ON THE PROBLEM OF CHARACTERIZING MULTIPLIERS FOR THE DRURY-ARVESON SPACE

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Abstract. Let H_n^2 be the Drury-Arveson space on the unit ball $\mathbf{B}$ in $\mathbf{C}^n$, and suppose that $n \geq 2$. Let k_z, $z \in \mathbf{B}$, be the normalized reproducing kernel for H_n^2. In this paper we consider the following rather basic question in the theory of the Drury-Arveson space: For $f \in H_n^2$, does the condition $\sup_{|z|<1} \|fk_z\| < \infty$ imply that f is a multiplier of H_n^2? We show that the answer is negative. We further show that the analogue of the familiar norm inequality $\|H_\varphi\| \leq C\|\varphi\|_{\mathrm{BMO}}$ for Hankel operators fails in the Drury-Arveson space.

1. Introduction

Let $\mathbf{B}$ be the open unit ball in $\mathbf{C}^n$. Throughout the paper, the complex dimension n is always assumed to be greater than or equal to 2. Recall that the Drury-Arveson space H_n^2 is the Hilbert space of analytic functions on $\mathbf{B}$ that has the function

$$\frac{1}{1 - \langle \zeta, z \rangle}$$

as its reproducing kernel [3,9]. Equivalently, H_n^2 can be described as the Hilbert space of analytic functions on $\mathbf{B}$ where the inner product is given by

$$\langle h, g \rangle = \sum_{\alpha \in \mathbf{Z}_+^n} \frac{\alpha!}{|\alpha|!} a_\alpha \overline{b_\alpha}$$

for

$$h(\zeta) = \sum_{\alpha \in \mathbf{Z}_+^n} a_\alpha \zeta^\alpha \quad \text{and} \quad g(\zeta) = \sum_{\alpha \in \mathbf{Z}_+^n} b_\alpha \zeta^\alpha.$$

Here and throughout the paper, we use the standard multi-index notation [17,page 3].

A newcomer in the family of reproducing-kernel Hilbert spaces, the Drury-Arveson space has been the subject of intense study [1-8,10-15,18] in recent years. Perhaps this intense interest in H_n^2 is mainly due to its close connection with a number of important topics, such as the von Neumann inequality for commuting row contractions, the corona theorem, and the Arveson conjecture. But this interest in H_n^2 is also attributable to the fascinating (some might say mysterious) properties of the space itself. For example, the finiteness of the H^∞-norm $\|f\|_\infty = \sup_{z \in \mathbf{B}} |f(z)|$ of an analytic function f on $\mathbf{B}$ does not

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guarantee $f \in H^2_n$, and the tuple of multiplication operators $(M_{\zeta_1}, \ldots, M_{\zeta_n})$ fails to be jointly subnormal on H^2_n [3].

One source of fascination with the Drury-Arveson space is its collection of *multipliers*. Recall that a function $f \in H^2_n$ is said to be a multiplier of the Drury-Arveson space if $fh \in H^2_n$ for every $h \in H^2_n$ [3]. We will write $\mathcal{M}$ for the collection of the multipliers of H^2_n. Also recall from [3] that if $f \in \mathcal{M}$, then the multiplication operator M_f is bounded on H^2_n. The operator norm $\|M_f\|$ on H^2_n is also called the *multiplier norm* of f. It is well known that the H^∞-norm $\|f\|_\infty$ does not dominated the multiplier norm of f [3]. What is more, for $f \in \mathcal{M}$, $\|f\|_\infty$ fails to dominate even the essential norm of M_f on H^2_n [12].

An enduring challenge in the theory of the Drury-Arveson space, since its very inception, has been the quest for a good characterization of the membership in $\mathcal{M}$. Let $k \in \mathbf{N}$ be such that $2k \geq n$. Then given any $f \in H^2_n$, one can define the measure $d\mu_f$ on $\mathbf{B}$ by the formula

$$d\mu_f(z) = |(R^k f)(z)|^2 (1 - |z|^2)^{2k-n} dv(z), \tag{1.1}$$

where dv is the normalized volume measure on $\mathbf{B}$ and R denotes the radial derivative $z_1\partial_1 + \cdots + z_n\partial_n$. Ortega and Fàbrega showed in [16] that f is a multiplier of the Drury-Arveson space if and only if $d\mu_f$ is an H^2_n-Carleson measure. That is, $f \in \mathcal{M}$ if and only if there is a C such that

$$\int |h(z)|^2 d\mu_f(z) \leq C\|h\|^2$$

for every $h \in H^2_n$. In [2], Arcozzi, Rochberg and Sawyer gave a characterization for all the H^2_n-Carleson measures on $\mathbf{B}$. See Theorem 34 in that paper.

For a given Borel measure on $\mathbf{B}$, the conditions in [2,Theorem 34] are not the easiest verify. More to the point, [2,Theorem 34] deals with all Borel measures on $\mathbf{B}$, not just the class of measures $d\mu_f$ of the form (1.1). Thus it is natural to ask, is there a simpler, or a more direct, characterization of the membership $f \in \mathcal{M}$?

Since the Drury-Arveson space is a reproducing-kernel Hilbert space, it is natural to turn to the reproducing kernel for possible answers. Recall that the normalized reproducing kernel for H^2_n is given by the formula

$$k_z(\zeta) = \frac{(1 - |z|^2)^{1/2}}{1 - \langle \zeta, z \rangle},$$

$z, \zeta \in \mathbf{B}$. One of the frequent tools in the study of reproducing-kernel Hilbert spaces is the *Berezin transform*. But for any $f \in H^2_n$, the Berezin transform

$$\langle f k_z, k_z \rangle$$

is none other than $f(z)$ itself. Given what we know about H^2_n, the boundedness of Berezin transform on $\mathbf{B}$ is not expected to guarantee the membership $f \in \mathcal{M}$. Here we use the phrase "not expected", because this is not an issue that has been settled in the literature.

Note that Arveson's example in [3] only shows that for an analytic function f on $\mathbf{B}$, the finiteness of $\|f\|_\infty$ does not guarantee $f \in H_n^2$. But if one starts with an $f \in H_n^2$, and then one assumes $\|f\|_\infty < \infty$, does it follow that $f \in \mathcal{M}$? In the literature one cannot find answer to this very simple question, although the answer is not expected to be affirmative.

Even if one accepts that for $f \in H_n^2$, the boundedness of the Berezin transform $\langle fk_z, k_z\rangle$ is not enough to guarantee the membership $f \in \mathcal{M}$, what about something stronger than the Berezin transform? For example, anyone who gives any thought about multipliers is likely to come up with the following natural and basic

Question 1.1. For $f \in H_n^2$, does the condition

$$\sup_{|z|<1} \|fk_z\| < \infty$$

imply the membership $f \in \mathcal{M}$?

Prima facie, one would think that there is at least a fair chance that the answer to Question 1.1 might be affirmative. And that was what we thought for quite a while. What makes this question particularly tempting is that an affirmative answer would give a very simple characterization of the membership $f \in \mathcal{M}$. But that would be too simple a characterization, as it turns out. After a long struggle, we have finally arrived at the conclusion that, tempting though the question may be, its answer is actually negative. The following is our main result:

Theorem 1.2. *There exists an $f \in H_n^2$ satisfying the conditions $f \notin \mathcal{M}$ and*

$$\sup_{|z|<1} \|fk_z\| < \infty.$$

As the reader will see, the proof of this theorem involves a construction that is quite technical. Indeed it involves numerous estimates and requires everything that we know about the Drury-Arveson space. As we will explain in the next section, the same construction also shows that the function-theoretic operator theory on the Drury-Arveson space is quite different from that on the more familiar reproducing-kernel Hilbert spaces, such as the Hardy space and the Bergman space. We hope that the techniques illustrated here will be useful for future investigations of the Drury-Arveson space.

