# COMMUTATORS AND LOCALIZATION ON THE DRURY-ARVESON SPACE 

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#### Abstract

Let $f$ be a multiplier for the Drury-Arveson space $H_{n}^{2}$ of the unit ball, and let $\zeta_{1}, \ldots$, $\zeta_{n}$ denote the coordinate functions. We show that for each $1 \leq i \leq n$, the commutator $\left[M_{f}^{*}, M_{\zeta_{i}}\right]$ belongs to the Schatten class $\mathcal{C}_{p}, p>2 n$. This leads to a localization result for multipliers.


## 1. Introduction

Let $\mathbf{B}$ denote the open unit ball $\{z:|z|<1\}$ in $\mathbf{C}^{n}$. Throughout the paper, the complex dimension $n$ is assumed to be greater than or equal to 2. A multivariable analogue of the classical Hardy space of the unit circle is the Drury-Arveson space $H_{n}^{2}$ on $\mathbf{B}[3,9]$. Because of its close relation to a number of topics in operator theory, among which we mention the von Neumann inequality for commuting row contractions, $H_{n}^{2}$ has been the subject of intense study of late $[2-7,10,12,13]$.

The space $H_{n}^{2}$ is a reproducing kernel Hilbert space with the kernel

$$
K(z, w)=\frac{1}{1-\langle z, w\rangle}, \quad z, w \in \mathbf{B}
$$

which is a multivariable generalization of the one-variable Szegö kernel. An orthonormal basis of $H_{n}^{2}$ is given by $\left\{e_{\alpha}: \alpha \in \mathbf{Z}_{+}^{n}\right\}$, where

$$
e_{\alpha}(\zeta)=\sqrt{\frac{|\alpha|!}{\alpha!}} \zeta^{\alpha}
$$

In this paper we use the standard multi-index notation: For $\alpha=\left(\alpha_{1}, \ldots, \alpha_{n}\right) \in \mathbf{Z}_{+}^{n}$,

$$
\alpha!=\alpha_{1}!\alpha_{2}!\cdots \alpha_{n}!, \quad|\alpha|=\alpha_{1}+\cdots+\alpha_{n}, \quad \zeta^{\alpha}=\zeta_{1}^{\alpha_{1}} \cdots \zeta_{n}^{\alpha_{n}} .
$$

For functions $f, g \in H_{n}^{2}$ with Taylor expansions

$$
f(\zeta)=\sum_{\alpha \in \mathbf{Z}_{+}^{n}} c_{\alpha} \zeta^{\alpha} \quad \text { and } \quad g(\zeta)=\sum_{\alpha \in \mathbf{Z}_{+}^{n}} d_{\alpha} \zeta^{\alpha}
$$

the inner product is given by

$$
\langle f, g\rangle=\sum_{\alpha \in \mathbf{Z}_{+}^{n}} \frac{\alpha!}{|\alpha|!} c_{\alpha} \overline{d_{\alpha}}
$$

Throughout the paper, we let $M_{\zeta_{1}}, \ldots, M_{\zeta_{n}}$ denote the operators of multiplication by the coordinate functions $\zeta_{1}, \ldots, \zeta_{n}$ on $H_{n}^{2}$. With the identification of each $\zeta_{i}$ with each $M_{\zeta_{i}}, H_{n}^{2}$ is often called the Drury-Arveson module over the polynomial ring $\mathbf{C}\left[\zeta_{1}, \ldots, \zeta_{n}\right]$.

A holomorphic function $f$ on $\mathbf{B}$ is called a multiplier for the space $H_{n}^{2}$ if $f H_{n}^{2} \subset H_{n}^{2}$. If $f$ is a multiplier, then the multiplication operator $M_{f}$ defined by $M_{f}(g)=f g$ is necessarily bounded on
$H_{n}^{2}$ [3], and the multiplier norm of $f$ is defined to be the operator norm of $M_{f}$. In [3], Arveson showed that, when $n \geq 2$, the collection of multipliers of $H_{n}^{2}$ is strictly smaller than $H^{\infty}$. On $H_{n}^{2}$, multipliers can be used to express orthogonal projections. Suppose that $\mathcal{E}$ is a submodule of the Drury-Arveson module, i.e., $\mathcal{E}$ is a closed linear subspace of $H_{n}^{2}$ which is invariant under $M_{\zeta_{1}}, \ldots, M_{\zeta_{n}}$. Then there exist multipliers $\left\{f_{1}, \ldots, f_{k}, \ldots\right\}$ of $H_{n}^{2}$ such that the operator

$$
M_{f_{1}} M_{f_{1}}^{*}+\cdots+M_{f_{k}} M_{f_{k}}^{*}+\cdots
$$

is the orthogonal projection from $H_{n}^{2}$ onto $\mathcal{E}$ (see page 191 in [4]).
Among the recent results related to multipliers, we would like to mention the following developments. Interpolation problems for multipliers and model theory related to the Drury-Arveson space also have been intensely studied over the past decade or so $[5,6,10,12,13]$. Recently, Arcozzi, Rochberg and Sawyer gave a characterization of the multipliers in terms of Carleson measures for $H_{n}^{2}$ [2]. In another study, Costea, Sawyer and Wick [7] proved a corona theorem for the Drury-Averson space multipliers.

Since $H_{n}^{2}$ is a natural analogue of the Hardy space, it is natural to take a list of Hardy-space results and try to determine which ones have analogues on $H_{n}^{2}$ and which ones do not. Commutators are certainly very high on any such list. One prominent part of the theory of the Hardy space is the Toeplitz operators on it. Since there is no $L^{2}$ associated with $H_{n}^{2}$, the only analogue of Toeplitz operators on $H_{n}^{2}$ are the multipliers. In this paper we are interested in the commutators of the form $\left[M_{f}^{*}, M_{\zeta_{i}}\right]$, where $f$ is a multiplier for the Drury-Arveson space. Since the story about the commutators of the form $\left[M_{f}^{*}, M_{\zeta_{i}}\right]$ is well known on the Hardy space, one would certainly like to know the analogous story on $H_{n}^{2}$.

Recall that for each $1 \leq p<\infty$, the Schatten class $\mathcal{C}_{p}$ consists of operators $A$ satisfying the condition $\|A\|_{p}<\infty$, where the $p$-norm is given by the formula

$$
\|A\|_{p}=\left\{\operatorname{tr}\left(\left(A^{*} A\right)^{p / 2}\right)\right\}^{1 / p}
$$

Arveson showed in his seminal paper [3] that commutators of the form $\left[M_{\zeta_{j}}^{*}, M_{\zeta_{i}}\right]$ all belong to $\mathcal{C}_{p}$, $p>n$. As the logical next step, one certainly expects a Schatten class result for commutators on $H_{n}^{2}$ involving multipliers other than the simplest coordinate functions. The following is the main result of the paper:

Theorem 1.1. Let $f$ be a multiplier for the Drury-Arveson space $H_{n}^{2}$. For each $1 \leq i \leq n$, the commutator $\left[M_{f}^{*}, M_{\zeta_{i}}\right.$ ] belongs to the Schatten class $\mathcal{C}_{p}, p>2 n$. Moreover, for each $2 n<p<\infty$, there is a constant $C$ which depends only on $p$ and $n$ such that

$$
\left\|\left[M_{f}^{*}, M_{\zeta_{i}}\right]\right\|_{p} \leq C\left\|M_{f}\right\|
$$

for every multiplier $f$ of $H_{n}^{2}$ and every $1 \leq i \leq n$.
This Schatten-class result has $C^{*}$-algebraic implications.
Throughout the paper, we denote the unit sphere $\left\{z \in \mathbf{C}^{n}:|z|=1\right\}$ in $\mathbf{C}^{n}$ by $S$.
Let $\mathcal{T}_{n}$ be the $C^{*}$-algebra generated by $M_{\zeta_{1}}, \cdots, M_{\zeta_{n}}$ on $H_{n}^{2}$. Recall that $\mathcal{T}_{n}$ was introduced by Arveson in [3]. In more ways than one, $\mathcal{T}_{n}$ is the analogue of the $C^{*}$-algebra generated by Toeplitz
operators with continuous symbols. Indeed Arveson showed that there is an exact sequence

$$
\begin{equation*}
\{0\} \rightarrow \mathcal{K} \rightarrow \mathcal{T}_{n} \xrightarrow{\tau} C(S) \rightarrow\{0\} \tag{1.1}
\end{equation*}
$$

where $\mathcal{K}$ is the collection of compact operators on $H_{n}^{2}$. But there is another natural $C^{*}$-algebra on $H_{n}^{2}$ which is also related to "Toeplitz operators", where the symbols are not necessarily continuous. We define

$$
\mathcal{T} \mathcal{M}_{n}=\text { the } C^{*} \text {-algebra generated by }\left\{M_{f}: f H_{n}^{2} \subset H_{n}^{2}\right\}
$$

Theorem 1.1 tells us that $\mathcal{T}_{n}$ is contained in the essential center of $\mathcal{T} \mathcal{M}_{n}$, in analogy with the classic situation on the Hardy space of the unit sphere $S$. This opens the door for us to use the classic localization technique [8] to analyze multipliers.

Recall that the essential norm of a bounded operator $A$ on a Hilbert space $\mathcal{H}$ is

$$
\|A\|_{\mathcal{Q}}=\inf \{\|A+K\|: K \text { is compact on } \mathcal{H}\}
$$

Alternately, $\|A\|_{\mathcal{Q}}=\|\pi(A)\|$, where $\pi$ denotes the quotient map from $\mathcal{B}(\mathcal{H})$ to the Calkin algebra $\mathcal{Q}=\mathcal{B}(\mathcal{H}) / \mathcal{K}(\mathcal{H})$.

To state our localization result, we need to introduce a class of Schur multipliers. For each $z \in \mathbf{B}$, let

$$
\begin{equation*}
s_{z}(\zeta)=\frac{1-|z|}{1-\langle\zeta, z\rangle} \tag{1.2}
\end{equation*}
$$

The reason we call $s_{z}$ a Schur multiplier is that the norm of the operator $M_{s_{z}}$ on $H_{n}^{2}$ is 1 , as we will see in Section 2. Using Theorem 1.1, we will prove

Theorem 1.2. Let $A \in \mathcal{T} \mathcal{M}_{n}$. Then for each $\xi \in S$, the limit

$$
\begin{equation*}
\lim _{r \uparrow 1}\left\|A M_{s_{r \xi}}\right\| \tag{1.3}
\end{equation*}
$$

exists. Moreover, we have

$$
\|A\|_{\mathcal{Q}}=\sup _{\xi \in S} \lim _{r \uparrow 1}\left\|A M_{s_{r \xi}}\right\| .
$$

The $C^{*}$-algebraic meaning of the "localized limit" (1.3) will be explained in Section 6. Alternately, we can state Theorem 1.2 in a version which may be better suited for applications:
Theorem 1.3. For each $A \in \mathcal{T} \mathcal{M}_{n}$, we have

$$
\|A\|_{\mathcal{Q}}=\lim _{r \uparrow 1} \sup _{r \leq|z|<1}\left\|A M_{s_{z}}\right\| .
$$

The rest of the paper is organized as follows. Section 2 begins with an orthogonal decomposition of $H_{n}^{2}$. This decomposition allows us to obtain the subnormality of certain multipliers. We then use this decomposition to make a number of norm estimates. In Section 3 we derive a "quasi-resolution" of the identity operator of $H_{n}^{2}$, which plays the key role in the proof of Theorem 1.1. In Section 4 we estimate the Schatten $p$-norm and the operator norm of certain finite-rank operators which arise from the "quasi-resolution". With this preparation, the proof of Theorem 1.1 is completed in Section 5. Section 6 deals with localization and proves Theorems 1.2 and 1.3.

In terms of techniques, the reader will notice that this paper is quite different from previous works on the Drury-Arveson space. This is highlighted by the fact that the unit sphere $S$ and the spherical measure $d \sigma$ play a prominent role in our estimates. Many of the techniques we use in this
paper are inspired by our earlier work on Hankel operators [11]. The best example to illustrate this is the idea of using "quasi-resolution" of the identity operator. This interchangeability of techniques serves to show that there is indeed much in common between the Hardy space and the Drury-Arveson space. This view was one of the motivating factors which started this investigation.

## 2. Estimates for Certain Multipliers

First of all, let us introduce the subset $\mathcal{B}=\left\{\left(0, \beta_{2}, \ldots, \beta_{n}\right): \beta_{2}, \ldots, \beta_{n} \in \mathbf{Z}_{+}\right\}$of $\mathbf{Z}_{+}^{n}$. As we indicated in Section 1, we denote the components of $\zeta$ by $\zeta_{1}, \ldots, \zeta_{n}$. For each $\beta \in \mathcal{B}$, define the closed linear subspace

$$
H_{\beta}=\overline{\operatorname{span}\left\{\zeta_{1}^{k} \zeta^{\beta}: k \geq 0\right\}}
$$

of $H_{n}^{2}$. Then we have the orthogonal decompostion

$$
H_{n}^{2}=\bigoplus_{\beta \in \mathcal{B}} H_{\beta}
$$

For each $\beta \in \mathcal{B}$, we have an orthonormal basis $\left\{e_{k, \beta}: k \geq 0\right\}$ for $H_{\beta}$, where

$$
\begin{equation*}
e_{k, \beta}(\zeta)=\sqrt{\frac{(k+|\beta|)!}{k!\beta!}} \zeta_{1}^{k} \zeta^{\beta} . \tag{2.1}
\end{equation*}
$$

It is well known that $H_{0}=H_{1}^{2}$, the Hardy space associated with the unit circle $T$. For our proofs, we need to identify each $H_{\beta}, \beta \neq 0$, as a weighted Bergman space on the unit disc.

Denote $D=\{z \in \mathbf{C}:|z|<1\}$, the open unit disc in the complex plane. Let $d A$ be the area measure on $D$ with the normalization $A(D)=1$. For each integer $m \geq 0$, let

$$
\begin{equation*}
B^{(m)}=L_{a}^{2}\left(D,\left(1-|z|^{2}\right)^{m} d A(z)\right) \tag{2.2}
\end{equation*}
$$

the usual weighted Bergman space of weight $m$. It is well known that the standard orthonormal basis for $B^{(m)}$ is $\left\{e_{k}^{(m)}: k \in \mathbf{Z}_{+}\right\}$, where

$$
\begin{equation*}
e_{k}^{(m)}(z)=\sqrt{\frac{(k+m+1)!}{k!m!}} z^{k} \tag{2.3}
\end{equation*}
$$

For each $\beta \in \mathcal{B} \backslash\{0\}$, define the unitary operator $W_{\beta}: H_{\beta} \rightarrow B^{(|\beta|-1)}$ by the formula

$$
\begin{equation*}
W_{\beta} e_{k, \beta}=e_{k}^{(|\beta|-1)}, \quad k \in \mathbf{Z}_{+} . \tag{2.4}
\end{equation*}
$$

Using (2.1) and (2.3), it is straightforward to verify that the weighted shift $M_{\zeta_{1}} \mid H_{\beta}$ is unitarily equivalent to $M_{z}$ on $B^{(|\beta|-1)}$. More precisely, if $\beta \in \mathcal{B} \backslash\{0\}$, then

$$
\begin{equation*}
W_{\beta} M_{\zeta_{1}} h_{\beta}=M_{z} W_{\beta} h_{\beta} \quad \text { for every } \quad h_{\beta} \in H_{\beta} . \tag{2.5}
\end{equation*}
$$

The operator $M_{\zeta_{1}} \mid H_{0}$ is, of course, the unilateral shift.
Lemma 2.1. For each individual $i \in\{1, \ldots, n\}$, the multiplication operator $M_{\zeta_{i}}$ is subnormal on $H_{n}^{2}$. Moreover, each $M_{\zeta_{i}}$ has a normal extension of norm 1.

