theta \in I_{\ell,k}. \tag{7.18}$$

Let K be the collection of $k \in \{0, 1, \ldots, \ell\}$ such that

$$E(e^{i\theta}\xi, 2r) \cap \mathcal{Z}(q) \neq \emptyset \quad \text{for some} \quad \theta \in I_{\ell,k}.$$

Then by (7.18) and Lemma 7.1, we have $\operatorname{card}(K) \leq M$. Combining (7.18) with (7.17), we see that for every $k \in K$, there is an $x_k \in F$ such that

$$\bigcup_{\theta \in I_{\ell,k}} E(e^{i\theta}\xi, 2r) \subset E(x_k, r(x_k)).$$

Let $0 < \rho \leq r$ also be given. For each $k \in K$, define

$$H(x_k; \rho) = \left\{ z \in E(x_k, r(x_k)) : \inf_{\zeta \in \mathcal{Z}(q)} |z - \zeta| < \rho \right\} \quad \text{and}$$
$$A_k = \left\{ w \in E(\xi, r) : \inf_{\zeta \in \mathcal{Z}(q)} |e^{i\theta} w - \zeta| < \rho \text{ for some } \theta \in I_{\ell,k} \right\}.$$

Also, define $\xi_k = e^{\frac{2k\pi i}{\ell+1}}\xi$, $k \in K$. Since $1/(\ell+1) \le r$, there is a constant C_2 such that

$$e^{i\theta}E(\xi, r) \subset E(\xi_k, C_2 r)$$

for every $\theta \in I_{\ell,k}$. Therefore for every $k \in K$ we have

$$A_k \subset \bigcup_{\theta \in I_{\ell,k}} e^{-i\theta}\{H(x_k;\rho) \cap E(\xi_k, C_2 r)\}. \tag{7.19}$$

Fix a $k \in K$ for the moment. By (7.15), there is an $h = h_k$ associated with $x_k \in \mathcal{Z}(q) \cap S$ such that

$$H(x_k;\rho) \cap E(\xi_k, C_2 r) \subset \bigcup_{\gamma \in \Gamma_\rho} \{E(A\gamma + Bh(\gamma), C_1\rho) \cap E(\xi_k, C_2 r)\}, \tag{7.20}$$

where $\Gamma_\rho \subset a + \rho \mathbf{Z}^{2n-2}$ for some $a = a_k \in \mathbf{R}^{2n-2}$.

Suppose that there is a $\gamma_0 \in \Gamma_\rho$ such that $E(A\gamma_0 + Bh(\gamma_0), C_1\rho) \cap E(\xi_k, C_2 r) \neq \emptyset$. If $\gamma \in \Gamma_\rho$ also has the property that $E(A\gamma + Bh(\gamma), C_1\rho) \cap E(\xi_k, C_2 r) \neq \emptyset$, then

$$|\gamma - \gamma_0| \le |(A\gamma + Bh(\gamma)) - (A\gamma_0 + Bh(\gamma_0))| \le 2C_1\rho + 2C_2 r \le C_3 r,$$

where $C_3 = 2C_1 + 2C_2$. Thus $\gamma = \gamma_0 + \rho(j_1, \dots, j_{2n-2})$, where each j_ν is an integer satisfying the condition $|j_\nu| \le C_3(r/\rho)$. For each $(j_2, \dots, j_{2n-2})$ satisfying the condition $|j_\nu| \le C_3(r/\rho)$ for every $2 \le \nu \le 2n-2$, let $\tilde{G}(j_2, \dots, j_{2n-2})$ be the set of $\gamma \in \Gamma_\rho$ satisfying the conditions

$$\gamma = \gamma_0 + \rho(j_1, j_2, \dots, j_{2n-2}) \quad \text{and} \quad E(A\gamma + Bh(\gamma), C_1\rho) \cap E(\xi_k, C_2 r) \neq \emptyset.$$

Furthermore, let $G(j_2, \dots, j_{2n-2}) = \{\gamma \in \tilde{G}(j_2, \dots, j_{2n-2}) : E(A\gamma + Bh(\gamma), C_1\rho) \cap S \neq \emptyset\}$. Then it follows from (7.13) and (7.14) that

$$\operatorname{card}(G(j_2, \dots, j_{2n-2})) \le 2\{1 + (C_1/c)\},$$

where $c = \min\{c(x) : x \in F\}$. On the other hand, by (7.20), we have

$$S \cap H(x_k;\rho) \cap E(\xi_k, C_2 r) \subset \bigcup_{\substack{|j_\nu| \le C_3(r/\rho) \\ 2 \le \nu \le 2n-2}} \bigcup_{\gamma \in G(j_2, \dots, j_{2n-2})} E(A\gamma + Bh(\gamma), C_1\rho)$$