2. An alternate statement

For notational convenience, let us introduce

Definition 2.1. (a) For each $f \in \mathcal{M}$, we write $\|f\|_{\mathcal{M}}$ for its multiplier norm. In other words, $\|f\|_{\mathcal{M}}$ denotes the norm of the multiplication operator M_f on H_n^2.
(b) For each $h \in H_n^2$, denote

$$\|h\|' = \sup_{|z|<1} \|hk_z\|.$$

Obviously, we have $\|f\|' \le \|f\|_{\mathcal{M}}$ for every $f \in \mathcal{M}$. But the reverse domination fails:

Theorem 2.2. *There does not exist any constant* $0 < C < \infty$ *such that*

$$\|f\|_{\mathcal{M}} \le C\|f\|'$$

for every $f \in \mathcal{M}$.

We can interpret Theorem 2.2 in terms of Berezin transform. For $f \in \mathcal{M}$, we have

$$\|fk_z\|^2 = \langle M_f^* M_f k_z, k_z\rangle,$$

$z \in \mathbf{B}$. Thus Theorem 2.2 tells us that the supremum of the Berezin transform of $M_f^* M_f$ on $\mathbf{B}$ does not dominate the operator norm of $M_f^* M_f$ on H_n^2.

The work of this paper will be done in the form of proving Theorem 2.2. But given Theorem 2.2, we immediately have

Proof of Theorem 1.2. By Theorem 2.2, for every $j \in \mathbf{N}$ there is a $\varphi_j \in \mathcal{M}$ such that

$$\|\varphi_j\|' \le 2^{-j} \quad \text{and} \quad \|\varphi_j\|_{\mathcal{M}} \ge j.$$

For each subset J of $\mathbf{N}$, define

$$f_J = \sum_{j \in J} \varphi_j.$$

Then obviously we have $\|f_J\|' \le 1$ for every $J \subset \mathbf{N}$. Thus the proof of Theorem 1.2 will be complete if we can find a $J_0 \subset \mathbf{N}$ such that $f_{J_0} \notin \mathcal{M}$.

Let $\mathcal{L}$ denote the linear span of $\{k_z : z \in \mathbf{B}\}$. Note that the condition $\|\varphi_j\|' \le 2^{-j}$ implies that

$$\lim_{j\to\infty} \|\varphi_j g\| = 0 \tag{2.1}$$

for every $g \in \mathcal{L}$. On the other hand, for each $j \in \mathbf{N}$, since M_{φ_j} is a bounded operator on H_n^2 and since $\mathcal{L}$ is dense in H_n^2, there is a $g_j \in \mathcal{L}$ with $\|g_j\| = 1$ such that

$$\|\varphi_j g_j\| \ge \|\varphi_j\|_{\mathcal{M}} - 1 \ge j - 1. \tag{2.2}$$

Let $j_1 = 1$. Suppose that $\nu \ge 1$ and that we have selected natural numbers $j_1 < j_2 < \cdots < j_\nu$. By (2.1) and (2.2), there is a natural number $j_{\nu+1} > j_\nu$ such that

$$\|\varphi_{j_{\nu+1}} g_{j_k}\| \le 2^{-\nu} \quad \text{for every} \ \ 1 \le k \le \nu \tag{2.3}$$

and

$$\|\varphi_{j_{\nu+1}} g_{j_{\nu+1}}\| \ge \nu + 1 + \sum_{k=1}^{\nu} \|\varphi_{j_k}\|_{\mathcal{M}}. \tag{2.4}$$

Thus, inductively we obtain a sequence $j_1 < j_2 < \cdots < j_\nu < \cdots$, giving us a subset

$$J_0 = \{j_1, j_2, \dots j_\nu, \dots\}$$

of $\mathbf{N}$. Let us verify that this J_0 has the property $f_{J_0} \notin \mathcal{M}$. Indeed for each $\nu \in \mathbf{N}$,

$$\|f_{J_0} g_{j_{\nu+1}}\| \geq \|\varphi_{j_{\nu+1}} g_{j_{\nu+1}}\| - \sum_{k=1}^{\nu} \|\varphi_{j_k} g_{j_{\nu+1}}\| - \sum_{i=1}^{\infty} \|\varphi_{j_{\nu+i+1}} g_{j_{\nu+1}}\|.$$

Since $\|\varphi_{j_k} g_{j_{\nu+1}}\| \leq \|\varphi_{j_k}\|_{\mathcal{M}}$, applying (2.4) and (2.3), we obtain

$$\|f_{J_0} g_{j_{\nu+1}}\| \geq \nu + 1 + \sum_{k=1}^{\nu} \|\varphi_{j_k}\|_{\mathcal{M}} - \sum_{k=1}^{\nu} \|\varphi_{j_k}\|_{\mathcal{M}} - \sum_{i=1}^{\infty} 2^{-\nu-i} \geq \nu.$$

Since $\|g_{j_{\nu+1}}\| = 1$ for every $\nu \in \mathbf{N}$, this inequality implies that $f_{J_0} \notin \mathcal{M}$. This completes the proof of Theorem 1.2. □

As it turns out, the construction we use for the proof Theorem 2.2 also gives us the following negative results as bonus:

Theorem 2.3. *There does not exist any constant* $0 < C < \infty$ *such that the inequality*

$$\|M_f^* M_f - M_f M_f^*\| \leq C(\|f\|')^2$$

holds for every $f \in \mathcal{M}$, *where* M_f *is the operator of multiplication by* f *on* H_n^2.

If $f \in \mathcal{M}$, then

$$\|(f - \langle f k_z, k_z\rangle) k_z\|^2 = \|f k_z\|^2 - |f(z)|^2 \leq \|f k_z\|^2$$

for every $z \in \mathbf{B}$. Hence Theorem 2.3 immediately implies

Corollary 2.4. *There does not exist any constant* $0 < C < \infty$ *such that the inequality*

$$\|M_f^* M_f - M_f M_f^*\| \leq C \sup_{|z|<1} \|(f - \langle f k_z, k_z\rangle) k_z\|^2$$

holds for every $f \in \mathcal{M}$, *where* M_f *is the operator of multiplication by* f *on* H_n^2.

Obviously, $\|(f - \langle f k_z, k_z\rangle) k_z\|$ is the "mean oscillation" of $f \in \mathcal{M}$ with respect to the normalized reproducing kernel of the Drury-Arveson space. And, for those who are familiar with Hankel operators, the commutator $M_f^* M_f - M_f M_f^*$ is the Drury-Arveson space analogue of

$$H_{\bar{f}}^* H_{\bar{f}}.$$

In the study of Hankel operators, we are all too familiar with the norm inequality

$$\|H_\varphi\| \leq C \|\varphi\|_{\mathrm{BMO}}, \tag{2.5}$$

which holds in the setting of either the Hardy space of the unit sphere or the Bergman space of the unit ball [20,21]. In contrast, Corollary 2.4 tells us that the Drury-Arveson space analogue of (2.5) fails. This uncovers another aspect of the Drury-Arveson that is quite different from the Hardy space and the Bergman space.

3. Outline of our approach

The rest of the paper is taken up by the proofs of Theorems 2.2 and 2.3. To help the reader navigate through the details involved in the proofs, let us first give an outline, which serves as a roadmap for the rest of the paper.