Proof. This is actually a known fact. See [1]. But this fact also follows from (2.5) for $M_{\zeta_{1}}$. By the obvious symmetry, the entire lemma follows from (2.5).

For each $z \in \mathbf{B}$, define the multiplier

$$
\begin{equation*}
m_{z}(\zeta)=\frac{1-|z|^{2}}{1-\langle\zeta, z\rangle} \tag{2.6}
\end{equation*}
$$

Obviously, $m_{z}$ is just a minor modification of the Schur multiplier $s_{z}$ defined in (1.2). For many purposes, it is easier to work with $m_{z}$ than $s_{z}$, as we will see. The proof of Theorem 1.1 involves the subnormality of $M_{m_{z}^{k}}$ and an estimate for $\left\|M_{m_{z} m_{w}}\right\|$.

Let $\mathcal{U}$ denote the collection of unitary transformations on $\mathbf{C}^{n}$. It is obvious that if $f$ is a multiplier for $H_{n}^{2}$ and if $U \in \mathcal{U}$, then the function $f \circ U$ is also a multiplier for $H_{n}^{2}$. Moreover, the multiplication operators

$$
M_{f} \quad \text { and } \quad M_{f \circ U}
$$

are unitarily equivalent on $H_{n}^{2}$. This fact will be used several times.
Corollary 2.2. For all $k \in \mathbf{Z}_{+}$and $z \in \mathbf{B}$, the operator $M_{m_{z}^{k}}$ is subnormal on $H_{n}^{2}$.
Proof. Given a $z \in \mathbf{B}$, pick a $U \in \mathcal{U}$ such that

$$
U^{*} z=(|z|, 0, \ldots, 0)
$$

Then for each $k \in \mathbf{Z}_{+}$we have

$$
m_{z}^{k}(U \zeta)=m_{U^{*} z}^{k}(\zeta)=\left(\frac{1-|z|^{2}}{1-|z| \zeta_{1}}\right)^{k}
$$

By Lemma 2.1 and the above-mentioned unitary equivalence, $M_{m_{z}^{k}}$ has a normal extension.
The following lemma provides a key estimate:
Lemma 2.3. If $0<s<1$, then the norm of the operator of multiplication by the function

$$
\frac{\zeta_{2}}{1-s \zeta_{1}}
$$

on $H_{n}^{2}$ does not exceed

$$
\frac{2}{\sqrt{1-s}}
$$

Proof. Consider an arbitrary $h_{\beta} \in H_{\beta}$, where $\beta=\left(0, \beta_{2}, \ldots, \beta_{n}\right)$. Then

$$
h_{\beta}(\zeta)=\sum_{k=0}^{\infty} c_{k} \zeta_{1}^{k} \zeta^{\beta} .
$$

First we assume that $\beta \neq 0$. By (2.4), we have

$$
\left(W_{\beta} h_{\beta}\right)(z)=\sqrt{\frac{\beta!}{(|\beta|-1)!}} \sum_{k=0}^{\infty} c_{k} z^{k}, \quad z \in D
$$

which is a vector in $B^{(|\beta|-1)}$. Denote $e_{2}=(0,1,0, \ldots, 0)$. Since $\zeta_{2} \zeta^{\beta}=\zeta^{\beta+e_{2}}$, we have

$$
\left(W_{\beta+e_{2}} \zeta_{2} h_{\beta}\right)(z)=\sqrt{\frac{\left(\beta+e_{2}\right)!}{|\beta|!}} \sum_{k=0}^{\infty} c_{k} z^{k}, \quad z \in D
$$

which is a vector in $B^{(|\beta|)}$. Now suppose that

$$
h_{\beta}(\zeta)=\left(1-s \zeta_{1}\right)^{-1} f_{\beta}(\zeta),
$$

where

$$
f_{\beta}(\zeta)=\sum_{k=0}^{\infty} a_{k} \zeta_{1}^{k} \zeta^{\beta}
$$

For $z \in D$ and $0<s<1$, we have $|1-s z| \geq 1-|z|$ and $|1-s z| \geq 1-s$. Thus the above yields

$$
\begin{aligned}
\left\|\zeta_{2}\left(1-s \zeta_{1}\right)^{-1} f_{\beta}\right\|_{H_{n}^{2}}^{2} & =\left\|\zeta_{2} h_{\beta}\right\|_{H_{n}^{2}}^{2}=\left\|W_{\beta+e_{2}} \zeta_{2} h_{\beta}\right\|_{B^{(|\beta|)}}^{2} \\
& =\frac{\left(\beta+e_{2}\right)!}{|\beta|!} \int_{D}\left|\sum_{k=0}^{\infty} c_{k} z^{k}\right|^{2}\left(1-|z|^{2}\right)^{|\beta|} d A(z) \\
& =\frac{\left(\beta+e_{2}\right)!}{|\beta|!} \int_{D}\left|\frac{1}{1-s z} \sum_{k=0}^{\infty} a_{k} z^{k}\right|^{2}\left(1-|z|^{2}\right)^{|\beta|} d A(z) \\
& \leq \frac{2}{1-s} \cdot \frac{\left(\beta+e_{2}\right)!}{|\beta|!} \int_{D}\left|\sum_{k=0}^{\infty} a_{k} z^{k}\right|^{2}\left(1-|z|^{2}\right)^{|\beta|-1} d A(z) \\
& =\frac{2}{1-s} \cdot \frac{\beta_{2}+1}{|\beta|} \cdot \frac{\beta!}{(|\beta|-1)!} \int_{D}\left|\sum_{k=0}^{\infty} a_{k} z^{k}\right|^{2}\left(1-|z|^{2}\right)^{|\beta|-1} d A(z) \\
& =\frac{2}{1-s} \cdot \frac{\beta_{2}+1}{|\beta|}\left\|W_{\beta} f_{\beta}\right\|_{B}^{2}(|\beta|-1) \\
& =\frac{2}{1-s} \cdot \frac{\beta_{2}+1}{|\beta|}\left\|f_{\beta}\right\|_{H_{n}^{2}}^{2} \leq \frac{4}{1-s}\left\|f_{\beta}\right\|_{H_{n}^{2}}^{2} .
\end{aligned}
$$

Thus we have shown that for $\beta \neq 0$, the norm of the restriction of the operator of multiplication by $\zeta_{2}\left(1-s \zeta_{1}\right)^{-1}$ to $H_{\beta}$ does not exceed $2(1-s)^{-1 / 2}$. Next we consider the case where $\beta=0$.

We know that $H_{0}=H_{1}^{2}$, the Hardy space on the unit circle $T$. Let $h \in H_{0}$. Then

$$
h(\zeta)=\sum_{k=0}^{\infty} c_{k} \zeta_{1}^{k}
$$

We have

$$
\left(W_{e_{2}} \zeta_{2} h\right)(z)=\sum_{k=0}^{\infty} c_{k} z^{k}, \quad z \in D
$$

which is a vector in the unweighted Bergman space $B^{(0)}$. Now suppose

$$
h(\zeta)=\left(1-s \zeta_{1}\right)^{-1} f(\zeta)
$$

for some

$$
f(\zeta)=\sum_{k=0}^{\infty} a_{k} \zeta_{1}^{k}
$$

Using the polar decomposition of $d A$, we see that

$$
\begin{aligned}
\left\|\zeta_{2}\left(1-s \zeta_{1}\right)^{-1} f\right\|_{H_{n}^{2}}^{2} & =\left\|W_{e_{2}} \zeta_{2} h\right\|_{B^{(0)}}^{2}=\int_{D}\left|\sum_{k=0}^{\infty} c_{k} z^{k}\right|^{2} d A(z)=\int_{D}\left|\frac{1}{1-s z} \sum_{k=0}^{\infty} a_{k} z^{k}\right|^{2} d A(z) \\
& =2 \int_{0}^{1} r \int_{T}\left|\frac{1}{1-s r \tau} \sum_{k=0}^{\infty} a_{k}(r \tau)^{k}\right|^{2} d m(\tau) d r \\
& \leq 2 \int_{0}^{1} \frac{1}{(1-s r)^{2}} d r \sum_{k=0}^{\infty}\left|a_{k}\right|^{2}=2 \int_{0}^{1} \frac{1}{(1-s r)^{2}} d r\|f\|_{H_{n}^{2}}^{2} \\
& =\frac{2}{1-s}\|f\|_{H_{n}^{2}}^{2} .
\end{aligned}
$$

Thus we have shown that the norm of the restriction of the operator of multiplication by $\zeta_{2}(1-$ $\left.s \zeta_{1}\right)^{-1}$ to $H_{0}$ does not exceed $\sqrt{2}(1-s)^{-1 / 2}$.

Obviously, if $f_{\beta} \in H_{\beta}, f_{\beta^{\prime}} \in H_{\beta^{\prime}}$ and $\beta \neq \beta^{\prime}$, then

$$
\frac{\zeta_{2}}{1-s \zeta_{1}} f_{\beta} \perp \frac{\zeta_{2}}{1-s \zeta_{1}} f_{\beta^{\prime}}
$$

Thus it follows from the above two paragraphs that the norm of $M_{\zeta_{2} /\left(1-s \zeta_{1}\right)}$ on the entire $H_{n}^{2}$ does not exceed $2(1-s)^{-1 / 2}$. This completes the proof.

The proof of Theorem 1.1 involves Möbius transform. For each $z \in \mathbf{B} \backslash\{0\}$, let

$$
\begin{equation*}
\varphi_{z}(\zeta)=\frac{1}{1-\langle\zeta, z\rangle}\left\{z-\frac{\langle\zeta, z\rangle}{|z|^{2}} z-\left(1-|z|^{2}\right)^{1 / 2}\left(\zeta-\frac{\langle\zeta, z\rangle}{|z|^{2}} z\right)\right\} \tag{2.7}
\end{equation*}
$$

Then $\varphi_{z}$ is an involution, i.e., $\varphi_{z} \circ \varphi_{z}=\mathrm{id}$. Recall that

$$
\begin{equation*}
k_{z}(\zeta)=\frac{\left(1-|z|^{2}\right)^{1 / 2}}{1-\langle\zeta, z\rangle}, \quad z, \zeta \in \mathbf{B} \tag{2.8}
\end{equation*}
$$

is the normalized reproducing kernel for $H_{n}^{2}$. Define the operator $U_{z}$ by the formula

$$
\begin{equation*}
\left(U_{z} f\right)(\zeta)=f\left(\varphi_{z}(\zeta)\right) k_{z}(\zeta), \quad f \in H_{n}^{2} \tag{2.9}
\end{equation*}
$$

for each $z \in \mathbf{B} \backslash\{0\}$. Using Theorem 2.2.2 in [14], it is straightforward to verify that

$$
\left\langle U_{z} k_{x}, U_{z} k_{y}\right\rangle=\frac{\left(1-|x|^{2}\right)^{1 / 2}\left(1-|y|^{2}\right)^{1 / 2}}{1-\langle y, x\rangle}=\left\langle k_{x}, k_{y}\right\rangle
$$

for all $z \in \mathbf{B} \backslash\{0\}$ and $x, y \in \mathbf{B}$. Therefore each $U_{z}$ is a unitary operator on $H_{n}^{2}$.
Recall the elementary fact that if $c$ is a complex number with $|c| \leq 1$ and if $0<t<1$, then

$$
\begin{equation*}
2|1-t c| \geq|1-c| \tag{2.10}
\end{equation*}
$$

This equality will be used frequently in the sequel.
Lemma 2.4. Let $z, w \in \mathbf{B}$ be such that $|z|=|w|$. Then

$$
\left\|M_{m_{w} m_{z}}\right\| \leq 48 \frac{1-|z|^{2}}{|1-\langle z, w\rangle|}
$$

Proof. If $z=w$, then the conclusion is a trivial consequence of Lemma 2.1. So let us assume $z \neq w$. Using the unitary operator defined by (2.9), we see that

$$
\left\|M_{m_{w} m_{z}}\right\|=\left\|M_{\left(m_{w} m_{z}\right) \circ \varphi_{z}}\right\| .
$$

Thus we only need to estimate the norm of $M_{\left(m_{w} m_{z}\right) \circ \varphi_{z}}$. By Theorem 2.2.2 in [14],

$$
1-\left\langle\varphi_{z}(\zeta), z\right\rangle=1-\left\langle\varphi_{z}(\zeta), \varphi_{z}(0)\right\rangle=\frac{1-|z|^{2}}{1-\langle\zeta, z\rangle}
$$

which leads to

$$
m_{z}\left(\varphi_{z}(\zeta)\right)=1-\langle\zeta, z\rangle
$$

Write $\lambda=\varphi_{z}(w)$. Then $w=\varphi_{z}(\lambda)$. Using the above-cited theorem,

$$
1-\left\langle\varphi_{z}(\zeta), w\right\rangle=1-\left\langle\varphi_{z}(\zeta), \varphi_{z}(\lambda)\right\rangle=\frac{\left(1-|z|^{2}\right)(1-\langle\zeta, \lambda\rangle)}{(1-\langle\zeta, z\rangle)(1-\langle z, \lambda\rangle)}
$$

Since $1-|w|^{2}=1-|z|^{2}$, this gives us

$$
\begin{equation*}
m_{w}\left(\varphi_{z}(\zeta)\right) m_{z}\left(\varphi_{z}(\zeta)\right)=(1-\langle z, \lambda\rangle) \frac{(1-\langle\zeta, z\rangle)^{2}}{1-\langle\zeta, \lambda\rangle} \tag{2.11}
\end{equation*}
$$

Since we know that

$$
1-\langle z, \lambda\rangle=1-\left\langle\varphi_{z}(0), \varphi_{z}(w)\right\rangle=\frac{1-|z|^{2}}{1-\langle z, w\rangle}
$$

we only need to consider the operator of multiplication by $F(\zeta)=(1-\langle\zeta, z\rangle)^{2} /(1-\langle\zeta, \lambda\rangle)$.
Write $s=|\lambda|=\left|\varphi_{z}(w)\right|$. Let $U: \mathbf{C}^{n} \rightarrow \mathbf{C}^{n}$ be a unitary transformation such that

$$
\begin{aligned}
U^{*} \lambda & =(s, 0,0, \ldots, 0) \quad \text { and } \\
U^{*} z & =(\bar{a}, \bar{b}, 0, \ldots, 0),
\end{aligned}
$$

where $\bar{a}=\langle z, \lambda / s\rangle$ and $|b|^{2}=|z|^{2}-|\langle z, \lambda / s\rangle|^{2}$. Since $1-|w|^{2}=1-|z|^{2}$, we have

$$
\begin{equation*}
2(1-s) \geq 1-s^{2}=1-\left|\varphi_{z}(w)\right|^{2}=\frac{\left(1-|z|^{2}\right)^{2}}{|1-\langle w, z\rangle|^{2}} \tag{2.12}
\end{equation*}
$$