In other words, we have

$$S \cap H(x_k;\rho) \cap E(\xi_k, C_2 r) \subset \bigcup_{j \in J} \tilde{E}_j, \tag{7.21}$$

where each $\tilde{E}_j$ is a Euclidian ball of radius $C_1\rho$ and

$$\operatorname{card}(J) \le (1 + 2C_3(r/\rho))^{2n-3} \cdot \{1 + 2(C_1/c)\} \le C_4(r/\rho)^{2n-3},$$

where $C_4 = \{1 + 2(C_1/c)\}(1 + 2C_3)^{2n-3}$. Since the length of $I_{\ell,k}$ is $2\pi/(\ell+1)$ and since $1/(\ell+1) \le r$, it is elementary that there is a constant C_5 such that for each $j \in J$, there are Euclidian ball $E_{j,1}, \dots, E_{j,m}$ of radius $(C_1 + 2\pi)\rho$ such that

$$\bigcup_{\theta \in I_{\ell,k}} e^{-i\theta} \tilde{E}_j \subset E_{j,1} \cup \cdots \cup E_{j,m}$$

and such that $m \le C_5(r/\rho)$. Combining this with (7.21) and (7.19), we see that there is an index set $\mathcal{N}_k$ with

$$\text{card}(\mathcal{N}_k) \le C_4 C_5 (r/\rho)^{2n-2} \tag{7.22}$$

such that

$$A_k \cap S \subset \bigcup_{\nu \in \mathcal{N}_k} E_\nu, \tag{7.23}$$

where each E_ν is a Euclidian ball of radius $(C_1 + 2\pi)\rho$.

Since $\rho \le r$, if there are $w \in E(\xi, r)$ and $\theta \in [0, 2\pi)$ such that $|e^{i\theta} w - \zeta| < \rho$ for some $\zeta \in \mathcal{Z}(q)$, then $E(e^{i\theta}\xi, 2r) \cap \mathcal{Z}(q) \ne \emptyset$. Thus by the definition of K we have

$$\left\{ w \in E(\xi, r) : \inf_{\zeta \in \mathcal{Z}(q)} |e^{i\theta} w - \zeta| < \rho \text{ for some } \theta \in [0, 2\pi) \right\} \subset \bigcup_{k \in K} A_k.$$

Intersecting both sides by the sphere S and recalling (7.23), we obtain

$$\{w \in S(\xi, r) : \Delta(q; w) < \rho\} \subset \bigcup_{k \in K} \{S \cap A_k\} \subset \bigcup_{k \in K} \bigcup_{\nu \in \mathcal{N}_k} \{S \cap E_\nu\}.$$

Since each E_ν is a ball of radius $(C_1 + 2\pi)\rho$, the property of the spherical measure gives us

$$\sigma(S \cap E_\nu) \le C_6 \{(C_1 + 2\pi)\rho\}^{2n-1}.$$

We know that $\text{card}(K) \le M$ from Lemma 7.1. Thus, using (7.22), we find that

$$\begin{aligned} \sigma(\{w \in S(\xi, r) : \Delta(q; w) < \rho\}) &\le \sum_{k \in K} \text{card}(\mathcal{N}_k) C_6 \{(C_1 + 2\pi)\rho\}^{2n-1} \\ &\le M C_4 C_5 (r/\rho)^{2n-2} C_6 \{(C_1 + 2\pi)\rho\}^{2n-1} = M C_4 C_5 C_6 (C_1 + 2\pi)^{2n-1} r^{2n-2} \rho. \end{aligned}$$

This completes the proof of the lemma. □

Proposition 7.3. *If $q \in \mathcal{G}_n$, then $\sigma(\mathcal{B}(q)) = 0$.*

Proof. Given $q \in \mathcal{G}_n$, let $r_0 > 0$ be the number provided by Lemma 7.2 for this q. We then fix an $r \in (0, r_0)$. Recalling Definition 6.5, for all $0 < \rho \le r$ and $\xi \in S$ we have

$$S(\xi, r) \cap \mathcal{B}(q) \subset \{w \in S(\xi, r) : \Delta(q; w) < \rho\}.$$

There is a finite subset F of S such that $S = \cup_{\xi \in F} S(\xi, r)$. Hence

$$\mathcal{B}(q) \subset \bigcup_{\xi \in F} \{w \in S(\xi, r) : \Delta(q; w) < \rho\}.$$

Applying the estimate in Lemma 7.2, we obtain $\sigma(\mathcal{B}(q)) \leq \mathrm{card}(F) C r^{2n-2} \rho$. Since this holds for every $\rho \in (0, r]$, it follows that $\sigma(\mathcal{B}(q)) = 0$. $\square$

Below is the main result of the section, which says that for $q \in \mathcal{G}_n$, the growth rate of the measure $\mu_{q;\epsilon}$ is worse than that of σ by at most ϵ.