From now on, L will denote a natural number, one that is the main parameter in the proofs. Given any $L \in \mathbf{N}$, we will show that there exist $f_L \in \mathcal{M}$ and $h_L \in H_n^2$ satisfying the conditions

$$\|f_L\|' \le C_{\mathrm{a}}, \tag{3.1}$$

$$\|f_L\| \le C_{\mathrm{b}}, \tag{3.2}$$

$$\|h_L\| \le C_{\mathrm{c}} L^{1/2}, \tag{3.3}$$

$$|\langle f_L h_L, f_L\rangle| \ge \delta L, \tag{3.4}$$

where $C_{\mathrm{a}}, C_{\mathrm{b}}, C_{\mathrm{c}}, \delta \in (0,\infty)$ are constants. Since $|\langle f_L h_L, f_L\rangle| \le \|f_L h_L\|\|f_L\|$, (3.2) and (3.4) together imply

$$\|f_L h_L\| \ge (\delta/C_{\mathrm{b}})L.$$

Since $\|f_L h_L\| \le \|f_L\|_{\mathcal{M}}\|h_L\|$, combining the above with (3.3), we find that

$$\|f_L\|_{\mathcal{M}} \ge \frac{\delta}{C_{\mathrm{b}} C_{\mathrm{c}}} L^{1/2}.$$

Since $L \in \mathbf{N}$ is arbitrary, Theorem 2.2 follows from this inequality and (3.1).

We want to emphasize that the function f_L plays two different roles in (3.4): both as a multiplier of H_n^2 and as a "test function".

To prove Theorem 2.3, we will show that f_L, h_L have the additional property

$$\lim_{L\to\infty} \langle M_{f_L} M_{f_L}^* h_L, 1\rangle = 0. \tag{3.5}$$

This and (3.4) together imply that there is an $L_0 \in \mathbf{N}$ such that

$$|\langle (M_{f_L}^* M_{f_L} - M_{f_L} M_{f_L}^*) h_L, 1\rangle| \ge (\delta/2)L$$

for every $L \ge L_0$. Combining this inequality with (3.3) and the fact $\|1\| = 1$, we have

$$\|M_{f_L}^* M_{f_L} - M_{f_L} M_{f_L}^*\| \ge \frac{\delta}{2C_{\mathrm{c}}} L^{1/2},$$

$L \geq L_0$. Obviously, Theorem 2.3 follows from this inequality and (3.1).

Thus the proofs of Theorems 2.2 and 2.3 are now decomposed into five parts, i.e., the proofs of (3.1)-(3.5).

The main ingredient in the construction of the functions f_L and h_L is the multiplier m_z we introduced in [11]. Recall from [11] that for each $z \in \mathbf{B}$, we define

$$m_z(\zeta) = \frac{1-|z|^2}{1-\langle \zeta, z\rangle}. \tag{3.6}$$

As we will see, both f_L and h_L are in the linear span of $\{m_z^\kappa : z \in \mathbf{B}\}$, where $\kappa = 2n+2$. Moreover, each f_L involves only one single radial value $|z|$, whereas h_L involves L different values of $|z|$. But much preparation is required before we can precisely define f_L and h_L.

4. Preliminaries

It is elementary that if c is a complex number with $|c| \leq 1$ and if $0 \leq \rho \leq 1$, then

$$2|1-\rho c| \geq |1-c|. \tag{4.1}$$

This inequality will frequently be used without explicit reference.

Denote $S = \{\xi \in \mathbf{C}^n : |\xi| = 1\}$, the unit sphere in $\mathbf{C}^n$. Recall that the formula

$$d(\xi,\eta) = |1-\langle \xi,\eta\rangle|^{1/2}, \qquad \xi,\eta \in S,$$

defines a metric on S [17,page 66]. For the rest of the paper, we write

$$B(\xi,r) = \{x \in S : |1-\langle x,\xi\rangle|^{1/2} < r\}$$

for $\xi \in S$ and $r > 0$. Let σ be the positive, regular Borel measure on S that is invariant under the orthogonal group $O(2n)$, i.e., the group of isometries on $\mathbf{C}^n \cong \mathbf{R}^{2n}$ which fix 0. As usual, the measure σ is normalized in such a way that $\sigma(S) = 1$. There is a constant $2^{-n} < A_0 < \infty$ such that

$$2^{-n} r^{2n} \leq \sigma(B(\xi,r)) \leq A_0 r^{2n} \tag{4.2}$$

for all $\xi \in S$ and $0 < r \leq \sqrt{2}$ [17,Proposition 5.1.4].

For each $z \in \mathbf{B}\backslash\{0\}$, we have the Möbius transform

$$\varphi_z(w) = \frac{1}{1-\langle w,z\rangle}\left\{z - \frac{\langle w,z\rangle}{|z|^2}z - (1-|z|^2)^{1/2}\left(w - \frac{\langle w,z\rangle}{|z|^2}z\right)\right\} \tag{4.3}$$

of the unit ball $\mathbf{B}$ [17,page 25]. Also, we define $\varphi_0(w) = -w$. It is well known that the Bergman metric on $\mathbf{B}$ is given by the formula

$$\beta(z,w) = \frac{1}{2}\log\frac{1+|\varphi_z(w)|}{1-|\varphi_z(w)|}, \tag{4.4}$$

$z, w \in \mathbf{B}$. For each $z \in \mathbf{B}$ and each $a > 0$, we define the corresponding β-ball

$$D(z,a) = \{w \in \mathbf{B} : \beta(z,w) < a\}.$$

Definition 4.1. Let a be a positive number. A subset Γ of $\mathbf{B}$ is said to be a-separated if $D(z,a) \cap D(w,a) = \emptyset$ for all distinct elements z, w in Γ.

More directly, the construction of the functions f_L, h_L promised in Section 3 involves another kind of separation, a separation that is best descibed in terms of the radial-spherical decomposition of vectors $w \in \mathbf{B}$.

Definition 4.2. A subset E of $\mathbf{B}$ is said to be a *quasi-lattice* if it is contained in

$$\{\zeta \in \mathbf{C}^n : 3/4 \le |\zeta|^2 < 1\}$$

and has the property that for every $k \in \mathbf{N}$, if

$$z, z' \in \{\zeta \in E : 1 - 2^{-2k} \le |\zeta|^2 < 1 - 2^{-2k-2}\}$$

and if $z \ne z'$, then there are $\xi, \xi' \in S$ satisfying the condition $d(\xi, \xi') \ge 2^{-k}$ such that $z = |z|\xi$ and $z' = |z'|\xi'$.

We need the following relation between these two kinds of separations:

Lemma 4.3. *There exists an $a_0 > 0$ such that if E is any quasi-lattice in $\mathbf{B}$, then it admits a partition $E = E_0 \cup E_1$ where both E_0 and E_1 are a_0-separated.*

Proof. By (4.4), it suffices to show that every quasi-lattice E admits a partition $E = E_0 \cup E_1$ with the property that, for each $i \in \{0,1\}$, if z and w are distinct elements in E_i, then $|\varphi_z(w)| \ge 1/12$. This will follow from the elementary argument below.

First recall from [17,Theorem 2.2.2] that for any $z, w \in \mathbf{B}$, we have

$$1 - |\varphi_z(w)|^2 = \frac{(1-|z|^2)(1-|w|^2)}{|1 - \langle w, z\rangle|^2}. \tag{4.5}$$

Given a quasi-lattice E, define $E_0 = \cup_{\ell=1}^{\infty} F_{2\ell}$ and $E_1 = \cup_{\ell=0}^{\infty} F_{2\ell+1}$, where

$$F_k = \{\zeta \in E : 1 - 2^{-2k} \le |\zeta|^2 < 1 - 2^{-2k-2}\},$$

$k \in \mathbf{N}$. Let $i \in \{0,1\}$. For any pair of $z \ne w$ in E_i, there are the following two possibilities:

(1) Suppose that $z \in F_{2\ell+i}$ and $w \in F_{2\nu+i}$ with $\ell \ne \nu$. Since $|\varphi_z(w)| = |\varphi_w(z)|$, we may assume $\ell > \nu$. Thus $1 - |z|^2 \le (1/4)(1-|w|^2)$, which implies $1 - |z| \le (1/2)(1-|w|)$. In this case we have

$$\begin{aligned} |\varphi_z(w)|^2 &= 1 - \frac{(1-|z|^2)(1-|w|^2)}{|1-\langle w,z\rangle|^2} \ge 1 - \frac{(1-|z|^2)(1-|w|^2)}{(1-|w||z|)^2} = \frac{(|z|-|w|)^2}{(1-|w||z|)^2} \\ &\ge \left(\frac{(1-|w|)-(1-|z|)}{1-|w|+1-|z|}\right)^2 = \left(\frac{1-\{(1-|z|)/(1-|w|)\}}{1+\{(1-|z|)/(1-|w|)\}}\right)^2 \\ &\ge \left(\frac{1-(1/2)}{1+(1/2)}\right)^2 = \frac{1}{9}. \end{aligned}$$

(2) Suppose that there is a $k = 2\ell + i$ such that $z, w \in F_k$. In this case, by Definition 4.2, there are $\xi, \eta \in S$ with $d(\xi, \eta) \geq 2^{-k}$ such that $z = |z|\xi$ and $w = |w|\eta$. Since $|\varphi_z(w)| = |\varphi_w(z)|$, we may assume that $|z| \geq |w|$. By (4.3), we have

$$|\varphi_z(w)| \geq \frac{|z|}{|1 - \langle w, z\rangle|}\left|1 - \frac{\langle w, z\rangle}{|z|^2}\right| \geq \frac{(1/2)}{1 - |w|^2 + 1 - |z|^2 + |1 - \langle \xi, \eta\rangle|} \cdot \frac{1}{2}|1 - \langle \xi, \eta\rangle|$$
$$= \frac{(1/4)}{\{(1 - |w|^2)/d^2(\xi, \eta)\} + \{(1 - |z|^2)/d^2(\xi, \eta)\} + 1}.$$

Since in this case we have $1 - |w|^2 \leq d^2(\xi, \eta)$ and $1 - |z|^2 \leq d^2(\xi, \eta)$, it follows that $|\varphi_z(w)| \geq 1/12$. This completes the proof. □

5. Almost orthogonality in the Drury-Arveson space

Although the definition of f_L, h_L will come much later, let us first do the work that ensures inequalities (3.2) and (3.3). That is, in this section we estimate the norms of vectors of a certain kind in H_n^2. One of the facts that we use repeatedly throughout the paper is the rotation invariance of the inner product (hence the norm) in H_n^2. In other words, if $U : \mathbf{C}^n \to \mathbf{C}^n$ is a unitary transformation, then $\langle h \circ U, g \circ U\rangle = \langle h, g\rangle$ for all $g, h \in H_n^2$.

In addition to the m_z defined by (3.6), for convenience let us introduce a modified version of the normalized reproducing kernel. Let $0 < t < \infty$. For each $z \in \mathbf{B}$, define

$$\psi_{z,t}(\zeta) = \frac{(1 - |z|^2)^{(1/2)+t}}{(1 - \langle \zeta, z\rangle)^{1+t}}.$$

Then obviously we have the relations

$$\text{(5.1)} \qquad \psi_{z,t} = m_z^t k_z \quad \text{and} \quad (1 - |z|^2)^{1/2}\psi_{z,t} = m_z^{1+t}.$$

Lemma 5.1. *Given any positive number $0 < t < \infty$, there is a constant $C_{5.1}(t)$ that depends only on t such that the inequality*

$$|\langle \psi_{z,t}, \psi_{w,t}\rangle| \leq C_{5.1}(t)\left(\frac{(1 - |z|^2)^{1/2}(1 - |w|^2)^{1/2}}{|1 - \langle w, z\rangle|}\right)^{1+t}$$

holds for all $z, w \in \mathbf{B}$.

Proof. For each $z \in \mathbf{B}$, let us write

$$g_z(\zeta) = \langle \zeta, z\rangle,$$

which is an element in H_n^2. Given $0 < t < \infty$, let us also define

$$\Psi_{z,t}(\zeta) = \frac{1}{(1 - \langle \zeta, z\rangle)^{1+t}},$$

$z \in \mathbf{B}$. Let $\mathbf{T}$ denote the unit circle $\{\tau \in \mathbf{C} : |\tau| = 1\}$ and let dm be the Lebesgue measure on $\mathbf{T}$ with the normalization $m(\mathbf{T}) = 1$. It is elementary that for the given t, there is a constant $C(t)$ such that

$$\int \frac{dm(\tau)}{|1-\rho\tau|^{1+t}} \leq \frac{C(t)}{(1-\rho^2)^t} \tag{5.2}$$

for all $0 \leq \rho < 1$. See, e.g., [17,Proposition 1.4.10]. Given $z, w \in \mathbf{B}$, there is a $v \in \mathbf{C}$ with $|v| < 1$ such that

$$\langle w, z\rangle = v^2. \tag{5.3}$$

On the open unit disc $\{u \in \mathbf{C} : |u| < 1\}$, we have the power series expansion

$$\frac{1}{(1-u)^{1+t}} = \sum_{j=0}^{\infty} b_j u^j.$$

If $j \neq k$, then g_z^j and g_w^k are orthogonal to each other in H_n^2. Therefore

$$\langle \Psi_{z,t}, \Psi_{w,t}\rangle = \sum_{j=0}^{\infty} b_j^2 \langle g_z^j, g_w^j\rangle.$$

Suppose that $z = |z|\xi$, where $\xi \in S$. Then we can write

$$w = \langle w, \xi\rangle\xi + w^{\perp},$$

where $\langle w^{\perp}, \xi\rangle = 0$. Thus by an obvious change of variables we have

$$\langle g_z^j, g_w^j\rangle = \langle w, z\rangle^j.$$

Combining this with (5.3), we obtain $\langle g_z^j, g_w^j\rangle = v^{2j}$. Thus

$$\langle \Psi_{z,t}, \Psi_{w,t}\rangle = \sum_{j=0}^{\infty} (b_j v^j) \cdot (b_j v^j) = \int \frac{dm(\tau)}{(1-v\tau)^{1+t}(1-v\bar{\tau})^{1+t}},$$

and consequently

$$|\langle \Psi_{z,t}, \Psi_{w,t}\rangle| \leq \int \frac{dm(\tau)}{|1-v\tau|^{1+t}|1-v\bar{\tau}|^{1+t}}. \tag{5.4}$$

Using (5.3) again, for each $\tau \in \mathbf{T}$ we have

$$|1 - \langle w, z\rangle| = |1 - v^2| = |1 - v\tau \cdot v\bar{\tau}| \leq |1 - v\tau| + |1 - v\bar{\tau}|.$$

Thus if we set

$$A = \{\tau \in \mathbf{T} : |1 - v\tau| \geq (1/2)|1 - \langle w, z\rangle|\} \quad \text{and}$$
$$B = \{\tau \in \mathbf{T} : |1 - v\bar{\tau}| \geq (1/2)|1 - \langle w, z\rangle|\},$$

then $A \cup B = \mathbf{T}$. Hence it follows from (5.4) that

$$|\langle \Psi_{z,t}, \Psi_{w,t}\rangle| \leq \frac{2^{1+t}}{|1 - \langle w, z\rangle|^{1+t}} \left(\int_A \frac{dm(\tau)}{|1 - v\bar{\tau}|^{1+t}} + \int_B \frac{dm(\tau)}{|1 - v\tau|^{1+t}} \right).$$

Applying (5.2) and the rotation-invariance of dm, we have

$$\int \frac{dm(\tau)}{|1 - v\bar{\tau}|^{1+t}} \leq \frac{C(t)}{(1 - |v|^2)^t} \quad \text{and} \quad \int \frac{dm(\tau)}{|1 - v\tau|^{1+t}} \leq \frac{C(t)}{(1 - |v|^2)^t}.$$

But $|v|^2 = |\langle w, z\rangle| \leq |w||z|$. Therefore $1-|v|^2 \geq (1/2)(1-|z|^2)$ and $1-|v|^2 \geq (1/2)(1-|w|^2)$. Combining the above, we find that

$$|\langle \Psi_{z,t}, \Psi_{w,t}\rangle| \leq \frac{2^{1+t} \cdot 2^{t+1} C(t)}{|1 - \langle w, z\rangle|^{1+t}(1 - |z|^2)^{t/2}(1 - |w|^2)^{t/2}}.$$

Since

$$\psi_{z,t} = (1 - |z|^2)^{(1/2)+t}\Psi_{z,t} \quad \text{and} \quad \psi_{w,t} = (1 - |w|^2)^{(1/2)+t}\Psi_{w,t},$$

the lemma follows. □

Corollary 5.2. *Let* $0 < t < \infty$. *Then*

$$|\langle \psi_{z,t}, \psi_{w,t}\rangle| \leq 2^{1+t} C_{5.1}(t) e^{-(1+t)\beta(z,w)}$$

for all $z, w \in \mathbf{B}$, *where* $C_{5.1}(t)$ *is the constant provided by Lemma* 5.1.