Since

$$
1-s a=1-\langle\lambda, z\rangle=\frac{1-|z|^{2}}{1-\langle w, z\rangle}
$$

(2.10) gives us

$$
\begin{equation*}
|1-a| \leq 2|1-s a|=2 \frac{1-|z|^{2}}{|1-\langle w, z\rangle|} \tag{2.13}
\end{equation*}
$$

Also,

$$
\begin{equation*}
|b|^{2} \leq 1-|\langle z, \lambda\rangle|^{2} \leq 2(1-|\langle z, \lambda\rangle|) \leq 2 \frac{1-|z|^{2}}{|1-\langle z, w\rangle|} \tag{2.14}
\end{equation*}
$$

Since $\left\|M_{F}\right\|=\left\|M_{F \circ U}\right\|$, it suffices to estimate the latter. We have

$$
\begin{align*}
\frac{(1-\langle U \zeta, z\rangle)^{2}}{1-\langle U \zeta, \lambda\rangle} & =\frac{\left(1-a \zeta_{1}-b \zeta_{2}\right)^{2}}{1-s \zeta_{1}}=\frac{\left(1-a \zeta_{1}\right)^{2}}{1-s \zeta_{1}}-2 b \frac{\left(1-a \zeta_{1}\right) \zeta_{2}}{1-s \zeta_{1}}+\frac{b^{2} \zeta_{2}^{2}}{1-s \zeta_{1}} \\
& =G_{1}(\zeta)-2 b G_{2}(\zeta)+G_{3}(\zeta) \tag{2.15}
\end{align*}
$$

Write the first term in (2.15) as

$$
\begin{aligned}
G_{1}(\zeta) & =\frac{(1-a)^{2}}{1-s \zeta_{1}}+2 a(1-a) \frac{1-\zeta_{1}}{1-s \zeta_{1}}+a^{2} \frac{\left(1-\zeta_{1}\right)^{2}}{1-s \zeta_{1}} \\
& =G_{11}(\zeta)+2 a(1-a) G_{12}(\zeta)+a^{2} G_{13}(\zeta)
\end{aligned}
$$

By (2.10) and Lemma 2.1, we have $\left\|M_{G_{12}}\right\| \leq 2$. Similarly, $\left\|M_{G_{13}}\right\| \leq 4$. For $G_{11}$, Lemma 2.1 yields

$$
\left\|M_{G_{11}}\right\| \leq \frac{|1-a|^{2}}{1-s} \leq 8
$$

where the second $\leq$ follows from (2.12) and (2.13). Therefore we conclude that

$$
\begin{equation*}
\left\|M_{G_{1}}\right\| \leq 20 \tag{2.16}
\end{equation*}
$$

For the second term in (2.15), we have

$$
G_{2}(\zeta)=\frac{\left(1-a \zeta_{1}\right) \zeta_{2}}{1-s \zeta_{1}}=(1-a) \frac{\zeta_{2}}{1-s \zeta_{1}}+a \frac{\left(1-\zeta_{1}\right)}{1-s \zeta_{1}} \zeta_{2}=G_{21}(\zeta)+G_{22}(\zeta)
$$

By Lemma 2.3, (2.12) and (2.13),

$$
\left\|M_{G_{21}}\right\| \leq \frac{2|1-a|}{\sqrt{1-s}} \leq 4 \sqrt{2}<8
$$

By (2.10) and Lemma 2.1, $\left\|M_{G_{22}}\right\| \leq 2$. Therefore

$$
\begin{equation*}
\left\|M_{G_{2}}\right\| \leq 10 \tag{2.17}
\end{equation*}
$$

Since

$$
G_{3}(\zeta)=b^{2} \frac{\zeta_{2}}{1-s \zeta_{1}} \cdot \zeta_{2}
$$

by Lemma 2.3, (2.12) and (2.14),

$$
\begin{equation*}
\left\|M_{G_{3}}\right\| \leq \frac{2|b|^{2}}{\sqrt{1-s}} \leq 8 \tag{2.18}
\end{equation*}
$$

Combining (2.16),(2.17),(2.18), and (2.15), we now have $\left\|M_{F}\right\|=\left\|M_{F \circ U}\right\| \leq 48$. Recalling (2.11), the proof is complete.

Lemma 2.5. For every $z \in \mathbf{B}$ and every $i \in\{1, \ldots, n\}$, the norm of the operator of multiplication by the function

$$
\left(\zeta_{i}-z_{i}\right) m_{z}(\zeta)
$$

on $H_{n}^{2}$ does not exceed

$$
3 n \sqrt{1-|z|^{2}}
$$

where $z_{i}$ is the $i$-th component of $z$.
Proof. Let $z \in \mathbf{B}$ and $i \in\{1, \ldots, n\}$ be given, and write $G(\zeta)=\left(\zeta_{i}-z_{i}\right) m_{z}(\zeta)$. Let $\hat{z}=$ $(|z|, 0, \ldots, 0)$. Then there is a unitary operator $U: \mathbf{C}^{n} \rightarrow \mathbf{C}^{n}$ such that $U^{*} z=\hat{z}$. Since $\left\|M_{G}\right\|=$ $\left\|M_{G o U}\right\|$, it suffices to estimate the latter. We have

$$
\begin{align*}
G(U \zeta) & =\left((U \zeta)_{i}-z_{i}\right) m_{z}(U \zeta)=\left((U \zeta)_{i}-(U \hat{z})_{i}\right) m_{\hat{z}}(\zeta)=(U(\zeta-\hat{z}))_{i} \frac{1-|z|^{2}}{1-|z| \zeta_{1}} \\
& =\left(u_{i 1}\left(\zeta_{1}-|z|\right)+u_{i 2} \zeta_{2}+\cdots+u_{i n} \zeta_{n}\right) \frac{1-|z|^{2}}{1-|z| \zeta_{1}} \tag{2.19}
\end{align*}
$$

where $\sum_{k=1}^{n}\left|u_{i k}\right|^{2}=1$. By Lemma 2.1, the norm of the operator of multiplication by $\left(\zeta_{1}-\right.$ $|z|) /\left(1-|z| \zeta_{1}\right)$ does not exceed 1. By Lemma 2.3, for each $2 \leq j \leq n$, the norm of the operator of multiplication by $\zeta_{j} /\left(1-|z| \zeta_{1}\right)$ does not exceed $2(1-|z|)^{-1 / 2}$. Therefore

$$
\left\|M_{G \circ U}\right\| \leq\left(1-|z|^{2}\right)+(n-1)\left(1-|z|^{2}\right) \cdot \frac{2}{\sqrt{1-|z|}} \leq 3 n \sqrt{1-|z|^{2}}
$$

This completes the proof.
The next lemma will be needed in Section 6 when we deal with localization.
Lemma 2.6. For each $h \in H_{n}^{2}$, we have

$$
\lim _{|z| \uparrow 1}\left\|s_{z} h\right\|=0
$$

where $s_{z}$ was defined in (1.2).
Proof. Write

$$
b_{r}(\zeta)=\frac{1-r}{1-r \zeta_{1}}
$$

for each $0 \leq r<1$. We first show that for each $h \in H_{n}^{2}$,

$$
\begin{equation*}
\lim _{r \uparrow 1}\left\|b_{r} h\right\|=0 \tag{2.20}
\end{equation*}
$$

For this, we use the orthogonal decomposition $H_{n}^{2}=\oplus_{\beta \in \mathcal{B}} H_{\beta}$ introduced at the beginning of the section. First consider any

$$
h_{0}(\zeta)=\sum_{k=0}^{\infty} c_{k} \zeta_{1}^{k}
$$

in $H_{0}$. Then

$$
\left\|b_{r} h_{0}\right\|^{2}=\int_{T}\left|\frac{1-r}{1-r \tau} \sum_{k=0}^{\infty} c_{k} \tau^{k}\right|^{2} d m(\tau)
$$

As $r \uparrow 1,(1-r) /(1-r \tau) \rightarrow 0$ for every $\tau \in T \backslash\{1\}$. Thus it follows from the dominated convergence theorem that

$$
\begin{equation*}
\lim _{r \uparrow 1}\left\|b_{r} h_{0}\right\|=0 . \tag{2.21}
\end{equation*}
$$

Next we consider an $h_{\beta} \in H_{\beta}$, where $\beta \in \mathcal{B} \backslash\{0\}$. Suppose that

$$
h_{\beta}(\zeta)=\sum_{k=0}^{\infty} a_{k} \zeta_{1}^{k} \zeta^{\beta}
$$

As we saw in the proof of Lemma 2.3,

$$
\left\|b_{r} h_{\beta}\right\|^{2}=\frac{\beta!}{(|\beta|-1)!} \int_{D}\left|\frac{1-r}{1-r z} \sum_{k=0}^{\infty} a_{k} z^{k}\right|^{2}\left(1-|z|^{2}\right)^{|\beta|-1} d A(z)
$$

As $r \uparrow 1,(1-r) /(1-r z) \rightarrow 0$ for every $z \in D$. Thus it follows from the dominated convergence theorem that

$$
\begin{equation*}
\lim _{r \uparrow 1}\left\|b_{r} h_{\beta}\right\|=0 . \tag{2.22}
\end{equation*}
$$

For each $\beta \in \mathcal{B}, b_{r} H_{\beta} \subset H_{\beta}$. Therefore (2.20) follows from (2.21) and (2.22).

Recall that we denote the collection of unitary transformations on $\mathbf{C}^{n}$ by $\mathcal{U}$. For each $h \in H_{n}^{2}$, the collection of vectors $\{h \circ U: U \in \mathcal{U}\}$ is a compact subset of $H_{n}^{2}$. Therefore (2.20) implies that

$$
\begin{equation*}
\lim _{r \uparrow 1} \sup _{U \in \mathcal{U}}\left\|b_{r} \cdot h \circ U\right\|=0 . \tag{2.23}
\end{equation*}
$$

For each $z \in \mathbf{B}$, there is a $V_{z} \in \mathcal{U}$ such that $V_{z}^{*} z=(|z|, 0, \ldots, 0)$. Hence

$$
\left\|s_{z} h\right\|=\left\|\left(s_{z} h\right) \circ V_{z}\right\|=\left\|s_{V_{z}^{*} z} \cdot h \circ V_{z}\right\|=\left\|b_{|z|} \cdot h \circ V_{z}\right\| .
$$

The lemma obviously follows from this identity and (2.23).

## 3. A Quasi-Resolution of the Identity Operator

Let $N$ be an integer greater than or equal to $n / 2$. For each $z \in \mathbf{B}$, define the function

$$
\psi_{z, N}(\zeta)=\frac{\left(1-|z|^{2}\right)^{(1 / 2)+N}}{(1-\langle\zeta, z\rangle)^{1+N}}
$$

Then we have the relation

$$
\psi_{z, N}=m_{z}^{N} k_{z},
$$

where $m_{z}$ and $k_{z}$ were given by (2.6) and (2.8) respectively. In this sense $\psi_{z, N}$ is a modified version of $k_{z}$. The main difference between these two functions is that $\psi_{z, N}$ "decays much faster". The reader will clearly see the meaning of this statement in the subsequent proofs.

Let $d \lambda$ be the Möbius invariant measure on $\mathbf{B}$. That is,

$$
d \lambda(z)=\frac{d v(z)}{\left(1-|z|^{2}\right)^{n+1}},
$$

where $d v$ is the volume measure on $\mathbf{B}$ with the normalization $v(\mathbf{B})=1$. Let $d \sigma$ be the positive, regular Borel measure on the unit sphere $S$ which is invariant under the orthogonal group $O(2 n)$, i.e., the group of isometries on $\mathbf{C}^{n} \cong \mathbf{R}^{2 n}$ which fix 0 . We normalize $\sigma$ such that $\sigma(S)=1$.

Theorem 3.1. Let $N$ be an integer greater than or equal to $n / 2$. Then the self-adjoint operator

$$
R_{N}=\int \psi_{z, N} \otimes \psi_{z, N} d \lambda(z)
$$

is both bounded and invertible on the Drury-Arveson space $H_{n}^{2}$. In other words, there exist constants $0<a(N) \leq b(N)<\infty$ which only depend on $N$ and the complex dimension $n$ such that

$$
a(N) \leq R_{N} \leq b(N)
$$

on $H_{n}^{2}$.
Proof. For each $z \in \mathbf{B}$, define the function $g_{z} \in H_{n}^{2}$ by the formula

$$
g_{z}(\zeta)=\langle\zeta, z\rangle .
$$

Write $C_{k}^{m}$ for the binomial coefficient $m!/(k!(m-k)!)$ as usual. Then

$$
\psi_{z, N}=\left(1-|z|^{2}\right)^{(1 / 2)+N} \sum_{k=0}^{\infty} C_{k}^{k+N} g_{z}^{k}
$$

and consequently

$$
\psi_{z, N} \otimes \psi_{z, N}=\left(1-|z|^{2}\right)^{1+2 N} \sum_{j, k=0}^{\infty} C_{k}^{k+N} C_{j}^{j+N} g_{z}^{k} \otimes g_{z}^{j}
$$

For each $0<\rho<1$, define $\mathbf{B}_{\rho}=\{z:|z|<\rho\}$. Since both $d \lambda$ and $\mathbf{B}_{\rho}$ are invariant under the substitution $z \rightarrow e^{i \theta} z, \theta \in \mathbf{R}$, we have

$$
\begin{aligned}
\int_{\mathbf{B}_{\rho}}\left(1-|z|^{2}\right)^{1+2 N} g_{z}^{k} \otimes g_{z}^{j} d \lambda(z) & =\int_{\mathbf{B}_{\rho}}\left(1-\left|e^{i \theta} z\right|^{2}\right)^{1+2 N} g_{e^{i \theta} z}^{k} \otimes g_{e^{i \theta} z}^{j} d \lambda(z) \\
& =e^{i(j-k) \theta} \int_{\mathbf{B}_{\rho}}\left(1-|z|^{2}\right)^{1+2 N} g_{z}^{k} \otimes g_{z}^{j} d \lambda(z)
\end{aligned}
$$

This implies that

$$
\int_{\mathbf{B}_{\rho}}\left(1-|z|^{2}\right)^{1+2 N} g_{z}^{k} \otimes g_{z}^{j} d \lambda(z)=0 \quad \text { if } \quad k \neq j
$$

Therefore

$$
\int_{\mathbf{B}_{\rho}} \psi_{z, N} \otimes \psi_{z, N} d \lambda(z)=\sum_{k=0}^{\infty}\left(C_{k}^{k+N}\right)^{2} \int_{\mathbf{B}_{\rho}}\left(1-|z|^{2}\right)^{1+2 N} g_{z}^{k} \otimes g_{z}^{k} d \lambda(z)
$$

Since

$$
g_{z}^{k}(\zeta)=\langle\zeta, z\rangle^{k}=\sum_{|\alpha|=k} \frac{k!}{\alpha!} \bar{z}^{\alpha} \zeta^{\alpha},
$$

we have

$$
g_{z}^{k} \otimes g_{z}^{k}=\sum_{|\alpha|=|\beta|=k} \frac{(k!)^{2}}{\alpha!\beta!} \bar{z}^{\alpha} z^{\beta} \zeta^{\alpha} \otimes \zeta^{\beta} .
$$