Proposition 7.4. *Let $q \in \mathcal{G}_n$. Then for every $0 < \epsilon < 1$, there is a constant C such that $\mu_{q;\epsilon}(S(\xi, r)) \leq C r^{2n-1-\epsilon}$ for all $\xi \in S$ and $r > 0$.*

Proof. Given $q \in \mathcal{G}_n$, let $r_0 > 0$ be the number provided by Lemma 7.2 for this q. Note that $\Delta(q; w) = 0$ if and only if $w \in \mathcal{B}(q)$. Thus for any $0 < r < r_0$ and $\xi \in S$, we have

$$S(\xi, r) \backslash \mathcal{B}(q) = A_0 \cup A_1 \cup \cdots \cup A_k \cup \cdots,$$

where

$$\begin{aligned} A_0 &= \{w \in S(\xi, r) : \Delta(q; w) \geq r\} \quad \text{and} \\ A_k &= \{w \in S(\xi, r) : 2^{-k} r \leq \Delta(q; w) < 2^{-k+1} r\}, \quad k \geq 1. \end{aligned}$$

By Lemma 7.2, we have $\sigma(A_k) \leq C 2^{-k+1} r^{2n-1}$ for $k \geq 1$. Of course, $\sigma(A_0) \leq \sigma(S(\xi, r)) \leq C_1 r^{2n-1}$. Set $C_2 = \max\{2C, C_1\}$. Then, by Definition 6.6, for each $0 < \epsilon < 1$ we have

$$\mu_{q;\epsilon}(S(\xi, r)) \leq \sum_{k=0}^{\infty} (2^{-k} r)^{-\epsilon} \sigma(A_k) \leq C_2 r^{2n-1-\epsilon} \sum_{k=0}^{\infty} 2^{-(1-\epsilon)k}.$$

Thus there is a C_3 such that $\mu_{q;\epsilon}(S(\xi, r)) \leq C_3 r^{2n-1-\epsilon}$ if $0 < r < r_0$. There is a finite subset F of S such that $\cup_{x \in F} S(x, r_0/2) = S$. Hence $\mu_{q;\epsilon}(S) < \infty$. If we set $C_4 = (2/r_0)^{2n-1-\epsilon} \mu_{q;\epsilon}(S)$, then $\mu_{q;\epsilon}(S(\xi, r)) \leq C_4 r^{2n-1-\epsilon}$ for all $r \geq r_0/2$ and $\xi \in S$. Hence if we set $C_5 = \max\{C_3, C_4\}$, then $\mu_{q;\epsilon}(S(\xi, r)) \leq C_5 r^{2n-1-\epsilon}$ for all $r > 0$ and $\xi \in S$. $\square$

8. Consequence of the growth condition

Once we have Proposition 7.4, we only need to work with the growth condition given there. In other words, given the growth condition, we no longer need to be concerned with the underlying polynomial q.

Proposition 8.1. *Let μ be a Borel measure on S and suppose that $0 < \epsilon < 1/2$. If there is a constant C such that $\mu(S(\xi, r)) \leq C r^{2n-1-2\epsilon}$ for all $\xi \in S$ and $r > 0$, then there is a constant $C_{8.1}$ such that*

$$\int\int \left| \frac{1}{(1 - \langle z\xi, w\rangle)^{n+1-\epsilon}} \right|^2 dA(z) d\mu(\xi) \leq \frac{C_{8.1}}{(1 - |w|^2)^{n+1-\epsilon}} \tag{8.1}$$

for every $w \in \mathbf{B}$.

Proof. First of all, on the unit disc D we have the power-series expansion

$$\frac{1}{(1-z)^{n+1-\epsilon}} = \sum_{k=0}^{\infty} b_k z^k, \quad \text{where} \quad b_k = \frac{1}{k!}\prod_{j=1}^{k}(n+j-\epsilon) \quad \text{for} \quad k \geq 1.$$

Given any $w \in \mathbf{B}$, we write it in the form $w = |w|\eta$, where $\eta \in S$. Then

$$\int\int \left|\frac{1}{(1-\langle z\xi, w\rangle)^{n+1-\epsilon}}\right|^2 dA(z)d\mu(\xi) = \int\int \left|\frac{1}{(1-|w|\langle \xi, \eta\rangle z)^{n+1-\epsilon}}\right|^2 dA(z)d\mu(\xi)$$

(8.2)

$$= \int \sum_{k=0}^{\infty} \frac{b_k^2}{k+1}|w|^{2k}|\langle \xi, \eta\rangle|^{2k} d\mu(\xi) = \sum_{k=0}^{\infty} \frac{b_k^2}{k+1}|w|^{2k} \int |\langle \xi, \eta\rangle|^{2k} d\mu(\xi).$$