Proof. It is elementary that if $0 \leq x < 1$, then $1 - x^2 \leq 4\exp(-\log\{(1 + x)/(1 - x)\})$. Recalling (4.5) and (4.4), we have

$$\left(\frac{(1 - |z|^2)^{1/2}(1 - |w|^2)^{1/2}}{|1 - \langle w, z\rangle|} \right)^{1+t} \leq 2^{1+t} e^{-(1+t)\beta(z,w)}$$

for all $z, w \in \mathbf{B}$ and $t > 0$. Combining this with Lemma 5.1, the corollary follows. □

Lemma 5.3. [19,Lemma 4.1] *Let X be a set and let E be a subset of $X \times X$. Suppose that m is a natural number such that*

$$\operatorname{card}\{y \in X : (x, y) \in E\} \leq m \quad \textit{and} \quad \operatorname{card}\{y \in X : (y, x) \in E\} \leq m$$

for every $x \in X$. Then there exist pairwise disjoint subsets E_1, E_2, ..., E_{2m} of E such that

$$E = E_1 \cup E_2 \cup \cdots \cup E_{2m}$$

and such that for each $1 \le j \le 2m$, *the conditions* $(x, y), (x', y') \in E_j$ *and* $(x, y) \neq (x', y')$ *imply both* $x \neq x'$ *and* $y \neq y'$.

Proposition 5.4. *Given any* $t > 2n - 1$ *and* $a > 0$, *there exists a constant* $C_{5.4}(t, a)$ *such that the inequality*

$$\left\| \sum_{z\in\Gamma} c_z \psi_{z,t} \right\| \le C_{5.4}(t, a) \left(\sum_{z\in\Gamma} |c_z|^2 \right)^{1/2}$$

holds for every finite, a-separated set Γ *in* $\mathbf{B}$, *where* $c_z \in \mathbf{C}$ *for every* $z \in \Gamma$.

Proof. Let λ be the standard Möbius-invariant measure on $\mathbf{B}$. That is,

$$d\lambda(z) = \frac{dv(z)}{(1 - |z|^2)^{n+1}},$$

where dv is the volume measure with the normalization $v(\mathbf{B}) = 1$. Using the Möbius invariance of both $d\lambda$ and β, it is easy to verify that there is a constant C such that $\lambda(D(\zeta, r)) \le Ce^{2nr}$ for all $\zeta \in \mathbf{B}$ and $r > 0$. Let $a > 0$ be given. Then there is a $C_1 = C_1(a)$ such that for each a-separated subset Γ of $\mathbf{B}$, the inequality

$$\operatorname{card}(\Gamma \cap D(\zeta, r)) \le C_1 e^{2nr} \tag{5.5}$$

holds for all $\zeta \in \mathbf{B}$ and $r > 0$.

Suppose that $t > 2n - 1$ and that Γ is an a-separated finite set. Let c_z, $z \in \Gamma$, be complex numbers. Then

$$\left\| \sum_{z\in\Gamma} c_z \psi_{z,t} \right\|^2 = \sum_{k=0}^{\infty} \sum_{(z,w)\in E^{(k)}} c_z \bar{c}_w \langle \psi_{z,t}, \psi_{w,t} \rangle, \tag{5.6}$$

where

$$E^{(k)} = \{(z, w) \in \Gamma \times \Gamma : k \le \beta(z, w) < k + 1\},$$

$k \in \mathbf{Z}_+$. Write $C_2 = 2^{1+t} C_{5.1}(t)$. By Corollary 5.2, if $(z, w) \in E^{(k)}$, then $|\langle \psi_{z,t}, \psi_{w,t} \rangle| \le C_2 e^{-(1+t)k}$. Therefore for every $k \in \mathbf{Z}_+$ we have

$$\left| \sum_{(z,w)\in E^{(k)}} c_z \bar{c}_w \langle \psi_{z,t}, \psi_{w,t} \rangle \right| \le C_2 e^{-(1+t)k} \sum_{(z,w)\in E^{(k)}} |c_z||c_w|. \tag{5.7}$$

By Lemma 5.3 and (5.5), for each $k \in \mathbf{Z}_+$, $E^{(k)}$ admits a partition

$$E^{(k)} = E_1^{(k)} \cup E_2^{(k)} \cup \cdots \cup E_{2m(k)}^{(k)}$$

with $m(k) \le C_1 e^{2n(k+1)}$ such that for each $1 \le j \le 2m(k)$, the set $E_j^{(k)}$ has the property that for $(z, w), (z', w') \in E_j^{(k)}$, the condition $(z, w) \neq (z', w')$ implies both $z \neq z'$ and

$w \neq w'$. In other words, the projections $(z, w) \mapsto z$ and $(z, w) \mapsto w$ are both injective on $E_j^{(k)}$. By this injectivity and the Cauchy-Schwarz inequality, we have

$$\sum_{(z,w)\in E_j^{(k)}} |c_z||c_w| \leq \sum_{z\in\Gamma} |c_z|^2.$$

Hence

$$\sum_{(z,w)\in E^{(k)}} |c_z||c_w| = \sum_{j=1}^{2m(k)} \sum_{(z,w)\in E_j^{(k)}} |c_z||c_w| \leq 2m(k) \sum_{z\in\Gamma} |c_z|^2 \leq 2C_1 e^{2n(k+1)} \sum_{z\in\Gamma} |c_z|^2.$$

Combining this with (5.7), we obtain

$$\left| \sum_{(z,w)\in E^{(k)}} c_z \bar{c}_w \langle \psi_{z,t}, \psi_{w,t} \rangle \right| \leq 2e^{2n} C_1 C_2 e^{-(1+t-2n)k} \sum_{z\in\Gamma} |c_z|^2.$$

Recalling (5.6), we find that

$$\left\| \sum_{z\in\Gamma} c_z \psi_{z,t} \right\|^2 \leq 2e^{2n} C_1 C_2 \sum_{k=0}^{\infty} e^{-(1+t-2n)k} \sum_{z\in\Gamma} |c_z|^2.$$

Since $1 + t > 2n$, the proposition follows from this inequality. $\square$

Combining Proposition 5.4 and Lemma 4.3, we immediately obtain

Corollary 5.5. *Given any $t > 2n-1$, there exists a constant $C_{5.5}(t)$ such that the inequality*

$$\left\| \sum_{z\in E} c_z \psi_{z,t} \right\| \leq C_{5.5}(t) \left(\sum_{z\in E} |c_z|^2 \right)^{1/2}$$

holds for every finite quasi-lattice E in $\mathbf{B}$, where $c_z \in \mathbf{C}$ for every $z \in E$.