By the radial-spherical decomposition of $d \lambda$, it is obvious that

$$
\int_{\mathbf{B}_{\rho}}\left(1-|z|^{2}\right)^{1+2 N} \bar{z}^{\alpha} z^{\beta} d \lambda(z)=0 \quad \text { if } \quad \alpha \neq \beta
$$

Therefore

$$
\int_{\mathbf{B}_{\rho}}\left(1-|z|^{2}\right)^{1+2 N} g_{z}^{k} \otimes g_{z}^{k} d \lambda(z)=\sum_{|\alpha|=k} \frac{(k!)^{2}}{(\alpha!)^{2}} \int_{\mathbf{B}_{\rho}}\left(1-|z|^{2}\right)^{1+2 N}\left|z^{\alpha}\right|^{2} d \lambda(z) \zeta^{\alpha} \otimes \zeta^{\alpha}
$$

Consequently

$$
\begin{equation*}
\int_{\mathbf{B}_{\rho}} \psi_{z, N} \otimes \psi_{z, N} d \lambda(z)=\sum_{k=0}^{\infty}\left(C_{k}^{k+N}\right)^{2} \sum_{|\alpha|=k} \frac{(k!)^{2}}{(\alpha!)^{2}} \int_{\mathbf{B}_{\rho}}\left(1-|z|^{2}\right)^{1+2 N}\left|z^{\alpha}\right|^{2} d \lambda(z) \zeta^{\alpha} \otimes \zeta^{\alpha} \tag{3.1}
\end{equation*}
$$

Notice that if $|\alpha|=k$, then

$$
\begin{align*}
\int_{\mathbf{B}_{\rho}}\left(1-|z|^{2}\right)^{1+2 N}\left|z^{\alpha}\right|^{2} d \lambda(z) & =\int_{\mathbf{B}_{\rho}}\left(1-|z|^{2}\right)^{2 N-n}\left|z^{\alpha}\right|^{2} d v(z) \\
& =\int_{0}^{\rho}\left(1-r^{2}\right)^{2 N-n} 2 n r^{2 k+2 n-1} d r \int_{S}\left|\xi^{\alpha}\right|^{2} d \sigma(\xi) \\
& =\int_{0}^{\rho}\left(1-r^{2}\right)^{2 N-n} 2 n r^{2 k+2 n-1} d r \frac{(n-1)!\alpha!}{(n-1+k)!} \tag{3.2}
\end{align*}
$$

where the third $=$ follows from Proposition 1.4.9 in [14]. Since $2 N-n \geq 0$, we can integrate by parts to obtain

$$
2 \int_{0}^{1}\left(1-r^{2}\right)^{2 N-n} r^{2 k+2 n-1} d r=\int_{0}^{1}(1-x)^{2 N-n} x^{n-1+k} d x=\frac{(2 N-n)!(n-1+k)!}{(2 N+k)!} .
$$

Letting $\rho \uparrow 1$ in (3.1) and (3.2), we see that

$$
\begin{equation*}
\int \psi_{z, N} \otimes \psi_{z, N} d \lambda(z)=\sum_{k=0}^{\infty} b_{k, N} \sum_{|\alpha|=k} \frac{k!}{\alpha!} \zeta^{\alpha} \otimes \zeta^{\alpha}, \tag{3.3}
\end{equation*}
$$

where

$$
b_{k, N}=\left(C_{k}^{k+N}\right)^{2} k!\frac{(2 N-n)!n!}{(2 N+k)!}=\frac{(2 N-n)!n!}{(N!)^{2}} \cdot \frac{\{(k+N)!\}^{2}}{k!(2 N+k)!}
$$

Using Stirling's formula, it is straightforward to verify that there exist $0<a(N) \leq b(N)<\infty$ which depend only on $N$ and $n$ such that

$$
\begin{equation*}
a(N) \leq b_{k, N} \leq b(N) \tag{3.4}
\end{equation*}
$$

for every $k \geq 0$. Since we can write the identity operator on $H_{n}^{2}$ as

$$
1=\sum_{k=0}^{\infty} \sum_{|\alpha|=k} \frac{k!}{\alpha!} \zeta^{\alpha} \otimes \zeta^{\alpha}
$$

the lemma follows from (3.3) and (3.4).

## 4. Three Lemmas

It is well known that the formula

$$
d(x, y)=|1-\langle x, y\rangle|^{1 / 2}, \quad x, y \in S
$$

defines a metric on the unit sphere $S$ [14]. Throughout the paper, we write

$$
B(x, r)=\left\{y \in S:|1-\langle x, y\rangle|^{1 / 2}<r\right\}
$$

for $x \in S$ and $r>0$. By Proposition 5.1.4 in [14], there is a constant $A_{0} \in\left(2^{-n}, \infty\right)$ such that

$$
\begin{equation*}
2^{-n} r^{2 n} \leq \sigma(B(x, r)) \leq A_{0} r^{2 n} \tag{4.1}
\end{equation*}
$$

for all $x \in S$ and $0<r \leq \sqrt{2}$. Note that the upper bound actually holds for all $r>0$.
Before getting to the main estimates of the section, let us recall:
Lemma 4.1 (Lemma 4.1 in [15]). Let $X$ be a set and let $E$ be a subset of $X \times X$. Suppose that $m$ is a natural number such that

$$
\operatorname{card}\{y \in X:(x, y) \in E\} \leq m \quad \text { and } \quad \operatorname{card}\{y \in X:(y, x) \in E\} \leq m
$$

for every $x \in X$. Then there exist pairwise disjoint subsets $E_{1}, E_{2}, \ldots, E_{2 m}$ of $E$ such that

$$
E=E_{1} \cup E_{2} \cup \ldots \cup E_{2 m}
$$

and such that for each $1 \leq j \leq 2 m$, the conditions $(x, y),\left(x^{\prime}, y^{\prime}\right) \in E_{j}$ and $(x, y) \neq\left(x^{\prime}, y^{\prime}\right)$ imply both $x \neq x^{\prime}$ and $y \neq y^{\prime}$.

For each $z \in \mathbf{B}$, define the functions

$$
\begin{equation*}
u_{z}(\zeta)=m_{z}^{n+3}(\zeta)=\left(\frac{1-|z|^{2}}{1-\langle\zeta, z\rangle}\right)^{n+3} \quad \text { and } \quad v_{z}(\zeta)=m_{z}^{n+4}(\zeta)=\left(\frac{1-|z|^{2}}{1-\langle\zeta, z\rangle}\right)^{n+4} \tag{4.2}
\end{equation*}
$$

The proofs of our next three lemmas have much in common. More specifically, they all use a counting argument based on Lemma 4.1. However, because the estimates involved vary in details,
it is difficult to reduce them to one. Therefore we present all three proofs.
It should be pretty clear from Lemma 2.1 that $\left\|M_{s_{z}}\right\|=1$ for each $z \in \mathbf{B}$. Therefore $\left\|M_{m_{z}}\right\|=$ $1+|z|$. This fact will be used several times in this section.
Lemma 4.2. Let $2 n<p<\infty$. Then there is a $C_{4.2}(p)$ which depends only on $p$ and $n$ such that the following estimate holds: Suppose that $0<t<1$ and that $\left\{\xi_{j}: j \in J\right\}$ is a subset of $S$ satisfying the condition

$$
\begin{equation*}
B\left(\xi_{i}, t\right) \cap B\left(\xi_{j}, t\right)=\emptyset \quad \text { for all } i \neq j \tag{4.3}
\end{equation*}
$$

Define $z_{j}=\left(1-t^{2}\right)^{1 / 2} \xi_{j}, j \in J$. Let $\left\{f_{j}: j \in J\right\}$ be a set of vectors in $H_{n}^{2}$ with norm at most 1 , and let $\left\{e_{j}: j \in J\right\}$ be an orthonormal set. For each $\nu \in\{1, \ldots, n\}$, define the operator

$$
E_{\nu}=\sum_{j \in J}\left(M_{\zeta_{\nu}-\left(z_{j}\right)_{\nu}}^{*} v_{z_{j}} f_{j}\right) \otimes e_{j}
$$

where $\left(z_{j}\right)_{\nu}$ denotes the $\nu$-th component of $z_{j}$. Then $\left\|E_{\nu}\right\|_{p} \leq C_{4.2}(p) t^{1-(2 n / p)}$.
Proof. Let $\nu \in\{1, \ldots, n\}$ be given. By Lemma 2.1, $M_{\zeta_{\nu}}$ has a normal extension. More precisely, there is a Hilbert space $L_{\nu}$ containing $H_{n}^{2}$ and a normal operator $M_{\nu}$ on $L_{\nu}$ such that

$$
\begin{equation*}
M_{\nu} h=M_{\zeta_{\nu}} h, \quad \text { for each } h \in H_{n}^{2} \tag{4.4}
\end{equation*}
$$

Let $P_{\nu}: L_{\nu} \rightarrow H_{n}^{2}$ be the orthogonal projection. Define the operator

$$
\tilde{E}_{\nu}=\sum_{j \in J}\left\{\left(M_{\nu}^{*}-\overline{\left(z_{j}\right)_{\nu}}\right) v_{z_{j}} f_{j}\right\} \otimes e_{j}
$$

Since $M_{\zeta_{\nu}}^{*}=P_{\nu} M_{\nu}^{*} \mid H_{n}^{2}$, we have

$$
E_{\nu}=P_{\nu} \tilde{E}_{\nu}
$$

Thus it suffices to estimate $\left\|\tilde{E}_{\nu}\right\|_{p}$.
For the convenience of the reader, we will denote the inner product and the norm on $L_{\nu}$ by $\langle\cdot, \cdot\rangle_{L_{\nu}}$ and $\|\cdot\|_{L_{\nu}}$ respectively, whereas those on the subspace $H_{n}^{2}$ will still be denoted by $\langle\cdot, \cdot\rangle$ and $\|\cdot\|$. We have

$$
\begin{equation*}
\tilde{E}_{\nu}^{*} \tilde{E}_{\nu}=\sum_{i, j \in J}\left\langle\left(M_{\nu}^{*}-\overline{\left(z_{j}\right)_{\nu}}\right) v_{z_{j}} f_{j},\left(M_{\nu}^{*}-\overline{\left(z_{i}\right)_{\nu}}\right) v_{z_{i}} f_{i}\right\rangle_{L_{\nu}} e_{i} \otimes e_{j}=B+\sum_{k=0}^{\infty} Y_{k}, \tag{4.5}
\end{equation*}
$$

where

$$
B=\sum_{j \in J}\left\|\left(M_{\nu}^{*}-\overline{\left(z_{j}\right)_{\nu}}\right) v_{z_{j}} f_{j}\right\|_{L_{\nu}}^{2} e_{j} \otimes e_{j}
$$

and

$$
Y_{k}=\sum_{2^{k} t \leq d\left(\xi_{i}, \xi_{j}\right)<2^{k+1} t}\left\langle\left(M_{\nu}^{*}-\overline{\left(z_{j}\right)_{\nu}}\right) v_{z_{j}} f_{j},\left(M_{\nu}^{*}-\overline{\left(z_{i}\right)_{\nu}}\right) v_{z_{i}} f_{i}\right\rangle_{L_{\nu}} e_{i} \otimes e_{j}
$$

$k \in \mathbf{Z}_{+}$. Next we estimate $\|B\|_{p / 2}$ and $\left\|Y_{k}\right\|_{p / 2}$.
For $\|B\|_{p / 2}$, note that by the normality of $M_{\nu}$ and (4.4), we have

$$
\begin{aligned}
\left\|\left(M_{\nu}^{*}-\overline{\left(z_{j}\right)_{\nu}}\right) v_{z_{j}} f_{j}\right\|_{L_{\nu}} & =\left\|\left(M_{\nu}-\left(z_{j}\right)_{\nu}\right) v_{z_{j}} f_{j}\right\|_{L_{\nu}}=\left\|M_{\zeta_{\nu}-\left(z_{j}\right)_{\nu}} v_{z_{j}} f_{j}\right\| \\
& =\left\|M_{\zeta_{\nu}-\left(z_{j}\right)_{\nu}} M_{m_{z_{j}}} u_{z_{j}} f_{j}\right\| \leq 2^{n+3}\left\|M_{\left(\zeta_{\nu}-\left(z_{j}\right)_{\nu}\right) m_{z_{j}}}\right\| .
\end{aligned}
$$

Applying Lemma 2.5, the above yields

$$
\left\|\left(M_{\nu}^{*}-\overline{\left(z_{j}\right)_{\nu}}\right) v_{z_{j}} f_{j}\right\|_{L_{\nu}} \leq 2^{n+3} 3 n \sqrt{1-\left|z_{j}\right|^{2}}=2^{n+3} 3 n t
$$

By (4.1) and (4.3), $\operatorname{card}(J) \leq 2^{n} t^{-2 n}$. Therefore

$$
\|B\|_{p / 2}^{p / 2}=\sum_{j \in J}\left\|\left(M_{\nu}^{*}-\overline{\left(z_{j}\right)_{\nu}}\right) v_{z_{j}} f_{j}\right\|_{L_{\nu}}^{p} \leq\left(2^{n+3} 3 n t\right)^{p} \cdot \operatorname{card}(J) \leq\left(2^{n+3} 3 n\right)^{p} 2^{n} t^{p-2 n}
$$

If we set $C=\left(2^{n+3} 3 n\right)^{2} 2^{2 n / p}$, then

$$
\begin{equation*}
\|B\|_{p / 2} \leq\left(\left(2^{n+3} 3 n\right)^{p} 2^{n} t^{p-2 n}\right)^{2 / p}=C t^{2(1-(2 n / p))} \tag{4.6}
\end{equation*}
$$

For $\left\|Y_{k}\right\|_{p / 2}$, note that by the normality of $M_{\nu}$ and (4.4), we have

$$
\begin{aligned}
\left\langle\left(M_{\nu}^{*}-\overline{\left(z_{j}\right)_{\nu}}\right) v_{z_{j}} f_{j}\right. & \left.,\left(M_{\nu}^{*}-\overline{\left(z_{i}\right)_{\nu}}\right) v_{z_{i}} f_{i}\right\rangle_{L_{\nu}} \\
& =\left\langle\left(M_{\nu}-\left(z_{i}\right)_{\nu}\right) v_{z_{j}} f_{j},\left(M_{\nu}-\left(z_{j}\right)_{\nu}\right) v_{z_{i}} f_{i}\right\rangle_{L_{\nu}} \\
& =\left\langle M_{\zeta_{\nu}-\left(z_{i}\right)_{\nu}} v_{z_{j}} f_{j}, M_{\zeta_{\nu}-\left(z_{j}\right)_{\nu}} v_{z_{i}} f_{i}\right\rangle \\
& =\left\langle M_{\zeta_{\nu}-\left(z_{i}\right)_{\nu}} v_{z_{j}} f_{j}, M_{u_{z_{i}}} M_{\zeta_{\nu}-\left(z_{j}\right)_{\nu}} m_{z_{i}} f_{i}\right\rangle \\
& =\left\langle M_{u_{z_{i}}}^{*} M_{\zeta_{\nu}-\left(z_{i}\right)_{\nu}} v_{z_{j}} f_{j}, M_{\zeta_{\nu}-\left(z_{j}\right)_{\nu}} m_{z_{i}} f_{i}\right\rangle .
\end{aligned}
$$