Obviously, $|\langle \xi, \eta\rangle| \leq 1$ for every $\xi \in S$. Thus by [16,Lemma I.4.1], for each $k \geq 1$ we have

$$\int |\langle \xi, \eta\rangle|^{2k} d\mu(\xi) = 2k \int_0^1 x^{2k-1}\mu(\{\xi \in S : |\langle \xi, \eta\rangle| > x\})dx. \tag{8.3}$$

For the right-hand side, we only need to consider $0 < x < 1$. Since $\eta \in S$, there are η_2, $\dots$, $\eta_n \in S$ such that the set $\{\eta, \eta_2, \dots, \eta_n\}$ is an orthonormal basis for $\mathbf{C}^n$. Thus if $\xi \in S$ is such that $|\langle \xi, \eta\rangle| > x$, then we have $\xi = ae^{i\theta}\eta + a_2\eta_2 + \cdots + a_n\eta_n$ with $a > x$ and $\theta \in [0, 2\pi)$. We have, of course, that $|a_2|^2 + \cdots + |a_n|^2 = 1 - a^2 < 1 - x^2$. Hence

$$\{\xi \in S : |\langle \xi, \eta\rangle| > x\} \subset \bigcup_{0 \leq \theta < 2\pi} S(e^{i\theta}\eta, 2\sqrt{1-x^2}). \tag{8.4}$$

As we saw in the proof of Lemma 7.2, there is a constant C_1 such that

$$\bigcup_{0 \leq \theta < 2\pi} S(e^{i\theta}\eta, 2\sqrt{1-x^2}) \subset \bigcup_{j=0}^{m} S(e^{\frac{2j\pi i}{m+1}}\eta, C_1\sqrt{1-x^2}), \tag{8.5}$$

where m is the natural number satisfying the condition $1/(m+1) \leq \sqrt{1-x^2} < 1/m$.

By (8.4), (8.5) and the growth condition on μ, we have

$$\mu(\{\xi \in S : |\langle \xi, \eta\rangle| > x\}) \leq \sum_{j=0}^{m} \mu(S(e^{\frac{2j\pi i}{m+1}}\eta, C_1\sqrt{1-x^2})) \leq (m+1)C\{C_1\sqrt{1-x^2}\}^{2n-1-2\epsilon}.$$

Since $m + 1 \leq 2m < 2(1-x^2)^{-1/2}$, we see that there is a constant C_2 such that

$$\mu(\{\xi \in S : |\langle \xi, \eta\rangle| > x\}) \leq C_2(1-x^2)^{n-1-\epsilon}.$$

Substitute this in (8.3), we find that

$$\int |\langle \xi, \eta\rangle|^{2k} d\mu(\xi) \le 2C_2 k \int_0^1 x^{2k-1}(1-x^2)^{n-1-\epsilon}dx = C_2 k \int_0^1 x^{k-1}(1-x)^{n-1-\epsilon}dx.$$

Integrating by parts, for each natural number $k \ge 1$ we have

$$\int_0^1 x^{k-1}(1-x)^{n-1-\epsilon}dx = \frac{(k-1)!}{\prod_{j=0}^{k-1}(n+j-\epsilon)} = \frac{n+k-\epsilon}{k(n-\epsilon)b_k}.$$

Hence there is a C_3 such that

$$\int |\langle \xi, \eta\rangle|^{2k} d\mu(\xi) \le C_3 \frac{k+1}{b_k}$$

for every $k \ge 0$. Substituting this in (8.2), we obtain

$$\int\int \left|\frac{1}{(1-\langle z\xi, w\rangle)^{n+1-\epsilon}}\right|^2 dA(z)d\mu(\xi) \le C_3 \sum_{k=0}^{\infty} b_k |w|^{2k} = \frac{C_3}{(1-|w|^2)^{n+1-\epsilon}}.$$

This completes the proof. □

For each $0 < \epsilon < 1$, consider the weighted measure

$$dv_{-\epsilon}(\zeta) = a_{n,-\epsilon}(1-|\zeta|^2)^{-\epsilon}dv(\zeta)$$

on $\mathbf{B}$ (see (2.1)). Then the normalized reproducing kernel for the weighted Bergman space $\mathcal{H}^{(-\epsilon)} = L_a^2(\mathbf{B}, dv_{-\epsilon})$ is given by the formula

$$k_w^{(-\epsilon)}(\zeta) = \frac{(1-|w|^2)^{(n+1-\epsilon)/2}}{(1-\langle \zeta, w\rangle)^{n+1-\epsilon}},$$

$w, \zeta \in \mathbf{B}$. Obviously, (8.1) is equivalent to

$$\int\int |k_w^{(-\epsilon)}(z\xi)|^2 dA(z)d\mu(\xi) \le C_{8.1},$$

$w \in \mathbf{B}$, which is a "Carleson condition" for the space $\mathcal{H}^{(-\epsilon)}$. Accordingly, one expects the consequent boundedness:

Proposition 8.2. *Let μ be a Borel measure on S and suppose that $0 < \epsilon < 1$. If there is a constant C such that*

$$\int\int \left|\frac{1}{(1-\langle z\xi, w\rangle)^{n+1-\epsilon}}\right|^2 dA(z)d\mu(\xi) \le \frac{C}{(1-|w|^2)^{n+1-\epsilon}} \tag{8.6}$$

for every $w \in \mathbf{B}$, then there is a constant $C_{8.2}$ such that

$$\int\int |h(z\xi)|^2 dA(z)d\mu(\xi) \le C_{8.2}\int |h(\zeta)|^2(1-|\zeta|^2)^{-\epsilon}dv(\zeta)$$

for every $h \in \mathbf{C}[z_1,\dots,z_n]$.

Proof. First of all, (8.6) implies that $\mu(S) < \infty$. Let $C(\overline{\mathbf{B}})$ be the collection of continuous functions on the closed unit ball $\overline{\mathbf{B}} = \mathbf{B}\cup S$. Since $\mu(S) < \infty$, the formula

$$\Phi(f) = \int\int f(z\xi)dA(z)d\mu(\xi), \quad f \in C(\overline{\mathbf{B}}),$$

defines a bounded, positive linear functional on $C(\overline{\mathbf{B}})$. By the Riesz representation theorem, there is a regular Borel measure ϕ on $\overline{\mathbf{B}}$ such that

$$\int f(\zeta)d\phi(\zeta) = \Phi(f) = \int\int f(z\xi)dA(z)d\mu(\xi) \tag{8.7}$$

for every $f \in C(\overline{\mathbf{B}})$. In particular, for every continuous function u on $[0,1]$ we have

$$\int u(|\zeta|)d\phi(\zeta) = \int\int u(|z|)dA(z)d\mu(\xi) = 2\mu(S)\int_0^1 u(r)rdr.$$

From this it is easy to deduce that $\phi(S) = 0$. That is, the measure ϕ is actually concentrated on the open unit ball $\mathbf{B}$.

By the discussion preceding the proposition, (8.6) implies

$$\int |k_w^{(-\epsilon)}(\zeta)|^2 d\phi(\zeta) = \int\int |k_w^{(-\epsilon)}(z\xi)|^2 dA(z)d\mu(\xi) \le C, \quad w \in \mathbf{B}.$$

From this it is an elementary exercise to show that there is a constant C_1 such that the Carleson condition

$$\phi(\{\zeta \in \mathbf{B} : |1-\langle\zeta,\xi\rangle| < r\}) \le C_1 v_{-\epsilon}(\{\zeta \in \mathbf{B} : |1-\langle\zeta,\xi\rangle| < r\})$$

holds for all $\xi \in S$ and $0 < r < 1$. It is well known (see, e.g., [7,Theorem 1] or [24,Corollary 47]) that this condition implies that ϕ is a Carleson measure for $\mathcal{H}^{(-\epsilon)}$. That is, there is a C_2 such that

$$\int |g(\zeta)|^2 d\phi(\zeta) \le C_2\int |g(\zeta)|^2(1-|\zeta|^2)^{-\epsilon}dv(\zeta)$$

for every $g \in \mathcal{H}^{(-\epsilon)}$. Given a polynomial $h \in \mathbf{C}[z_1,\dots,z_n]$, if we apply (8.7) to the case $f = |h|^2$, we obtain

$$\int\int |h(z\xi)|^2 dA(z)d\mu(\xi) = \int |h(\zeta)|^2 d\phi(\zeta) \le C_2\int |h(\zeta)|^2(1-|\zeta|^2)^{-\epsilon}dv(\zeta).$$

This completes the proof. □

Proposition 8.3. *For every pair of $n \geq 3$ and $0 < \epsilon < 1/2$ we have $\mathcal{G}_n \subset \mathcal{P}_n(-3;\epsilon)$.*

Proof. Let $0 < \epsilon < 1/2$. If $q \in \mathcal{G}_n$, $n \geq 3$, then Proposition 7.3 tells us that $\sigma(\mathcal{B}(q)) = 0$, i.e., condition (1) in Proposition 6.7 is satisfied. By Proposition 7.4, there is a C such that $\mu_{q;2\epsilon}(S(\xi, r)) \leq Cr^{2n-1-2\epsilon}$ for all $\xi \in S$ and $r > 0$. Thus Proposition 8.1 gives us

$$\int\int \left|\frac{1}{(1-\langle z\xi, w\rangle)^{n+1-\epsilon}}\right|^2 dA(z)d\mu_{q;2\epsilon}(\xi) \leq \frac{C_{8.1}}{(1-|w|^2)^{n+1-\epsilon}}$$

for every $w \in \mathbf{B}$. By Proposition 8.2, this implies that

$$\int\int |h(z\xi)|^2 dA(z)d\mu_{q;2\epsilon}(\xi) \leq C_{8.2}\int |h(\zeta)|^2(1-|\zeta|^2)^{-\epsilon}dv(\zeta)$$

for every $h \in \mathbf{C}[z_1, \ldots, z_n]$. That is, condition (2) in Proposition 6.7 is also satisfied. Thus it follows from Proposition 6.7 that $q \in \mathcal{P}_n(-3;\epsilon)$. □