6. Constants N and M

To simplify our notation, we write

$$\kappa = 2n + 2$$

for the rest of the paper. The proof of (3.4) will be based on estimates of inner products of the form $\langle m_w^\kappa m_z^\kappa, m_z^\kappa \rangle$, where $0 \leq |w| \leq |z| < 1$. We begin with a special case:

Lemma 6.1. *There exist constants $0 < \alpha_0 < 1$ and $N_0 \in \mathbf{N}$ such that the following statement holds true: Let $z, w \in \mathbf{B}$ and suppose that*

$$1 - |z|^2 \leq 2^{-2N_0}(1 - |w|^2).$$

Furthermore, suppose that there is a $\xi \in S$ *such that* $z = |z|\xi$ *and* $w = |w|\xi$*. Then*

$$\langle m_w^\kappa m_z^\kappa, m_z^\kappa \rangle = p_0(z,w) + c_0(z,w)$$

where $p_0(z,w)$ *is a positive number satisfying the inequality*

$$p_0(z,w) \geq \alpha_0(1-|z|^2) \tag{6.1}$$

and $c_0(z,w)$ *is a complex number such that*

$$|c_0(z,w)| \leq (\alpha_0/8)(1-|z|^2). \tag{6.2}$$

Proof. Again, let dm be the Lebesgue measure on the unit circle $\mathbf{T}$ with the normalization $m(\mathbf{T}) = 1$. First note that there is an $0 < \alpha_0 < 1$ such that

$$\int \left(\frac{1-\rho^2}{|1-\rho\tau|} \right)^{2\kappa} dm(\tau) \geq \alpha_0(1-\rho^2) \tag{6.3}$$

for every $0 \leq \rho < 1$. Indeed, to see this, consider $\theta \in [-1+\rho, 1-\rho]$. For such a θ we have

$$|1-\rho e^{i\theta}| \leq 1-\rho+\rho|1-e^{i\theta}| \leq 1-\rho+|\sin\theta|+\sin^2\theta \leq 3(1-\rho).$$

Thus the inequality

$$\left(\frac{1-\rho^2}{|1-\rho e^{i\theta}|} \right)^{2\kappa} \geq \frac{1}{3^{2\kappa}}$$

holds for every $\theta \in [-1+\rho, 1-\rho]$. This clearly implies (6.3).

Write $|z| = s$ and $|w| = r$. Then by assumption

$$z = s\xi \quad \text{and} \quad w = r\xi$$

for some $\xi \in S$. By the rotation invariance of the inner product in H_n^2, we have

$$\langle m_w^\kappa m_z^\kappa, m_z^\kappa \rangle = \langle m_{r\xi}^\kappa m_{s\xi}^\kappa, m_{s\xi}^\kappa \rangle = \int \left(\frac{1-r^2}{1-r\tau} \right)^{\kappa} \left(\frac{1-s^2}{|1-s\tau|} \right)^{2\kappa} dm(\tau).$$

Now we set

$$p_0(z,w) = \left(\frac{1-r^2}{1-r} \right)^{\kappa} \int \left(\frac{1-s^2}{|1-s\tau|} \right)^{2\kappa} dm(\tau) \quad \text{and}$$
$$c_0(z,w) = \int \left\{ \left(\frac{1-r^2}{1-r\tau} \right)^{\kappa} - \left(\frac{1-r^2}{1-r} \right)^{\kappa} \right\} \left(\frac{1-s^2}{|1-s\tau|} \right)^{2\kappa} dm(\tau).$$

Then $p_0(z,w) \in (0,\infty)$ and (6.1) follows from (6.3). Also, (6.2) follows from the following assertion: There exists an $N_0 \in \mathbf{N}$ such that the inequality

$$\int \left| \left(\frac{1-r^2}{1-r\tau} \right)^{\kappa} - \left(\frac{1-r^2}{1-r} \right)^{\kappa} \right| \left(\frac{1-s^2}{|1-s\tau|} \right)^{2\kappa} dm(\tau) \leq \frac{\alpha_0}{8}(1-s^2)$$

holds for all $r, s \in [0,1)$ satisfying the condition $1-s^2 \le 2^{-2N_0}(1-r^2)$. This assertion itself can be proved in two easy steps. First of all, there is an $N_1 \in \mathbf{N}$ such that the inequality

$$2^{\kappa+1}\left(\frac{1-s^2}{|1-se^{i\theta}|}\right)^{2\kappa-2} \le \frac{\alpha_0}{16}$$

holds whenever $2^{2N_1}(1-s^2) \le |\theta| \le \pi$. Second, there is an $N_1 \in \mathbf{N}$ such that

$$\kappa \cdot 2^{\kappa-1} \cdot \left|\frac{1-r^2}{1-re^{i\theta}} - \frac{1-r^2}{1-r}\right| \cdot 2^{2\kappa-2} \le \frac{\alpha_0}{16}$$

whenever $|\theta| \le 2^{-2N_2}(1-r^2)$. Then the above assertion holds for $N_0 = N_1 + N_2$. This completes the proof. □

Next we will replace the assumption $z = |z|\xi$ and $w = |w|\xi$ in Lemma 6.1 by the condition that $d(w/|w|, z/|z|)$ is small relative to $(1-|w|^2)^{1/2}$. For this we need to estimate certain multiplier norm. As usual, for each $i \in \{1, \ldots, n\}$, we write ζ_i for the i-th component of the variable $\zeta \in \mathbf{B}$.

Lemma 6.2. [11,Lemma 2.3] *If $0 < s < 1$, then the norm of the operator of multiplication by the function*

$$\frac{\zeta_2}{1-s\zeta_1}$$

on H^2_n does not exceed

$$\frac{2}{\sqrt{1-s}}.$$

Lemma 6.3. *For all $0 \le \rho < 1$ and $x, y \in S$ we have*

$$\|m_{\rho y} - m_{\rho x}\|_{\mathcal{M}} \le 8\frac{d(x,y)}{\sqrt{1-\rho^2}} + 4\left(\frac{d(x,y)}{\sqrt{1-\rho^2}}\right)^2$$

Proof. Given any $x, y \in S$, there is an $x^\perp \in S$ with $\langle x, x^\perp\rangle = 0$ such that $y = ax + bx^\perp$, where $a = \langle y, x\rangle$ and

$$|b| = \sqrt{1-|\langle y,x\rangle|^2} \le 2^{1/2}d(x,y).$$

Let $\rho \in [0,1)$. Since $\|\cdot\|_{\mathcal{M}}$ is rotation invariant, the multiplier norm of $m_{\rho y} - m_{\rho x}$ equals the multiplier norm of the function

$$F(\zeta) = \frac{1-\rho^2}{1-\rho\bar{a}\zeta_1 - \rho\bar{b}\zeta_2} - \frac{1-\rho^2}{1-\rho\zeta_1}.$$

Simple algebra then gives us $F = F_1 - F_2$, where

$$F_1(\zeta) = \frac{1-\rho^2}{1-\rho\bar{a}\zeta_1 - \rho\bar{b}\zeta_2} \cdot \frac{\rho\bar{b}\zeta_2}{1-\rho\zeta_1} \quad \text{and} \quad F_2(\zeta) = \frac{1-\rho^2}{1-\rho\bar{a}\zeta_1 - \rho\bar{b}\zeta_2} \cdot \frac{\rho(1-\bar{a})\zeta_1}{1-\rho\zeta_1}.$$

By Lemma 6.2, the multiplier norm of $\rho\bar{b}\zeta_2/(1-\rho\zeta_1)$ does not exceed $2|b|/\sqrt{1-\rho} \le 4d(x,y)/\sqrt{1-\rho^2}$. Therefore $\|F_1\|_{\mathcal{M}} \le 8d(x,y)/\sqrt{1-\rho^2}$. On the other hand, the multiplier norm of $(1-\bar{a})\zeta_1/(1-\rho\zeta_1)$ is at most

$$\frac{|1-\bar{a}|}{1-\rho} = \frac{d^2(x,y)}{1-\rho} \le 2\left(\frac{d(x,y)}{\sqrt{1-\rho^2}}\right)^2.$$