By the Cauchy-Schwarz inequality,

$$
\begin{equation*}
\mid\left\langle\left(M_{\nu}^{*}-\overline{\left(z_{j}\right)_{\nu}}\right) v_{z_{j}} f_{j},\left(M_{\nu}^{*}-\overline{\left.\left(z_{i}\right)_{\nu}\right)} v_{z_{i}} f_{i}\right\rangle_{L_{\nu}}\right| \leq\left\|M_{u_{z_{i}}}^{*} M_{\zeta_{\nu}-\left(z_{i}\right)_{\nu}} v_{z_{j}} f_{j}\right\|\left\|M_{\zeta_{\nu}-\left(z_{j}\right)_{\nu}} m_{z_{i}} f_{i}\right\| . \tag{4.7}
\end{equation*}
$$

The two norms above need to be estimated separately, which is the most subtle part of the proof.
For the first norm in (4.7), we use Corollary 2.2. Since $M_{u_{z_{i}}}$ is subnormal, it is hyponormal. Therefore

$$
\begin{aligned}
\left\|M_{u_{z_{i}}}^{*} M_{\zeta_{\nu}-\left(z_{i}\right)_{\nu}} v_{z_{j}} f_{j}\right\| & \leq\left\|M_{u_{z_{i}}} M_{\zeta_{\nu}-\left(z_{i}\right)_{\nu}} v_{z_{j}} f_{j}\right\| \\
& =\left\|M_{m_{z_{i}} m_{z_{j}}}^{n+2} M_{\left(\zeta_{\nu}-\left(z_{i}\right)_{\nu}\right) m_{z_{i}}} m_{z_{j}}^{2} f_{j}\right\| \\
& \leq 4\left\|M_{m_{z_{i}} m_{z_{j}}}\right\|^{n+2}\left\|M_{\left(\zeta_{\nu}-\left(z_{i}\right)_{\nu}\right) m_{z_{i}}}\right\| .
\end{aligned}
$$

Applying Lemma 2.4 to the first factor and Lemma 2.5 to the second factor, we have

$$
\begin{equation*}
\left\|M_{u_{z_{i}}}^{*} M_{\zeta_{\nu}-\left(z_{i}\right)_{\nu}} v_{z_{j}} f_{j}\right\| \leq 4(48)^{n+2}\left(\frac{t^{2}}{\left|1-\left\langle z_{i}, z_{j}\right\rangle\right|}\right)^{n+2} \cdot 3 n t \leq 12 n(96)^{n+2}\left(\frac{t^{2}}{\left|1-\left\langle\xi_{i}, \xi_{j}\right\rangle\right|}\right)^{n+2} t \tag{4.8}
\end{equation*}
$$

For the second norm in (4.7), we use Lemma 2.5 again:

$$
\begin{align*}
\left\|M_{\zeta_{\nu}-\left(z_{j}\right)_{\nu}} m_{z_{i}} f_{i}\right\| & \leq\left\|M_{\left(\zeta_{\nu}-\left(z_{i}\right)_{\nu}\right) m_{z_{i}}} f_{i}\right\|+\left\|M_{\left(\left(z_{i}\right)_{\nu}-\left(z_{j}\right)_{\nu}\right) m_{z_{i}}} f_{i}\right\| \\
& \leq 3 n \sqrt{1-\left|z_{i}\right|^{2}}+\left|\left(z_{i}\right)_{\nu}-\left(z_{j}\right)_{\nu}\right|\left\|M_{m_{z_{i}}}\right\| \\
& \leq 3 n t+2\left|z_{i}-z_{j}\right| \leq 3 n t+4\left|1-\left\langle\xi_{i}, \xi_{j}\right\rangle\right|^{1 / 2} \\
& =3 n t+4 \frac{\left|1-\left\langle\xi_{i}, \xi_{j}\right\rangle\right|^{1 / 2}}{t} \cdot t . \tag{4.9}
\end{align*}
$$

Bringing (4.8) and (4.9) into (4.7), we obtain

$$
\begin{aligned}
& \left|\left\langle\left(M_{\nu}^{*}-\overline{\left(z_{j}\right)_{\nu}}\right) v_{z_{j}} f_{j},\left(M_{\nu}^{*}-\overline{\left(z_{i}\right)_{\nu}}\right) v_{z_{i}} f_{i}\right\rangle_{L_{\nu}}\right| \\
& \leq C_{1}\left\{\left(\frac{t^{2}}{\left|1-\left\langle\xi_{i}, \xi_{j}\right\rangle\right|}\right)^{n+2}+\left(\frac{t^{2}}{\left|1-\left\langle\xi_{i}, \xi_{j}\right\rangle\right|}\right)^{n+(3 / 2)}\right\} t^{2}
\end{aligned}
$$

where $C_{1}=48 n^{2}(96)^{n+2}$. For any pair of $i, j$ such that $d\left(\xi_{i}, \xi_{j}\right) \geq 2^{k} t$, the above gives us

$$
\begin{equation*}
\left|\left\langle\left(M_{\nu}^{*}-\overline{\left(z_{j}\right)_{\nu}}\right) v_{z_{j}} f_{j},\left(M_{\nu}^{*}-\overline{\left(z_{i}\right)_{\nu}}\right) v_{z_{i}} f_{i}\right\rangle_{L_{\nu}}\right| \leq \frac{2 C_{1}}{2^{2 k(n+(3 / 2))}} t^{2} \tag{4.10}
\end{equation*}
$$

For each $i \in J$, if $d\left(\xi_{i}, \xi_{j}\right)<2^{k+1} t$, then $B\left(\xi_{j}, t\right) \subset B\left(\xi_{i}, 2^{k+2} t\right)$. By (4.3) and the fact that $\sigma(B(x, t))=\sigma(B(y, t))$ for all $x, y \in S$, for each $i \in J$ we have

$$
\begin{equation*}
\operatorname{card}\left\{j \in J: d\left(\xi_{i}, \xi_{j}\right)<2^{k+1} t\right\} \leq \frac{\sigma\left(B\left(\xi_{i}, 2^{k+2} t\right)\right)}{\sigma\left(B\left(\xi_{i}, t\right)\right)} \leq \frac{A_{0}\left(2^{k+2} t\right)^{2 n}}{2^{-n} t^{2 n}}=C_{2} 2^{2 n k} \tag{4.11}
\end{equation*}
$$

where $A_{0}$ is the constant that appears in (4.1) and $C_{2}=2^{5 n} A_{0}$. Set

$$
\begin{equation*}
\ell(k)=\min \left\{\ell \in \mathbf{N}: \ell \geq C_{2} 2^{2 n k}\right\} \tag{4.12}
\end{equation*}
$$

According to Lemma 4.1, we can decompose

$$
\mathcal{E}^{(k)}=\left\{(i, j) \in J \times J: 2^{k} t \leq d\left(\xi_{i}, \xi_{j}\right)<2^{k+1} t\right\}
$$

as the union of pairwise disjoint subsets

$$
\mathcal{E}_{1}^{(k)}, \ldots, \mathcal{E}_{2 \ell(k)}^{(k)}
$$

such that for each $m \in\{1, \ldots, 2 \ell(k)\}$, if $(i, j),\left(i^{\prime}, j^{\prime}\right) \in \mathcal{E}_{m}^{(k)}$ and if $(i, j) \neq\left(i^{\prime}, j^{\prime}\right)$, then we have both $i \neq i^{\prime}$ and $j \neq j^{\prime}$. This decomposition of $\mathcal{E}^{(k)}$ allows us to write

$$
\begin{equation*}
Y_{k}=Y_{k, 1}+\ldots+Y_{k, 2 \ell(k)}, \tag{4.13}
\end{equation*}
$$

where

$$
Y_{k, m}=\sum_{(i, j) \in \mathcal{E}_{m}^{(k)}}\left\langle\left(M_{\nu}^{*}-\overline{\left(z_{j}\right)_{\nu}}\right) v_{z_{j}} f_{j},\left(M_{\nu}^{*}-\overline{\left(z_{i}\right)_{\nu}}\right) v_{z_{i}} f_{i}\right\rangle_{L_{\nu}} e_{i} \otimes e_{j},
$$

$1 \leq m \leq 2 \ell(k)$.
The property of $\mathcal{E}_{m}^{(k)}$ simply means that the projection onto the first component, $(i, j) \mapsto i$, is injective on $\mathcal{E}_{m}^{(k)}$. Similarly, the projection onto the second component, $(i, j) \mapsto j$, is also injective on each $\mathcal{E}_{m}^{(k)}$. Combining these injectivities with the fact that $\left\{e_{j}: j \in J\right\}$ is an orthonormal set and with (4.10), we obtain

$$
\begin{aligned}
\left\|Y_{k, m}\right\|_{p / 2}^{p / 2} & =\sum_{(i, j) \in \mathcal{E}_{m}^{(k)}}\left|\left\langle\left(M_{\nu}^{*}-\overline{\left(z_{j}\right)_{\nu}}\right) v_{z_{j}} f_{j},\left(M_{\nu}^{*}-\overline{\left(z_{i}\right)_{\nu}}\right) v_{z_{i}} f_{i}\right\rangle_{L_{\nu}}\right|^{p / 2} \\
& \leq\left(\frac{2 C_{1}}{2^{2 k(n+(3 / 2))}} t^{2}\right)^{p / 2} \cdot \operatorname{card}(J) \\
& \leq\left(\frac{2 C_{1}}{2^{2 k(n+(3 / 2))}}\right)^{p / 2} \cdot t^{p} \cdot 2^{n} t^{-2 n}=\frac{C_{3}}{2^{p k(n+(3 / 2))}} t^{p-2 n}
\end{aligned}
$$

where $C_{3}=\left(2 C_{1}\right)^{p / 2} 2^{n}$. Setting $C_{4}=C_{3}^{2 / p}$, the above yields

$$
\left\|Y_{k, m}\right\|_{p / 2} \leq \frac{C_{4}}{2^{2 k(n+(3 / 2))}} t^{2(1-(2 n / p))}
$$

for each $m \in\{1, \ldots, 2 \ell(k)\}$. Recalling (4.12) and (4.13), we now have

$$
\left\|Y_{k}\right\|_{p / 2} \leq \frac{C_{4}}{2^{2 k(n+(3 / 2))}} t^{2(1-(2 n / p))} \cdot 2\left(1+C_{2} 2^{2 n k}\right) \leq \frac{2 C_{4}\left(1+C_{2}\right)}{2^{3 k}} t^{2(1-(2 n / p))}
$$

Combining this with (4.5) and (4.6), we see that

$$
\left\|\tilde{E}_{\nu}^{*} \tilde{E}_{\nu}\right\|_{p / 2} \leq\left(C+\sum_{k=0}^{\infty} \frac{2 C_{4}\left(1+C_{2}\right)}{2^{3 k}}\right) t^{2(1-(2 n / p))}
$$

Since $\left\|\tilde{E}_{\nu}\right\|_{p}=\left\|\tilde{E}_{\nu}^{*} \tilde{E}_{\nu}\right\|_{p / 2}^{1 / 2}$ and $\left\|E_{\nu}\right\|_{p} \leq\left\|\tilde{E}_{\nu}\right\|_{p}$, this completes the proof.
If we replace the operator $M_{\zeta_{\nu}-\left(z_{j}\right)_{\nu}}^{*}$ in the above lemma by $M_{\zeta_{\nu}-\left(z_{j}\right)_{\nu}}$, with an easier proof, we obtain the same type of estimate:
Lemma 4.3. Let $2 n<p<\infty$. Then there is a $C_{4.3}(p)$ which depends only on $p$ and $n$ such that the following estimate holds: Suppose that $0<t<1$ and that $\left\{\xi_{j}: j \in J\right\}$ is a subset of $S$ satisfying the condition

$$
\begin{equation*}
B\left(\xi_{i}, t\right) \cap B\left(\xi_{j}, t\right)=\emptyset \quad \text { for all } i \neq j \tag{4.14}
\end{equation*}
$$

Define $z_{j}=\left(1-t^{2}\right)^{1 / 2} \xi_{j}, j \in J$. Let $\left\{f_{j}: j \in J\right\}$ be a set of vectors in $H_{n}^{2}$ with norm at most 1 , and let $\left\{e_{j}: j \in J\right\}$ be an orthonormal set. For each $\nu \in\{1, \ldots, n\}$, define

$$
E_{\nu}=\sum_{j \in J}\left(M_{\zeta_{\nu}-\left(z_{j}\right)_{\nu}} v_{z_{j}} f_{j}\right) \otimes e_{j}
$$

where $\left(z_{j}\right)_{\nu}$ denotes the $\nu$-th component of $z_{j}$. Then $\left\|E_{\nu}\right\|_{p} \leq C_{4.3}(p) t^{1-(2 n / p)}$.
Proof. We have

$$
\begin{equation*}
E_{\nu}^{*} E_{\nu}=\sum_{i, j \in J}\left\langle M_{\zeta_{\nu}-\left(z_{j}\right)_{\nu}} v_{z_{j}} f_{j}, M_{\zeta_{\nu}-\left(z_{i}\right)_{\nu}} v_{z_{i}} f_{i}\right\rangle e_{i} \otimes e_{j}=B+\sum_{k=0}^{\infty} Y_{k}, \tag{4.15}
\end{equation*}
$$

where

$$
B=\sum_{j \in J}\left\|M_{\zeta_{\nu}-\left(z_{j}\right)_{\nu}} v_{z_{j}} f_{j}\right\|^{2} e_{j} \otimes e_{j}
$$

and

$$
Y_{k}=\sum_{2^{k} t \leq d\left(\xi_{i}, \xi_{j}\right)<2^{k+1} t}\left\langle M_{\zeta_{\nu}-\left(z_{j}\right)_{\nu}} v_{z_{j}} f_{j}, M_{\zeta_{\nu}-\left(z_{i}\right)_{\nu}} v_{z_{i}} f_{i}\right\rangle e_{i} \otimes e_{j},
$$

$k \in \mathbf{Z}_{+}$. As in the previous lemma, we need to estimate $\|B\|_{p / 2}$ and $\left\|Y_{k}\right\|_{p / 2}$.
For $\|B\|_{p / 2}$, by Lemma 2.5 we have

$$
\left\|M_{\zeta_{\nu}-\left(z_{j}\right)_{\nu}} v_{z_{j}} f_{j}\right\| \leq 2^{n+3}\left\|M_{\left(\zeta_{\nu}-\left(z_{j}\right)_{\nu}\right) m_{z_{j}}}\right\| \leq 2^{n+3} 3 n \sqrt{1-\left|z_{j}\right|^{2}}=2^{n+3} 3 n t
$$

By (4.1) and (4.14), $\operatorname{card}(J) \leq 2^{n} t^{-2 n}$. Therefore

$$
\|B\|_{p / 2}^{p / 2}=\sum_{j \in J}\left\|M_{\zeta_{\nu}-\left(z_{j}\right)_{\nu}} v_{z_{j}} f_{j}\right\|^{p} \leq\left(2^{n+3} 3 n t\right)^{p} \cdot \operatorname{card}(J) \leq\left(2^{n+3} 3 n\right)^{p} 2^{n} t^{p-2 n}
$$