Proof of Theorem 1.5. Let $q \in \mathcal{G}_n$, $n \geq 3$, be given, and suppose that $n < p < \infty$. We pick an $0 < \epsilon < 1/2$ such that $n/(1-\epsilon) < p$. Then the norm ideal $\mathcal{C}^+_{n/(1-\epsilon)}$ is contained in the Schatten class $\mathcal{C}_p$. By Proposition 8.3 we have $q \in \mathcal{P}_n(-3;\epsilon)$. Applying Proposition 4.4, we conclude that the submodule operators

$$Z^{(-3)}_{q,j} = M_{z_j}|[q]^{(-3)}, \quad 1 \leq j \leq n,$$

have the property $[Z^{(-3)*}_{q,j}, Z^{(-3)}_{q,i}] \in \mathcal{C}^+_{n/(1-\epsilon)}$ for all $j, i \in \{1, \ldots, n\}$. Since $\mathcal{C}^+_{n/(1-\epsilon)} \subset \mathcal{C}_p$, this means that the submodule $[q]^{(-3)}$ of $\mathcal{H}^{(-3)}$ is p-essentially normal as promised. □

Note that by the product rule for R, the set $\mathcal{P}_n(t;\epsilon)$ is multiplicative for all $-n \leq t < \infty$ and $0 \leq \epsilon < 1$. That is, for $q_1, \ldots, q_k \in \mathcal{P}_n(t;\epsilon)$, $k \geq 1$, we have $q_1 \cdots q_k \in \mathcal{P}_n(t;\epsilon)$. Of course, this fact is not significant in cases where we know that the equality $\mathcal{P}_n(t;\epsilon) = \mathbf{C}[z_1, \ldots, z_n]$ holds. But in cases where we do not yet know this equality for a fact, the multiplicativity of $\mathcal{P}_n(t;\epsilon)$ becomes significant. Indeed using this multiplicativity, from Propositions 8.3 and 4.4 we actually obtain

Corollary 8.4. *If $q_1, \ldots, q_k \in \mathcal{G}_n$, $n \geq 3$ and $k \geq 1$, then the submodule $[q_1 \cdots q_k]^{(-3)}$ of $\mathcal{H}^{(-3)}$ is p-essentially normal for every $p > n$.*

9. Polynomials in $\mathcal{G}_n$

The most prominent feature of $\mathcal{G}_n$ is that its membership is stable under small perturbation. To make this precise, we need to introduce a norm. For any function h that is analytic on an open set Ω containing the closed ball $\overline{\mathbf{B}}$, we define

$$\|h\|_\# = \max\left\{\max_{|z|\leq 1}|h(z)|, \max_{|z|\leq 1}|(\partial h)(z)|\right\},$$

where $|(\partial h)(z)|$ is the Euclidian length of the analytic gradient vector $(\partial h)(z) = ((\partial_1 h)(z), \ldots, (\partial_n h)(z))$. By the Cauchy-Schwarz inequality, $|(Rh)(z)| \le |(\partial h)(z)|$ whenever $|z| \le 1$.

Proposition 9.1. *For each $q \in \mathcal{G}_n$, there is a $\rho > 0$ such that for every $h \in \mathbf{C}[z_1, \ldots, z_n]$ satisfying the condition $\|h\|_\# \le \rho$, we have $q + h \in \mathcal{G}_n$.*

Proof. Recall from Section 7 that the membership $q \in \mathcal{G}_n$ is equivalent to the strict inequality (7.10). Since $\mathcal{Z}(q) \cap S$ is compact, (7.10) implies that there is a $c > 0$ such that

$$|(\partial q)(\xi)| \ge |(Rq)(\xi)| + c \quad \text{and} \quad |(Rq)(\xi)| \ge c$$

for every $\xi \in \mathcal{Z}(q) \cap S$. Let U be the collection of $\zeta \in S$ satisfying the conditions $|(\partial q)(\zeta)| > |(Rq)(\zeta)| + (c/2)$ and $|(Rq)(\zeta)| > c/2$ simultaneously. Then $U \supset \mathcal{Z}(q) \cap S$ and U is an open subset of S. Hence there is an $r > 0$ such that $|q(\zeta)| \ge r$ for every $\zeta \in S \backslash U$. Thus