Consequently $\|F_2\|_{\mathcal{M}} \le 4\{d(x,y)/\sqrt{1-\rho^2}\}^2$. This completes the proof. □

Lemma 6.4. *Let α_0 be the same as in Lemma 6.1. There exists a constant $N \in \mathbf{N}$ such that for all $z, w \in \mathbf{B}$ satisfying the conditions*

$$(6.4) \qquad 1-|z|^2 \le 2^{-2N}(1-|w|^2) \quad \textit{and} \quad d\left(\frac{z}{|z|}, \frac{w}{|w|}\right) \le 2^{-N}(1-|w|^2)^{1/2},$$

we have

$$\langle m_w^\kappa m_z^\kappa, m_z^\kappa\rangle = p(z,w) + c(z,w),$$

where $p(z,w)$ is a positive number satisfying the inequality

$$p(z,w) \ge \alpha_0(1-|z|^2)$$

and $c(z,w)$ is a complex number such that

$$|c(z,w)| \le (\alpha_0/4)(1-|z|^2).$$

Proof. First of all, we have

$$(6.5) \qquad \|m_u^\kappa\|^2 \le (2^{\kappa-1})^2(1-|u|^2)$$

for every $u \in \mathbf{B}$. Next, note that

$$(6.6) \qquad \|m_w^\kappa - m_v^\kappa\|_{\mathcal{M}} \le \kappa 2^{\kappa-1}\|m_w - m_v\|_{\mathcal{M}}$$

for all $w, v \in \mathbf{B}$. Let N_0 also be the same as in Lemma 6.1. Combining Lemma 6.3 with (6.5) and (6.6), we see that we can pick a natural number

$$N \ge N_0$$

such that if $\rho \in [0,1)$ and $x, y \in S$ satisfy the condition $d(x,y) \le 2^{-N}(1-\rho^2)^{1/2}$, then

$$(6.7) \qquad |\langle (m_{\rho y}^\kappa - m_{\rho x}^\kappa)m_u^\kappa, m_u^\kappa\rangle| \le (\alpha_0/8)(1-|u|^2)$$

for every $u \in \mathbf{B}$. Now suppose that $z, w \in \mathbf{B}$ satisfy the conditions in (6.4). Then there are $s, r \in [0,1)$ and $\xi, \eta \in S$ such that $z = s\xi$ and $w = r\eta$. Moreover, (6.4) translates to

$$1-s^2 \le 2^{-2N}(1-r^2) \quad \text{and} \quad d(\xi,\eta) \le 2^{-N}(1-r^2)^{1/2}.$$

Define $w' = r\xi$ and

$$\varphi(z, w) = \langle (m_w^\kappa - m_{w'}^\kappa) m_z^\kappa, m_z^\kappa \rangle.$$

Then it follows from (6.7) that

$$|\varphi(z, w)| \le (\alpha_0/8)(1 - |z|^2).$$

Note that $z = |z|\xi$, $w' = |w'|\xi$, and that $1 - |z|^2 \le 2^{-2N}(1 - |w'|^2)$. Since $N \ge N_0$, it follows from Lemma 6.1 that

$$\langle m_{w'}^\kappa m_z^\kappa, m_z^\kappa \rangle = p_0(z, w') + c_0(z, w'),$$

where $p_0(z, w')$ is a positive number satisfying the inequality $p_0(z, w') \ge \alpha_0(1 - |z|^2)$ and $c_0(z, w')$ is a complex number such that $|c_0(z, w')| \le (\alpha_0/8)(1 - |z|^2)$. Thus once we set

$$p(z, w) = p_0(z, w') \quad \text{and} \quad c(z, w) = \varphi(z, w) + c_0(z, w'),$$

the lemma follows. □

Lemma 6.5. *Let $z, w \in \mathbf{B}$ be such that $|w| \le |z|$ and $z \ne 0$. Then*

$$\|m_w m_z\|_{\mathcal{M}} \le 96 \frac{1 - |w|^2}{|1 - (|w|/|z|)\langle z, w \rangle|}.$$

Proof. Given $z, w \in \mathbf{B}$ satisfying the conditions $|w| \le |z|$ and $z \ne 0$, we set $\hat{z} = (|w|/|z|)z$. Since $|w| = |\hat{z}|$, by [11,Lemma 2.4] we have

$$\|m_w m_{\hat{z}}\|_{\mathcal{M}} \le 48 \frac{1 - |w|^2}{|1 - \langle \hat{z}, w \rangle|} = 48 \frac{1 - |w|^2}{|1 - (|w|/|z|)\langle z, w \rangle|}.$$

Since $m_w m_z = m_w m_{\hat{z}} \cdot m_{\hat{z}}^{-1} m_z$, it now suffices to show that $\|m_{\hat{z}}^{-1} m_z\|_{\mathcal{M}} \le 2$. To prove this, write $z = |z|\xi$, where $\xi \in S$. Then $\hat{z} = |w|\xi$. By an obvious change of coordinates, $\|m_{\hat{z}}^{-1} m_z\|_{\mathcal{M}}$ equals the multiplier norm of the function

$$\frac{1 - |w|\zeta_1}{1 - |w|^2} \cdot \frac{1 - |z|^2}{1 - |z|\zeta_1},$$

where ζ_1 denotes the first coordinate of $\zeta \in \mathbf{B}$. By the well-known von Neumann inequality for a single contraction, the multiplier norm of the above function does not exceed

$$\sup_{|\tau|=1} \frac{|1 - |w|\tau|}{1 - |w|^2} \cdot \frac{1 - |z|^2}{|1 - |z|\tau|}.$$

Using the condition $|w| \le |z|$, it is an easy exercise to show that the above supremum does not exceed $(1 + |z|)/(1 + |w|)$, which is less than 2. This completes the proof. □

Lemma 6.6. *There is a constant $C_{6.6}$ which depends only on the complex dimension n such that the following holds true: Given any $\rho > 0$, $k \in \mathbf{N}$ and $\eta \in S$, there exists a subset $E(\rho, k, \eta)$ of $B(\eta, 2^{k+1}\rho)\backslash B(\eta, 2^k\rho)$ with*

$$\operatorname{card}\{E(\rho, k, \eta)\} \le C_{6.6} 2^{2nk} \tag{6.8}$$

such that $\cup_{x\in E(\rho,k,\eta)} B(x, \rho) \supset B(\eta, 2^{k+1}\rho)\backslash B(\eta, 2^k\rho)$.

Proof. Given such ρ, k and η, let $E(\rho, k, \eta)$ be a subset of $B(\eta, 2^{k+1}\rho)\backslash B(\eta, 2^k\rho)$ that is maximal with respect to the property that if $x, y \in E(\rho, k, \eta)$ and $x \ne y$, then $B(x, \rho/2) \cap B(y, \rho/2) = \emptyset$. Then it follows easily from (4.2) that there is a $C_{6.6}$ which depends only on the complex dimension n such that (6.8) holds. The maximality of $E(\rho, k, \eta)$ ensures that $\cup_{x\in E(\rho,k,\eta)} B(x, \rho) \supset B(\eta, 2^{k+1}\rho)\backslash B(\eta, 2^k\rho)$. $\square$

Definition 6.7. If F is a finite subset of S and $\rho > 0$, we set

$$\mathcal{N}(F, \rho) = \sup_{x\in S} \operatorname{card}\{F \cap B(x, \rho)\}.$$

Lemma 6.8. *There exists a natural number $M \ge 4$ such that the inequality*

$$\sum_{\xi\in F\backslash B(\eta,2^M\sqrt{1-r^2})} |\langle m_{r\eta}^{\kappa} m_{s\xi}^{\kappa}, m_{s\xi}^{\kappa}\rangle| \le (\alpha_0/4)\mathcal{N}\left(F, \sqrt{1-r^2}\right)(1-s^2)$$

holds whenever F is a finite subset of S, $\eta \in S$, and $0 \le r \le s < 1$, where α_0 is the same as in Lemma 6.1.