Consequently,

$$
\begin{equation*}
\|B\|_{p / 2} \leq\left(\left(2^{n+3} 3 n\right)^{p} 2^{n} t^{p-2 n}\right)^{2 / p}=C t^{2(1-(2 n / p))} \tag{4.16}
\end{equation*}
$$

For $\left\|Y_{k}\right\|_{p / 2}$, note that

$$
\left\langle M_{\zeta_{\nu}-\left(z_{j}\right)_{\nu}} v_{z_{j}} f_{j}, M_{\zeta_{\nu}-\left(z_{i}\right)_{\nu}} v_{z_{i}} f_{i}\right\rangle=\left\langle M_{u_{z_{i}}}^{*} M_{\zeta_{\nu}-\left(z_{j}\right)_{\nu}} v_{z_{j}} f_{j}, M_{\zeta_{\nu}-\left(z_{i}\right)_{\nu}} m_{z_{i}} f_{i}\right\rangle
$$

By the Cauchy-Schwarz inequality,

$$
\begin{equation*}
\left|\left\langle M_{\zeta_{\nu}-\left(z_{j}\right)_{\nu}} v_{z_{j}} f_{j}, M_{\zeta_{\nu}-\left(z_{i}\right)_{\nu} v_{z_{i}}} f_{i}\right\rangle\right| \leq\left\|M_{u_{z_{i}}}^{*} M_{\zeta_{\nu}-\left(z_{j}\right)_{\nu}} v_{z_{j}} f_{j}\right\|\left\|M_{\zeta_{\nu}-\left(z_{i}\right)_{\nu}} m_{z_{i}} f_{i}\right\| . \tag{4.17}
\end{equation*}
$$

As before, we will estimate the two norms above separately.
For the first norm in (4.17), it follows from Corollary 2.2 that

$$
\begin{aligned}
\left\|M_{u_{z_{i}}}^{*} M_{\zeta_{\nu}-\left(z_{j}\right)_{\nu}} v_{z_{j}} f_{j}\right\| & \leq\left\|M_{u_{z_{i}}} M_{\zeta_{\nu}-\left(z_{j}\right)_{\nu}} v_{z_{j}} f_{j}\right\| \\
& =\left\|M_{m_{z_{i}} m_{z_{j}}}^{n+3} M_{\left(\zeta_{\nu}-\left(z_{j}\right)_{\nu}\right) m_{z_{j}}} f_{j}\right\| \\
& \leq\left\|M_{m_{z_{i}} m_{z_{j}}}\right\|^{n+3}\left\|M_{\left(\zeta_{\nu}-\left(z_{j}\right)_{\nu}\right) m_{z_{j}}}\right\| .
\end{aligned}
$$

Applying Lemma 2.4 to the first factor and Lemma 2.5 to the second factor, we have

$$
\begin{equation*}
\left\|M_{u_{z_{i}}}^{*} M_{\zeta_{\nu}-\left(z_{j}\right)_{\nu}} v_{z_{j}} f_{j}\right\| \leq(48)^{n+3}\left(\frac{t^{2}}{\left|1-\left\langle z_{i}, z_{j}\right\rangle\right|}\right)^{n+3} \cdot 3 n t \leq 3 n(96)^{n+3}\left(\frac{t^{2}}{\left|1-\left\langle\xi_{i}, \xi_{j}\right\rangle\right|}\right)^{n+3} t \tag{4.18}
\end{equation*}
$$

For the second norm in (4.17), we use Lemma 2.5 again:

$$
\begin{equation*}
\left\|M_{\zeta_{\nu}-\left(z_{i}\right)_{\nu}} m_{z_{i}} f_{i}\right\| \leq\left\|M_{\left(\zeta_{\nu}-\left(z_{i}\right)_{\nu}\right) m_{z_{i}}}\right\| \leq 3 n \sqrt{1-\left|z_{i}\right|^{2}}=3 n t . \tag{4.19}
\end{equation*}
$$

Bringing (4.18) and (4.19) into (4.17), we obtain

$$
\left|\left\langle M_{\zeta_{\nu}-\left(z_{j}\right)_{\nu}} v_{z_{j}} f_{j}, M_{\zeta_{\nu}-\left(z_{i}\right)_{\nu}} v_{z_{i}} f_{i}\right\rangle\right| \leq C_{1}\left(\frac{t^{2}}{\left|1-\left\langle\xi_{i}, \xi_{j}\right\rangle\right|}\right)^{n+3} t^{2}
$$

where $C_{1}=9 n^{2}(96)^{n+3}$. For any pair of $i, j$ such that $d\left(\xi_{i}, \xi_{j}\right) \geq 2^{k} t$, the above gives us

$$
\begin{equation*}
\left|\left\langle M_{\zeta_{\nu}-\left(z_{j}\right)_{\nu}} v_{z_{j}} f_{j}, M_{\zeta_{\nu}-\left(z_{i}\right)_{\nu}} v_{z_{i}} f_{i}\right\rangle\right| \leq \frac{C_{1}}{2^{2 k(n+3)}} t^{2} \tag{4.20}
\end{equation*}
$$

Set

$$
\ell(k)=\min \left\{\ell \in \mathbf{N}: \ell \geq C_{2} 2^{2 n k}\right\}
$$

where $C_{2}=2^{5 n} A_{0}$. Then, by (4.11),

$$
\begin{equation*}
\operatorname{card}\left\{j \in J: d\left(\xi_{i}, \xi_{j}\right)<2^{k+1} t\right\} \leq \ell(k) \tag{4.21}
\end{equation*}
$$

According to Lemma 4.1, we can decompose

$$
\mathcal{E}^{(k)}=\left\{(i, j) \in J \times J: 2^{k} t \leq d\left(\xi_{i}, \xi_{j}\right)<2^{k+1} t\right\}
$$

as the union of pairwise disjoint subsets

$$
\mathcal{E}_{1}^{(k)}, \ldots, \mathcal{E}_{2 \ell(k)}^{(k)}
$$

such that for each $m \in\{1, \ldots, 2 \ell(k)\}$, if $(i, j),\left(i^{\prime}, j^{\prime}\right) \in \mathcal{E}_{m}^{(k)}$ and if $(i, j) \neq\left(i^{\prime}, j^{\prime}\right)$, then we have both $i \neq i^{\prime}$ and $j \neq j^{\prime}$. This decomposition of $\mathcal{E}^{(k)}$ allows us to write

$$
\begin{equation*}
Y_{k}=Y_{k, 1}+\ldots+Y_{k, 2 \ell(k)} \tag{4.22}
\end{equation*}
$$

where

$$
Y_{k, m}=\sum_{(i, j) \in \mathcal{E}_{m}^{(k)}}\left\langle M_{\zeta_{\nu}-\left(z_{j}\right)_{\nu}} v_{z_{j}} f_{j}, M_{\zeta_{\nu}-\left(z_{i}\right)_{\nu}} v_{z_{i}} f_{i}\right\rangle e_{i} \otimes e_{j},
$$

$1 \leq m \leq 2 \ell(k)$.
By the property of $\mathcal{E}_{m}^{(k)}$ and (4.20), we have

$$
\begin{aligned}
\left\|Y_{k, m}\right\|_{p / 2}^{p / 2} & =\sum_{(i, j) \in \mathcal{E}_{m}^{(k)}}\left|\left\langle M_{\zeta_{\nu}-\left(z_{j}\right)_{\nu}} v_{z_{j}} f_{j}, M_{\zeta_{\nu}-\left(z_{i}\right)_{\nu}} v_{z_{i}} f_{i}\right\rangle\right|^{p / 2} \\
& \leq\left(\frac{C_{1}}{2^{2 k(n+3)}} t^{2}\right)^{p / 2} \cdot \operatorname{card}(J) \leq\left(\frac{C_{1}}{2^{2 k(n+3)}}\right)^{p / 2} \cdot t^{p} \cdot 2^{n} t^{-2 n}=\frac{C_{3}}{2^{p k(n+3)}} t^{p-2 n}
\end{aligned}
$$

where $C_{3}=C_{1}{ }^{p / 2} 2^{n}$. Setting $C_{4}=C_{3}^{2 / p}$, the above yields

$$
\left\|Y_{k, m}\right\|_{p / 2} \leq \frac{C_{4}}{2^{2 k(n+3)}} t^{2(1-(2 n / p))}
$$

for each $m \in\{1, \ldots, 2 \ell(k)\}$. Recalling (4.21) and (4.22), we now have

$$
\left\|Y_{k}\right\|_{p / 2} \leq \frac{C_{4}}{2^{2 k(n+3)}} t^{2(1-(2 n / p))} \cdot 2\left(1+C_{2} 2^{2 n k}\right) \leq \frac{2 C_{4}\left(1+C_{2}\right)}{2^{6 k}} t^{2(1-(2 n / p))}
$$

Combining this with (4.16) and (4.15), we see that

$$
\left\|E_{\nu}^{*} E_{\nu}\right\|_{p / 2} \leq\left(C+\sum_{k=0}^{\infty} \frac{2 C_{4}\left(1+C_{2}\right)}{2^{6 k}}\right) t^{2(1-(2 n / p))}
$$

Since $\left\|E_{\nu}\right\|_{p}=\left\|E_{\nu}^{*} E_{\nu}\right\|_{p / 2}^{1 / 2}$, this completes the proof.
The last lemma of this section is about operator norm.
Lemma 4.4. There is a $C_{4.4}$ which depends only on $n$ such that the following estimate holds: Suppose that $0<t<1$ and that $\left\{\xi_{j}: j \in J\right\}$ is a subset of $S$ satisfying the condition

$$
B\left(\xi_{i}, t\right) \cap B\left(\xi_{j}, t\right)=\emptyset \quad \text { for all } i \neq j
$$

Define $z_{j}=\left(1-t^{2}\right)^{1 / 2} \xi_{j}, j \in J$. Let $\left\{f_{j}: j \in J\right\}$ be a set of vectors in $H_{n}^{2}$ with norm at most 1 , and let $\left\{e_{j}: j \in J\right\}$ be an orthonormal set. Then the operator

$$
E=\sum_{j \in J}\left(v_{z_{j}} f_{j}\right) \otimes e_{j}
$$

satisfies the estimate $\|E\| \leq C_{4.4}$.
Proof. It suffices to estimate $\left\|E^{*} E\right\|$. We have

$$
\begin{equation*}
E^{*} E=\sum_{i, j \in J}\left\langle v_{z_{j}} f_{j}, v_{z_{i}} f_{i}\right\rangle e_{i} \otimes e_{j}=B+\sum_{k=0}^{\infty} Y_{k} \tag{4.23}
\end{equation*}
$$

where

$$
B=\sum_{j \in J}\left\|v_{z_{j}} f_{j}\right\|^{2} e_{j} \otimes e_{j}
$$

and

$$
Y_{k}=\sum_{2^{k} t \leq d\left(\xi_{i}, \xi_{j}\right)<2^{k+1} t}\left\langle v_{z_{j}} f_{j}, v_{z_{i}} f_{i}\right\rangle e_{i} \otimes e_{j},
$$

$k \in \mathbf{Z}_{+}$. By Lemma 2.1 and (4.2), $\left\|M_{v_{z}}\right\| \leq(1+|z|)^{n+4} \leq 2^{n+4}$ for each $z \in \mathbf{B}$. Since $\left\|f_{j}\right\| \leq 1$, we have $\left\|v_{z_{j}} f_{j}\right\| \leq 2^{n+4}, j \in J$. Since $\left\{e_{j}: j \in J\right\}$ is an orthonormal set, we conclude that

$$
\begin{equation*}
\|B\| \leq 4^{n+4} \tag{4.24}
\end{equation*}
$$

Next we estimate $\left\|Y_{k}\right\|$. For each $k \in \mathbf{Z}_{+}$, define

$$
\mathcal{E}^{(k)}=\left\{(i, j) \in J \times J: 2^{k} t \leq d\left(\xi_{i}, \xi_{j}\right)<2^{k+1} t\right\} .
$$

Now, since $\left\|f_{j}\right\| \leq 1$ and $\left\|f_{i}\right\| \leq 1$, from Corollary 2.2 we obtain

$$
\left|\left\langle v_{z_{j}} f_{j}, v_{z_{i}} f_{i}\right\rangle\right|=\left|\left\langle M_{v_{z_{i}}}^{*} M_{v_{z_{j}}} f_{j}, f_{i}\right\rangle\right| \leq\left\|M_{v_{z_{i}}}^{*} M_{v_{z_{j}}}\right\| \leq\left\|M_{v_{z_{i}}} M_{v_{z_{j}}}\right\|=\left\|M_{m_{z_{i}} m_{z_{j}}}^{n+4}\right\| .
$$

For each $(i, j) \in \mathcal{E}^{(k)}$, it follows from Lemma 2.4 and the condition $d\left(\xi_{i}, \xi_{j}\right) \geq 2^{k} t$ that

$$
\left\|M_{m_{z_{i}} m_{z_{j}}}^{n+4}\right\| \leq\left(48 \frac{1-\left|z_{i}\right|^{2}}{\left|1-\left\langle z_{i}, z_{j}\right\rangle\right|}\right)^{n+4} \leq\left(96 \frac{1-\left|z_{i}\right|^{2}}{\left|1-\left\langle\xi_{i}, \xi_{j}\right\rangle\right|}\right)^{n+4} \leq \frac{C_{1}}{2^{2 k(n+4)}}
$$

where $C_{1}=(96)^{n+4}$. Hence

$$
\begin{equation*}
\left|\left\langle v_{z_{j}} f_{j}, v_{z_{i}} f_{i}\right\rangle\right| \leq \frac{C_{1}}{2^{2 k(n+4)}} \quad \text { for each } \quad(i, j) \in \mathcal{E}^{(k)} \tag{4.25}
\end{equation*}
$$

Set $\ell(k)=\min \left\{\ell \in \mathbf{N}: \ell \geq C_{2} 2^{2 n k}\right\}$ as before, where $C_{2}=2^{5 n} A_{0}$. Then, by (4.11),

$$
\operatorname{card}\left\{j \in J: d\left(\xi_{i}, \xi_{j}\right)<2^{k+1} t\right\} \leq \ell(k)
$$

According to Lemma 4.1, we can decompose $\mathcal{E}^{(k)}$ as the union of pairwise disjoint subsets

$$
\mathcal{E}_{1}^{(k)}, \ldots, \mathcal{E}_{2 \ell(k)}^{(k)}
$$

such that for each $m \in\{1, \ldots, 2 \ell(k)\}$, if $(i, j),\left(i^{\prime}, j^{\prime}\right) \in \mathcal{E}_{m}^{(k)}$ and if $(i, j) \neq\left(i^{\prime}, j^{\prime}\right)$, then we have both $i \neq i^{\prime}$ and $j \neq j^{\prime}$. This decomposition of $\mathcal{E}^{(k)}$ allows us to write

$$
\begin{equation*}
Y_{k}=Y_{k, 1}+\ldots+Y_{k, 2 \ell(k)}, \tag{4.26}
\end{equation*}
$$

where

$$
Y_{k, m}=\sum_{(i, j) \in \mathcal{E}_{m}^{(k)}}\left\langle v_{z_{j}} f_{j}, v_{z_{i}} f_{i}\right\rangle e_{i} \otimes e_{j},
$$

$1 \leq m \leq 2 \ell(k)$. By the property of $\mathcal{E}_{m}^{(k)}$ and (4.25), we have

$$
\left\|Y_{k, m}\right\| \leq \frac{C_{1}}{2^{2 k(n+4)}}
$$

for each $m \in\{1, \ldots, 2 \ell(k)\}$. By (4.26) and the definition of $\ell(k)$,

$$
\left\|Y_{k}\right\| \leq \frac{C_{1}}{2^{2 k(n+4)}} \cdot 2 \ell(k) \leq \frac{C_{1}}{2^{2 k(n+4)}} \cdot 2\left(C_{2}+1\right) 2^{2 n k}=\frac{2 C_{1}\left(C_{2}+1\right)}{2^{8 k}}
$$

Combining this estimate with (4.23) and (4.24), we see that if we set

$$
C_{4.4}=\left\{4^{n+4}+2 C_{1}\left(C_{2}+1\right) \sum_{k=0}^{\infty} \frac{1}{2^{8 k}}\right\}^{1 / 2},
$$

then $\|E\| \leq C_{4.4}$.