$$U \supset \{\zeta \in S : |q(\zeta)| < r\}. \tag{9.1}$$

Set $\rho = \min\{(r/2), (c/8)\}$. Let us show that this ρ has the desired property. Suppose that $h \in \mathbf{C}[z_1, \ldots, z_n]$ and that $\|h\|_\# \le \rho$. If $\xi \in S$ is such that $q(\xi) + h(\xi) = 0$, then $|q(\xi)| = |h(\xi)| \le \rho$. By (9.1), we have $\mathcal{Z}(q + h) \cap S \subset U$. But for every $\zeta \in U$, we have

$$\begin{aligned}|(\partial(q+h))(\zeta)| &\ge |(\partial q)(\zeta)| - |(\partial h)(\zeta)| \ge |(Rq)(\zeta)| + (c/2) - \rho \\ &\ge |(Rq)(\zeta)| + |(Rh)(\zeta)| + (c/2) - 2\rho \ge |(R(q+h))(\zeta)| + (c/4).\end{aligned}$$

Similarly, if $\zeta \in U$, then $|(R(q+h))(\zeta)| \ge |(Rq)(\zeta)| - |(Rh)(\zeta)| \ge (c/2) - \rho \ge (3/8)c$. Since $c > 0$, it follows that

$$0 < |(R(q+h))(\xi)| < |(\partial(q+h))(\xi)|$$

whenever $\xi \in \mathcal{Z}(q+h) \cap S$. By the discussion in Section 7, this means $q + h \in \mathcal{G}_n$. □

For explicit polynomials, we can make the ρ above more explicit.

Example 9.2. Let $a \in \mathbf{C}$ be such that $|a| = 1/2$. If $h \in \mathbf{C}[z_1, \ldots, z_n]$ satisfies the condition $\|h\|_\# \le 1/8$, then the polynomial

$$q(z_1, \ldots, z_n) = z_1 - a + h(z_1, \ldots, z_n)$$

belongs to $\mathcal{G}_n$. Indeed if $\xi = (\xi_1, \ldots, \xi_n)$ belongs to $\mathcal{Z}(q) \cap S$, then we have $\xi_1 - a + h(\xi_1, \ldots, \xi_n) = 0$. Since $\|h\|_\# \le 1/8$, this implies $3/8 \le |\xi_1| \le 5/8$, and consequently

$$|(Rq)(\xi_1, \ldots, \xi_n)| = |\xi_1 - (Rh)(\xi_1, \ldots, \xi_n)| \le (5/8) + (1/8) = 3/4.$$

On the other hand, for every $\zeta \in S$ we have $|(\partial q)(\zeta)| \ge 1 - |(\partial h)(\zeta)| \ge 1 - (1/8) > 3/4$. Therefore $|(\partial q)(\xi)| > |(Rq)(\xi)|$ for every $\xi \in \mathcal{Z}(q) \cap S$. For $(\xi_1, \ldots, \xi_n) \in \mathcal{Z}(q) \cap S$, we also have

$$|(Rq)(\xi_1, \ldots, \xi_n)| \ge |\xi_1| - |(Rh)(\xi_1, \ldots, \xi_n)| \ge (3/8) - (1/8) > 0.$$

Hence $q \in \mathcal{G}_n$. Note that if there are $a_2, \ldots, a_n$ such that $(a, a_2, \ldots, a_n) \in \mathbf{B}$ and $h(a, a_2, \ldots, a_n) = 0$, then we also have $q(a, a_2, \ldots, a_n) = 0$.

Note that the set $\mathcal{G}_n$ is not closed under multiplication. Indeed for $q \in \mathbf{C}[z_1, \dots, z_n]$ with $\mathcal{Z}(q) \cap S \neq \emptyset$, we have $q^2 \notin \mathcal{G}_n$. Nevertheless, there is a conditional multiplicativity:

Proposition 9.3. *For $q_1, q_2 \in \mathcal{G}_n$, if $\mathcal{Z}(q_1) \cap \mathcal{Z}(q_2) \cap S = \emptyset$, then $q_1 q_2 \in \mathcal{G}_n$.*

Proof. Let such q_1, q_2 be given. By the product rule of differentiation, we have $R(q_1q_2) = q_2Rq_1 + q_1Rq_2$ and $\partial(q_1q_2) = q_2\partial q_1 + q_1\partial q_2$. If $\xi \in \mathcal{Z}(q_1) \cap S$, then $q_1(\xi) = 0$ and $q_2(\xi) \neq 0$. Hence $|(R(q_1q_2))(\xi)| = |q_2(\xi)||(Rq_1)(\xi)|$ and $|(\partial(q_1q_2))(\xi)| = |q_2(\xi)||(\partial q_1)(\xi)|$. Since $|q_2(\xi)| > 0$, from the strict inequality $0 < |(Rq_1)(\xi)| < |(\partial q_1)(\xi)|$ we obtain the strict inequality

$$0 < |(R(q_1q_2))(\xi)| < |(\partial(q_1q_2))(\xi)|.$$

Similarly, this also holds for $\xi \in \mathcal{Z}(q_2) \cap S$. Since $\mathcal{Z}(q_1q_2) \cap S = \{\mathcal{Z}(q_1) \cap S\} \cup \{\mathcal{Z}(q_2) \cap S\}$, the proposition follows. $\square$