## 5. Spherical Decomposition

Before we get to the proof of Theorem 1.1, we want to recall an elementary fact:
Lemma 5.1. Let $\mathcal{H}$ be a separable Hilbert space. Suppose that $\{\mathcal{X}, \mu\}$ is a measure space and that $A$ is a weakly measurable $\mathcal{B}(\mathcal{H})$-valued function on $\mathcal{X}$. If $A(x) \in \mathcal{C}_{p}$ for every $x, 1<p<\infty$, then

$$
\left\|\int_{\mathcal{X}} A(x) d \mu(x)\right\|_{p} \leq \int_{\mathcal{X}}\|A(x)\|_{p} d \mu(x) .
$$

This lemma follows easily from the duality between $\mathcal{C}_{p}$ and $\mathcal{C}_{p /(p-1)}$. We omit the details.

Proof of Theorem 1.1. Recall that for each integer $N \geq n / 2$, Theorem 3.1 provides an operator

$$
R_{N}=\int \psi_{z, N} \otimes \psi_{z, N} d \lambda(z)
$$

which is both bounded and invertible on $H_{n}^{2}$. We will only use the case where $N=n+4$. That is, for the rest of the section, we will denote

$$
R=R_{n+4} .
$$

Similarly, we write

$$
\psi_{z}=\psi_{z, n+4}
$$

This gives us the relation

$$
\begin{equation*}
\psi_{z}=v_{z} k_{z} \tag{5.1}
\end{equation*}
$$

where $v_{z}$ was given in (4.2). Next we express $R$ in a slightly different form, a form which is more convenient for subsequent estimates. Since

$$
R=\int_{0}^{1} 2 n r^{2 n-1} \int \psi_{r \xi} \otimes \psi_{r \xi} d \sigma(\xi) \frac{d r}{\left(1-r^{2}\right)^{n+1}}
$$

making the substitution $t=\left(1-r^{2}\right)^{1 / 2}$, we have

$$
\begin{equation*}
R=\int_{0}^{1} 2 n\left(1-t^{2}\right)^{n-1} T_{t} \frac{d t}{t} \tag{5.2}
\end{equation*}
$$

where

$$
\begin{equation*}
T_{t}=\frac{1}{t^{2 n}} \int \psi_{\left(1-t^{2}\right)^{1 / 2} \xi} \otimes \psi_{\left(1-t^{2}\right)^{1 / 2} \xi} d \sigma(\xi) \tag{5.3}
\end{equation*}
$$

$0<t<1$. We then decompose each $T_{t}$, which involves spherical decomposition.
Let a $0<t<1$ be given. Then there is a subset $\left\{x_{1}, \ldots, x_{m(t)}\right\}$ of $S$ which is maximal with respect to the property

$$
B\left(x_{i}, t / 2\right) \cap B\left(x_{j}, t / 2\right)=\emptyset \quad \text { whenever } i \neq j
$$

The maximality implies that

$$
\bigcup_{j=1}^{m(t)} B\left(x_{j}, t\right)=S
$$

There are Borel sets $G_{1}, \ldots, G_{m(t)}$ in $S$ such that

$$
\begin{align*}
& G_{j} \subset B\left(x_{j}, t\right) \text { for each } j \in\{1, \ldots, m(t)\}, \\
& G_{i} \cap G_{j}=\emptyset \quad \text { whenever } i \neq j, \text { and } \\
& \cup_{j=1}^{m(t)} G_{j}=S . \tag{5.4}
\end{align*}
$$

For any $i, j$, if $B\left(x_{i}, 2 t\right) \cap B\left(x_{j}, 2 t\right) \neq \emptyset$, then $d\left(x_{i}, x_{j}\right)<4 t$, which implies $B\left(x_{j}, t / 2\right) \subset B\left(x_{i}, 5 t\right)$. It follows that for each $i \in\{1, \ldots, m(t)\}$,

$$
\begin{align*}
\operatorname{card}\{j: 1 \leq j & \left.\leq m(t), B\left(x_{i}, 2 t\right) \cap B\left(x_{j}, 2 t\right) \neq \emptyset\right\} \\
& \leq \frac{\sigma\left(B\left(x_{i}, 5 t\right)\right)}{\sigma\left(B\left(x_{i}, t / 2\right)\right)} \leq \frac{A_{0}(5 t)^{2 n}}{2^{-n}(t / 2)^{2 n}}=2^{3 n} 5^{2 n} A_{0}=C_{1} \tag{5.5}
\end{align*}
$$

Let $L$ be the smallest integer which is greater than $C_{1}$. Then we have the decomposition

$$
\{1, \ldots, m(t)\}=J_{1} \cup \cdots \cup J_{L}
$$

where $J_{1}, \ldots, J_{L}$ are pairwise disjoint and, for each $1 \leq \ell \leq L$, $J_{\ell}$ has the property that

$$
\begin{equation*}
B\left(x_{i}, 2 t\right) \cap B\left(x_{j}, 2 t\right)=\emptyset \quad \text { if } \quad i, j \in J_{\ell} \quad \text { and } \quad i \neq j \tag{5.6}
\end{equation*}
$$

The $J_{\ell}$ 's are obtained through a well-known method. One starts with a maximal subset $J_{1}$ of $\{1, \ldots, m(t)\}$ which has property (5.6). If $\{1, \ldots, m(t)\} \backslash J_{1} \neq \emptyset$, one similarly picks a maximal subset $J_{2}$ of $\{1, \ldots, m(t)\} \backslash J_{1}$, and so on. The maximality of each $J_{\ell}$ and (5.5) ensure that this process stops after at most $L$ steps.

There exist an $x \in S$ and unitary transformations $U_{1}, \ldots, U_{m(t)}$ on $\mathbf{C}^{n}$ such that $x_{j}=U_{j} x$ for $j=1, \ldots, m(t)$. Then we can write

$$
\frac{1}{t^{2 n}}=\frac{C(t)}{\sigma(B(x, t))}, \quad \text { where } \quad C(t) \leq A_{0}
$$

by (4.1). Therefore by (5.3) and (5.4),

$$
\begin{align*}
T_{t} & =\frac{C(t)}{\sigma(B(x, t))} \sum_{j=1}^{m(t)} \int_{B\left(x_{j}, t\right)} \chi_{G_{j}}(\xi) \psi_{\left(1-t^{2}\right)^{1 / 2} \xi} \otimes \psi_{\left(1-t^{2}\right)^{1 / 2} \xi} d \sigma(\xi) \\
& =\frac{C(t)}{\sigma(B(x, t))} \int_{B(x, t)} \sum_{j=1}^{m(t)} \chi_{G_{j}}\left(U_{j} \xi\right) \psi_{\left(1-t^{2}\right)^{1 / 2} U_{j} \xi} \otimes \psi_{\left(1-t^{2}\right)^{1 / 2} U_{j} \xi} d \sigma(\xi) \\
& =\frac{C(t)}{\sigma(B(x, t))} \int_{B(x, t)} \sum_{\ell=1}^{L} Y_{\ell}(\xi) d \sigma(\xi), \tag{5.7}
\end{align*}
$$

where

$$
Y_{\ell}(\xi)=\sum_{j \in J_{\ell}} \chi_{G_{j}}\left(U_{j} \xi\right) \psi_{\left(1-t^{2}\right)^{1 / 2} U_{j} \xi} \otimes \psi_{\left(1-t^{2}\right)^{1 / 2} U_{j}} \xi
$$

If $\xi \in B(x, t)$, then $U_{j} \xi \in B\left(x_{j}, t\right)$. Therefore by (5.6), for each $\xi \in B(x, t)$ we have

$$
\begin{equation*}
B\left(U_{i} \xi, t\right) \cap B\left(U_{j} \xi, t\right)=\emptyset \quad \text { if } \quad i, j \in J_{\ell} \quad \text { and } \quad i \neq j . \tag{5.8}
\end{equation*}
$$

To ease the notation, let us denote

$$
\begin{equation*}
z_{j}(\xi)=\left(1-t^{2}\right)^{1 / 2} U_{j} \xi \tag{5.9}
\end{equation*}
$$

for $j=1, \ldots, m(t)$ and $\xi \in B(x, t)$. Thus

$$
\begin{equation*}
Y_{\ell}(\xi)=\sum_{j \in J_{\ell}} \chi_{G_{j}}\left(U_{j} \xi\right) \psi_{z_{j}(\xi)} \otimes \psi_{z_{j}(\xi)} \tag{5.10}
\end{equation*}
$$

Now let a multiplier $f$ of $H_{n}^{2}$ be given. Then by (5.1),

$$
M_{f} Y_{\ell}(\xi)=\sum_{j \in J_{\ell}} \chi_{G_{j}}\left(U_{j} \xi\right)\left(f \psi_{z_{j}(\xi)}\right) \otimes \psi_{z_{j}(\xi)}=\sum_{j \in J_{\ell}} \chi_{G_{j}}\left(U_{j} \xi\right)\left(v_{z_{j}(\xi)} f_{z_{j}(\xi)}\right) \otimes \psi_{z_{j}(\xi)}
$$

where $f_{z_{j}(\xi)}=f k_{z_{j}(\xi)}$. We have $\left\|f_{z_{j}(\xi)}\right\| \leq\left\|M_{f}\right\|$. Let a $\nu \in\{1, \ldots, n\}$ be given. Then

$$
\begin{align*}
{\left[M_{\zeta_{\nu}}^{*}, M_{f} Y_{\ell}(\xi)\right] } & =\sum_{j \in J_{\ell}} \chi_{G_{j}}\left(U_{j} \xi\right)\left(M_{\zeta_{\nu}-\left(z_{j}(\xi)\right)_{\nu}}^{*} v_{z_{j}(\xi)} f_{z_{j}(\xi)}\right) \otimes \psi_{z_{j}(\xi)} \\
& -\sum_{j \in J_{\ell}} \chi_{G_{j}}\left(U_{j} \xi\right)\left(v_{z_{j}(\xi)} f_{z_{j}(\xi)}\right) \otimes\left(M_{\zeta_{\nu}-\left(z_{j}(\xi)\right)_{\nu}} \psi_{z_{j}(\xi)}\right), \tag{5.11}
\end{align*}
$$

where $\left(z_{j}(\xi)\right)_{\nu}$ denotes the $\nu$-th component of $z_{j}(\xi)$. Let $2 n<p<\infty$ also be given. We will estimate the Schatten $p$-norm of the above two terms.

Let $\left\{e_{j}: j \in J_{\ell}\right\}$ be an orthonormal set. We have

$$
\sum_{j \in J_{\ell}} \chi_{G_{j}}\left(U_{j} \xi\right)\left(M_{\zeta_{\nu}-\left(z_{j}(\xi)\right)_{\nu}}^{*} v_{z_{j}(\xi)} f_{z_{j}(\xi)}\right) \otimes \psi_{z_{j}(\xi)}=E_{\nu} A^{*}
$$

where

$$
E_{\nu}=\sum_{j \in J_{\ell}}\left(M_{\zeta_{\nu}-\left(z_{j}(\xi)\right)_{\nu}}^{*} v_{z_{j}(\xi)} f_{z_{j}(\xi)}\right) \otimes e_{j}
$$

and

$$
A=\sum_{j \in J_{\ell}} \chi_{G_{j}}\left(U_{j} \xi\right) \psi_{z_{j}(\xi)} \otimes e_{j}=\sum_{j \in J_{\ell}} \chi_{G_{j}}\left(U_{j} \xi\right)\left(v_{z_{j}(\xi)} k_{z_{j}(\xi)}\right) \otimes e_{j} .
$$

Conditions (5.8) and (5.9) enable us to apply the lemmas in Section 4 here. By Lemma 4.2, we have $\left\|E_{\nu}\right\|_{p} \leq C_{4.2}(p) t^{1-(2 n / p)}\left\|M_{f}\right\|$. On the other hand, Lemma 4.4 tells us $\|A\| \leq C_{4.4}$. Therefore

$$
\begin{equation*}
\left\|\sum_{j \in J_{\ell}} \chi_{G_{j}}\left(U_{j} \xi\right)\left(M_{\zeta_{\nu}-\left(z_{j}(\xi)\right)_{\nu}}^{*} v_{z_{j}(\xi)} f_{z_{j}(\xi)}\right) \otimes \psi_{z_{j}(\xi)}\right\|_{p} \leq C_{4.4} C_{4.2}(p)\left\|M_{f}\right\| t^{1-(2 n / p)} \tag{5.12}
\end{equation*}
$$

Similarly,

$$
\sum_{j \in J_{\ell}} \chi_{G_{j}}\left(U_{j} \xi\right)\left(v_{z_{j}(\xi)} f_{z_{j}(\xi)}\right) \otimes\left(M_{\zeta_{\nu}-\left(z_{j}(\xi)\right)_{\nu}} \psi_{z_{j}(\xi)}\right)=B F_{\nu}^{*}
$$

where

$$
F_{\nu}=\sum_{j \in J_{\ell}}\left(M_{\zeta_{\nu}-\left(z_{j}(\xi)\right)_{\nu}} \psi_{z_{j}(\xi)}\right) \otimes e_{j}=\sum_{j \in J_{\ell}}\left(M_{\zeta_{\nu}-\left(z_{j}(\xi)\right)_{\nu}} v_{z_{j}(\xi)} k_{z_{j}(\xi)}\right) \otimes e_{j}
$$

and

$$
B=\sum_{j \in J_{\ell}} \chi_{G_{j}}\left(U_{j} \xi\right)\left(v_{z_{j}(\xi)} f_{z_{j}(\xi)}\right) \otimes e_{j}
$$

By Lemma 4.3, $\left\|F_{\nu}\right\|_{p} \leq C_{4.3}(p) t^{1-(2 n / p)}$. By Lemma 4.4, $\|B\| \leq C_{4.4}\left\|M_{f}\right\|$. Therefore

$$
\begin{equation*}
\left\|\sum_{j \in J_{\ell}} \chi_{G_{j}}\left(U_{j} \xi\right)\left(v_{z_{j}(\xi)} f_{z_{j}(\xi)}\right) \otimes\left(M_{\zeta_{\nu}-\left(z_{j}(\xi)\right)_{\nu}} \psi_{z_{j}(\xi)}\right)\right\|_{p} \leq C_{4.4} C_{4.3}(p)\left\|M_{f}\right\| t^{1-(2 n / p)} \tag{5.13}
\end{equation*}
$$