Let $b_1, \dots, b_m$ be *pairwise distinct* complex numbers satisfying the condition $2^{-1/2} < |b_j| < 1$, $j = 1, \dots, m$. It is easy to see that for every pair of $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, m\}$, the polynomial $z_i - b_j$ belongs to $\mathcal{G}_n$. Note that for each point $(\xi_1, \dots, \xi_n)$ in S, there is at most one $i \in \{1, \dots, n\}$ such that $\xi_i \in \{b_1, \dots, b_m\}$. Thus by the above proposition and a simple induction, the polynomial

$$\prod_{i=1}^{n} \prod_{j=1}^{m} (z_i - b_j)$$

belongs to $\mathcal{G}_n$.

References

1. W. Arveson, Subalgebras of C^*-algebras. III. Multivariable operator theory, Acta Math. **181** (1998), 159-228.

2. W. Arveson, The Dirac operator of a commuting d-tuple, J. Funct. Anal. **189** (2002), 53-79

3. W. Arveson, p-summable commutators in dimension d, J. Operator Theory **54** (2005), 101-117.

4. W. Arveson, Myhill Lectures, SUNY Buffalo, April 2006.

5. W. Arveson, Quotients of standard Hilbert modules, Trans. Amer. Math. Soc. **359** (2007), 6027-6055.

6. C. Bennett and R. Sharpley, Interpolation of Operators, Pure Appl. Math., vol. 129, Academic Press, Boston, 1988.

7. J. Cima and W. Wogen, A Carleson measure theorem for the Bergman space on the ball, J. Operator Theory **7** (1982), 157-165.

8. K. Davidson, The mathematical legacy of William Arveson, J. Operator Theory **68** (2012), 307-334.

9. R. Douglas, Essentially reductive Hilbert modules, J. Operator Theory **55** (2006), 117-133.

10. R. Douglas, A new kind of index theorem, Analysis, geometry and topology of elliptic operators, 369-382, World Sci. Publ., Hackensack, NJ, 2006.

11. R. Douglas, X. Tang and G. Yu, An analytic Grothendieck Riemann Roch theorem, preprint 2014, arXiv:1404.4396
12. R. Douglas and K. Wang, Essential normality of cyclic submodule generated by any polynomial, J. Funct. Anal. **261** (2011), 3155-3180.
13. M. Engliš and J. Eschmeier, Geometric Arveson-Douglas conjecture, Adv. Math. **274** (2015), 606-630.
14. J. Eschmeier, Essential normality of homogeneous submodules, Integr. Equ. Oper. Theory **69** (2011), 171-182.
15. Q. Fang and J. Xia, Essential normality of polynomial-generated submodules: Hardy space and beyond, J. Funct. Anal. **265** (2013), 2991-3008.
16. J. Garnett, Bounded analytic functions, Academic Press, New York-London, 1981.
17. I. Gohberg and M. Krein, Introduction to the theory of linear nonselfadjoint operators, Amer. Math. Soc. Translations of Mathematical Monographs **18**, Providence, 1969.
18. K. Guo, Defect operators for submodules of H_d^2, J. Reine Angew. Math. **573** (2004), 181-209.
19. K. Guo and K. Wang, Essentially normal Hilbert modules and K-homology, Math. Ann. **340** (2008), 907-934.
20. K. Guo and C. Zhao, p-essential normality of quasi-homogeneous Drury-Arveson submodules, J. Lond. Math. Soc. (2) **87** (2013), 899-916.
21. J. Munkres, Analysis on manifolds, Addison-Wesley Publishing Company, Advanced Book Program, Redwood City, CA, 1991.
22. W. Rudin, Function theory in the unit ball of $\mathbf{C}^n$, Springer-Verlag, New York, 1980.
23. O. Shalit, Stable polynomial division and essential normality of graded Hilbert modules, J. Lond. Math. Soc. (2) **83** (2011), 273-289.
24. R. Zhao and K. Zhu, Theory of Bergman spaces in the unit ball of $\mathbf{C}^n$, Mém. Soc. Math. Fr. (N.S.) **115** (2008), vi+103 pp.

Quanlei Fang
Department of Mathematics and Computer Science, Bronx Community College, CUNY, Bronx, NY 10453

E-mail: quanlei.fang@bcc.cuny.edu

Jingbo Xia
Department of Mathematics, State University of New York at Buffalo, Buffalo, NY 14260

E-mail: jxia@acsu.buffalo.edu