Let $C_{2}=C_{4.4} C_{4.2}(p)+C_{4.4} C_{4.3}(p)$. Then, combining (5.11), (5.12) and (5.13), we have

$$
\left\|\left[M_{\zeta_{\nu}}^{*}, M_{f} Y_{\ell}(\xi)\right]\right\|_{p} \leq C_{2}\left\|M_{f}\right\| t^{1-(2 n / p)}
$$

Thus

$$
\left\|\left[M_{\zeta_{\nu}}^{*}, M_{f} \sum_{\ell=1}^{L} Y_{\ell}(\xi)\right]\right\|_{p} \leq C_{2} L\left\|M_{f}\right\| t^{1-(2 n / p)}
$$

Recalling (5.7) and the fact that $C(t) \leq A_{0}$, and using Lemma 5.1, we obtain

$$
\left\|\left[M_{\zeta_{\nu}}^{*}, M_{f} T_{t}\right]\right\|_{p}=\frac{C(t)}{\sigma(B(x, t))}\left\|\int_{B(x, t)}\left[M_{\zeta_{\nu}}^{*}, M_{f} \sum_{\ell=1}^{L} Y_{\ell}(\xi)\right] d \sigma(\xi)\right\|_{p} \leq A_{0} C_{2} L\left\|M_{f}\right\| t^{1-(2 n / p)} .
$$

Recalling (5.2) and using Lemma 5.1 again, we find that

$$
\begin{align*}
\left\|\left[M_{\zeta_{\nu}}^{*}, M_{f} R\right]\right\|_{p} & =\left\|\int_{0}^{1} 2 n\left(1-t^{2}\right)^{n-1}\left[M_{\zeta_{\nu}}^{*}, M_{f} T_{t}\right] \frac{d t}{t}\right\|_{p} \\
& \leq \int_{0}^{1} 2 n\left(1-t^{2}\right)^{n-1}\left\|\left[M_{\zeta_{\nu}}^{*}, M_{f} T_{t}\right]\right\|_{p} \frac{d t}{t} \\
& \leq A_{0} C_{2} L\left\|M_{f}\right\| \int_{0}^{1} 2 n\left(1-t^{2}\right)^{n-1} t^{1-(2 n / p)} \frac{d t}{t}=C(n, p)\left\|M_{f}\right\| \tag{5.14}
\end{align*}
$$

Note that the condition $p>2 n$ ensures $C(n, p)<\infty$. The above in particular implies

$$
\begin{equation*}
\left\|\left[M_{\zeta_{\nu}}^{*}, R\right]\right\|_{p} \leq C(n, p) \tag{5.15}
\end{equation*}
$$

Getting to the commutator that we are interested in, we have

$$
\left[M_{\zeta_{\nu}}^{*}, M_{f}\right]=\left[M_{\zeta_{\nu}}^{*}, M_{f} R R^{-1}\right]=\left[M_{\zeta_{\nu}}^{*}, M_{f} R\right] R^{-1}+M_{f}\left[R, M_{\zeta_{\nu}}^{*}\right] R^{-1} .
$$

Now it follows from (5.14) and (5.15) that

$$
\left\|\left[M_{\zeta_{\nu}}^{*}, M_{f}\right]\right\|_{p} \leq C(n, p)\left\|M_{f}\right\|\left\|R^{-1}\right\|+\left\|M_{f}\right\| C(n, p)\left\|R^{-1}\right\|=2 C(n, p)\left\|M_{f}\right\|\left\|R^{-1}\right\| .
$$

Since Theorem 3.1 asserts that $\left\|R^{-1}\right\|<\infty$, this completes the proof of Theorem 1.1.

## 6. Localization

Let us recall Arveson's exact sequence (1.1), in particular the homomorphism $\tau$. According to Theorem 5.7 in [3],

$$
\begin{equation*}
\tau\left(M_{\zeta_{j}}\right)=\zeta_{j} \tag{6.1}
\end{equation*}
$$

for each $j \in\{1, \ldots, n\}$. Define the quotient $C^{*}$-algebras

$$
\hat{\mathcal{T}}_{n}=\mathcal{T}_{n} / \mathcal{K} \quad \text { and } \quad \widehat{\mathcal{T} \mathcal{M}_{n}}=\mathcal{T} \mathcal{M}_{n} / \mathcal{K}
$$

Let $\hat{\tau}: \hat{\mathcal{T}}_{n} \rightarrow C(S)$ be the isomorphism induced by $\tau$. Thus $S$ is the maximal ideal space of $\hat{\mathcal{T}}_{n}$. Theorem 1.1 asserts that $\hat{\mathcal{T}}_{n}$ is contained in the center of $\widehat{\mathcal{T} \mathcal{M}_{n}}$.

For each $\xi \in S$, let $\hat{\mathcal{I}_{\xi}}$ be the ideal in $\widehat{\mathcal{T} \mathcal{M}_{n}}$ generated by

$$
\begin{equation*}
\left\{b \in \hat{\mathcal{T}}_{n}: \hat{\tau}(b)(\xi)=0\right\} \tag{6.2}
\end{equation*}
$$

By Douglas' localization theorem (see Theorem 7.47 in [8]), we have

$$
\bigcap_{\xi \in S} \hat{\mathcal{I}}_{\xi}=\{0\}
$$

An elementary $C^{*}$-algebraic argument then yields

$$
\begin{equation*}
\|a\|=\sup _{\xi \in S}\left\|a+\hat{\mathcal{I}}_{\xi}\right\| \tag{6.3}
\end{equation*}
$$

for every $a \in \widehat{\mathcal{T} \mathcal{M}_{n}}$. Let $\pi: \mathcal{T} \mathcal{M}_{n} \rightarrow \widehat{\mathcal{T} \mathcal{M}_{n}}$ be the quotient map. For each $\xi \in S$, let $\mathcal{I}_{\xi}$ be the inverse image of $\hat{\mathcal{I}}_{\xi}$ under $\pi$. Since $\mathcal{T} \mathcal{M}_{n} \supset \mathcal{K}$ and since $\mathcal{I}_{\xi} \neq\{0\}$, we have $\mathcal{I}_{\xi} \supset \mathcal{K}$. By (6.2), $\mathcal{I}_{\xi}$ is the ideal in $\mathcal{T} \mathcal{M}_{n}$ generated by

$$
\begin{equation*}
\left\{B \in \mathcal{T}_{n}: \tau(B)(\xi)=0\right\} \tag{6.4}
\end{equation*}
$$

It follows from (6.3) that

$$
\begin{equation*}
\|A\|_{\mathcal{Q}}=\sup _{\xi \in S}\left\|A+\mathcal{I}_{\xi}\right\| \tag{6.5}
\end{equation*}
$$

for every $A \in \mathcal{T} \mathcal{M}_{n}$.
Lemma 6.1. Let $\xi \in S$. Then the linear span of operators of the form

$$
T M_{\zeta_{j}-\xi_{j}}+K
$$

where $j \in\{1, \ldots, n\}, \xi_{j}$ is the $j$-th component of $\xi, T \in \mathcal{T} \mathcal{M}_{n}$, and $K \in \mathcal{K}$, is dense in $\mathcal{I}_{\xi}$.
Proof. Let $Z \in \mathcal{I}_{\xi}$ and $\epsilon>0$ be given. By (6.4), there are $B_{1}, \ldots, B_{m} \in \mathcal{T}_{n}$ with $\tau\left(B_{1}\right)(\xi)=\cdots=$ $\tau\left(B_{m}\right)(\xi)=0$ and $X_{1}, \ldots, X_{m}, Y_{1}, \ldots, Y_{m} \in \mathcal{T} \mathcal{M}_{n}$ such that

$$
\left\|Z-\left(X_{1} B_{1} Y_{1}+\cdots+X_{m} B_{m} Y_{m}\right)\right\| \leq \epsilon
$$

But, by Theorem 1.1, there is a $K_{1} \in \mathcal{K}$ such that

$$
X_{1} B_{1} Y_{1}+\cdots+X_{m} B_{m} Y_{m}=X_{1} Y_{1} B_{1}+\cdots+X_{m} Y_{m} B_{m}+K_{1}
$$

Therefore

$$
\begin{equation*}
\left\|Z-\left(X_{1} Y_{1} B_{1}+\cdots+X_{m} Y_{m} B_{m}+K_{1}\right)\right\| \leq \epsilon \tag{6.6}
\end{equation*}
$$

Let $B \in \mathcal{T}_{n}$ be such that $\tau(B)(\xi)=0$. Then $\tau(B)$ lies in the ideal in $C(S)$ generated by $\zeta_{1}-\xi_{1}, \ldots, \zeta_{n}-\xi_{n}$. Let $\delta>0$ be given. By (6.1), there exist $T_{1}, \ldots, T_{n} \in \mathcal{T}_{n}$ and a $K \in \mathcal{K}$ such that

$$
\begin{equation*}
\left\|B-\left(T_{1} M_{\zeta_{1}-\xi_{1}}+\cdots+T_{n} M_{\zeta_{n}-\xi_{n}}+K\right)\right\| \leq \delta \tag{6.7}
\end{equation*}
$$

The conclusion of the lemma follows from (6.6) and (6.7).
Proposition 6.2. For every $A \in \mathcal{T} \mathcal{M}_{n}$ and every $\xi \in S$, we have

$$
\lim _{r \uparrow 1}\left\|A M_{s_{r \xi}}\right\|=\left\|A+\mathcal{I}_{\xi}\right\| .
$$

Proof. Let $\xi \in S$ be given. We first show that

$$
\begin{equation*}
\lim _{r \uparrow 1}\left\|W M_{s_{r \xi}}\right\|=0 \tag{6.8}
\end{equation*}
$$

for every $W \in \mathcal{I}_{\xi}$. Applying Lemma 2.5, we have

$$
\begin{equation*}
\lim _{r \uparrow 1}\left\|M_{\left(\zeta_{j}-\xi_{j}\right)} M_{s_{r \xi}}\right\|=\lim _{r \uparrow 1}\left\|M_{\left(\zeta_{j}-\xi_{j}\right) s_{r \xi}}\right\| \leq \lim _{r \uparrow 1}\left\{\left\|M_{\left(\zeta_{j}-r \xi_{j}\right) s_{r \xi}}\right\|+\left|r \xi_{j}-\xi_{j}\right|\left\|M_{s_{r \xi}}\right\|\right\}=0 \tag{6.9}
\end{equation*}
$$

for each $j \in\{1, \ldots, n\}$, where $\xi_{j}$ is the $j$-th component of $\xi$. Let $K$ be a compact operator. Then Corollary 2.2 gives us $\left\|K M_{s_{r \xi}}\right\|=\left\|M_{s_{r \xi}}^{*} K^{*}\right\| \leq\left\|M_{s_{r \xi}} K^{*}\right\|$. Since $K^{*}$ is also compact, it follows from Lemma 2.6 that

$$
\begin{equation*}
\lim _{r \uparrow 1}\left\|K M_{s_{r \xi}}\right\| \leq \lim _{r \uparrow 1}\left\|M_{s_{r \xi}} K^{*}\right\|=0 . \tag{6.10}
\end{equation*}
$$

Combining (6.9), (6.10) and Lemma 6.1, (6.8) is proved.
Let $A \in \mathcal{T} \mathcal{M}_{n}$ be given. Then by (6.8), for every $W \in \mathcal{I}_{\xi}$ we have

$$
\underset{r \uparrow 1}{\limsup }\left\|A M_{s_{r \xi}}\right\|=\underset{r \uparrow 1}{\limsup }\left\|(A+W) M_{s_{r \xi}}\right\| \leq\|A+W\| \text {. }
$$

Since this holds for every $W \in \mathcal{I}_{\xi}$, it follows that

$$
\begin{equation*}
\underset{r \uparrow 1}{\limsup }\left\|A M_{s_{r \xi}}\right\| \leq\left\|A+\mathcal{I}_{\xi}\right\| . \tag{6.11}
\end{equation*}
$$

Next we show that

$$
\begin{equation*}
\left\|A M_{s_{r \xi}}\right\| \geq\left\|A+\mathcal{I}_{\xi}\right\| \tag{6.12}
\end{equation*}
$$

for every $0<r<1$. Note that, since $|\xi|=1$,

$$
\begin{equation*}
1-s_{r \xi}(\zeta)=1-\frac{1-r}{1-r\langle\zeta, \xi\rangle}=\frac{r\langle\xi-\zeta, \xi\rangle}{1-r\langle\zeta, \xi\rangle} \tag{6.13}
\end{equation*}
$$

This and (6.1) together imply $1-M_{s_{r \xi}} \in \mathcal{I}_{\xi}$. Thus $A-A M_{s_{r \xi}} \in \mathcal{I}_{\xi}$, which clearly implies (6.12). The proposition follows from (6.11) and (6.12).

Proof of Theorem 1.2. It follows immediately from Proposition 6.2 and (6.5).
Proof of Theorem 1.3. Let $A \in \mathcal{T} \mathcal{M}_{n}$ be given. Then we obviously have

$$
\sup _{\xi \in S} \lim _{r \Uparrow 1}\left\|A M_{s_{r \xi}}\right\| \leq \lim _{r \Uparrow 1} \sup _{r \leq|z|<1}\left\|A M_{s_{z}}\right\| .
$$

By Theorem 1.2, the proof of Theorem 1.3 is reduced to the proof of the inequality

$$
\begin{equation*}
\lim _{r \uparrow 1} \sup _{r \leq|z|<1}\left\|A M_{s_{z}}\right\| \leq\|A\|_{\mathcal{Q}} . \tag{6.14}
\end{equation*}
$$

Let $K$ be a compact operator. By the subnormality of $M_{s_{z}}$ and Lemma 2.6, we have

$$
\lim _{|z| \uparrow 1}\left\|K M_{s_{z}}\right\|=\lim _{|z| \uparrow 1}\left\|M_{s_{z}}^{*} K^{*}\right\| \leq \lim _{|z| \uparrow 1}\left\|M_{s_{z}} K^{*}\right\|=0
$$

Therefore for each compact operator $K$ we have

$$
\lim _{r \uparrow 1} \sup _{r \leq|z|<1}\left\|A M_{s_{z}}\right\|=\lim _{r \uparrow 1} \sup _{r \leq|z|<1}\left\|(A+K) M_{s_{z}}\right\| \leq\|A+K\| .
$$

This clearly implies (6.14). Hence the theorem follows.

Remark 1. There is a "left version" for Theorems 1.2 and 1.3. That is, if we replace $A M_{s_{r \xi}}$ by $M_{s_{r \xi}} A$ in Theorem 1.2 (resp. $A M_{s_{z}}$ by $M_{s_{z}} A$ in Theorem 1.3), then the same statement holds. The point is that by Lemma 6.1 and Theorem 1.1, the linear span of operators of the form $M_{\zeta_{j}-\xi_{j}} T+K$ is also dense in $\mathcal{I}_{\xi}$. Using this "left version" of Lemma 6.1, the proof of the left version of Theorems 1.2 and 1.3 is the same as the right version.

Remark 2. Theorems 1.2 and 1.3 are better suited for application to the problem of determining compactness than their left version.

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