Proof. Suppose that $0 \le r \le s < 1$. By Lemma 6.5 and (4.1), we have

$$\|m_{r\eta} m_{s\xi}\|_{\mathcal{M}} \le 192 \frac{1-r^2}{d^2(\eta, \xi)}$$

for all $\eta \ne \xi$ in S. Consequently,

$$\|m_{r\eta}^{2n} m_{s\xi}^{2n}\|_{\mathcal{M}} = \|(m_{r\eta} m_{s\xi})^{2n}\|_{\mathcal{M}} \le \|m_{r\eta} m_{s\xi}\|_{\mathcal{M}}^{2n} \le 192^{2n} \left(\frac{\sqrt{1-r^2}}{d(\eta, \xi)}\right)^{4n}.$$

Obviously, $\|m_{s\xi}\| = \sqrt{1-s^2}$, $\|m_z\|_{\mathcal{M}} \le 2$ for every $z \in \mathbf{B}$, and $\|m_{s\xi}^{\kappa}\| \le 2^{\kappa-1}\sqrt{1-s^2}$. Since $\kappa = 2n + 2$, we have

$$|\langle m_{r\eta}^{\kappa} m_{s\xi}^{\kappa}, m_{s\xi}^{\kappa}\rangle| \le \|m_{r\eta}^{2n} m_{s\xi}^{2n}\|_{\mathcal{M}} \|m_{r\eta}^2 m_{s\xi}^2\| \|m_{s\xi}^{\kappa}\| \le 192^{2n} 2^{\kappa+2} \left(\frac{\sqrt{1-r^2}}{d(\eta, \xi)}\right)^{4n} (1-s^2). \tag{6.9}$$

Let $k \ge 4$ and consider the set

$$R_k = \{x \in S : 2^k\sqrt{1-r^2} \le d(x, \eta) < 2^{k+1}\sqrt{1-r^2}\}.$$

By Lemma 6.6, there is a subset E_k of R_k with $\mathrm{card}(E_k) \le C_{6.6}2^{2nk}$ such that

$$R_k \subset \bigcup_{x \in E_k} B(x, \sqrt{1-r^2}).$$

If $x \in E_k$ and $\xi \in B(x, \sqrt{1-r^2})$, then

$$d(\eta, \xi) \ge d(\eta, x) - d(x, \xi) \ge 2^k\sqrt{1-r^2} - \sqrt{1-r^2} \ge 2^{k-1}\sqrt{1-r^2}.$$

Applying (6.9), for every finite subset F of S we have

$$\begin{aligned}\sum_{\xi \in F \cap R_k} |\langle m^\kappa_{r\eta} m^\kappa_{s\xi}, m^\kappa_{s\xi}\rangle| &\le \sum_{x \in E_k} \sum_{\xi \in F \cap B(x, \sqrt{1-r^2})} |\langle m^\kappa_{r\eta} m^\kappa_{s\xi}, m^\kappa_{s\xi}\rangle| \\ &\le \mathrm{card}(E_k)\mathcal{N}\left(F, \sqrt{1-r^2}\right) 192^{2n}2^{\kappa+2}\left(\frac{1}{2^{k-1}}\right)^{4n}(1-s^2) \\ &\le C_{6.6}192^{2n}2^{\kappa+2}2^{4n}\mathcal{N}\left(F, \sqrt{1-r^2}\right)\left(\frac{2^{2nk}}{2^{4nk}}\right)(1-s^2).\end{aligned}$$

Write $C = C_{6.6}192^{2n}2^{\kappa+2}2^{4n}$. Then for every $M \ge 4$ we have

$$\sum_{k=M}^{\infty} \sum_{\xi \in F \cap R_k} |\langle m^\kappa_{r\eta} m^\kappa_{s\xi}, m^\kappa_{s\xi}\rangle| \le C \sum_{k=M}^{\infty} \frac{1}{2^{2nk}}\mathcal{N}\left(F, \sqrt{1-r^2}\right)(1-s^2).$$

To complete the proof, it suffices to pick an $M \ge 4$ such that $C\sum_{k=M}^{\infty} 2^{-2nk} \le \alpha_0/4$. $\square$

7. The functions f_L and h_L

From now on, N and M will always denote the constants obtained in Lemmas 6.4 and 6.8 respectively. With these constants in hand, we are almost ready to define the functions f_L and h_L promised in Section 3. We say almost ready, because we need to establish one more counting lemma based on N and M. We alert the reader that this counting lemma is the place where the assumption $n \ge 2$ enters our construction in an essential way. In fact, as the reader can see, this is one lemma that fails in complex dimension 1.

Lemma 7.1. *There exists a natural number $q > M + N + 3$ such that for every $k \in \mathbf{N}$ and every $\eta \in S$, there exist $x_1, \dots, x_{2^{2q}} \in B(\eta, 2^{-k-N-1})$ which have the following properties:*

(1) *$B(x_i, 2^{-k-q+M+1}) \cap B(x_j, 2^{-k-q+M+1}) = \emptyset$ for every pair of $1 \le i < j \le 2^{2q}$.*
(2) *$B(x_i, 2^{-k-q+M+1}) \subset B(\eta, 2^{-k-N-1})$ for every $1 \le i \le 2^{2q}$.*

Proof. Since $n \ge 2$, we can pick a natural number $q > M + N + 3$ such that

$$2^{2(n-1)q} \ge 2^{2n(M+N)}2^{9n}A_0, \tag{7.1}$$

where A_0 is the constant that appears in (4.2). Let us show that this q has the desired properties. Given any $k \in \mathbf{N}$ and $\eta \in S$, let E be a subset of $B(\eta, 2^{-k-N-2})$ that is maximal with respect to the property that

$$B(x, 2^{-k-q+M+1}) \cap B(y, 2^{-k-q+M+1}) = \emptyset$$

if $x, y \in E$ and $x \neq y$. Since $q - M - 1 > N + 2$, if $x \in E$ and $\xi \in B(x, 2^{-k-q+M+1})$, then we have $d(\xi, \eta) \leq d(\xi, x) + d(x, \eta) < 2^{-k-q+M+1} + 2^{-k-N-2} < 2^{-k-N-1}$. That is,

$$B(x, 2^{-k-q+M+1}) \subset B(\eta, 2^{-k-N-1})$$

for every $x \in E$. Thus the proof will be complete once we show that $\operatorname{card}(E) \geq 2^{2q}$. Indeed the maximality of E implies that

$$\bigcup_{x \in E} B(x, 2^{-k-q+M+2}) \supset B(\eta, 2^{-k-N-2}).$$

By (4.2), this implies

$$\operatorname{card}(E) A_0 \left(2^{-k-q+M+2}\right)^{2n} \geq 2^{-n} \left(2^{-k-N-2}\right)^{2n}.$$

Thus

$$\operatorname{card}(E) \geq \frac{2^{2nq}}{2^{2n(M+N)} 2^{9n} A_0} = \left(\frac{2^{2(n-1)q}}{2^{2n(M+N)} 2^{9n} A_0} \right) 2^{2q}.$$

By (7.1), we have $\operatorname{card}(E) \geq 2^{2q}$. This completes the proof. □

To define f_L and h_L, we also need a good labelling system to match what is essentially a tree structure in our construction. For each $j \in \mathbf{N}$, let W_j be the collection of words of length j with $\{1, 2, \ldots, 2^{2q}\}$ as the set of alphabet, where q is, of course, the natural number provided by Lemma 7.1. That is,

$$W_j = \{\gamma_1 \cdots \gamma_j : \gamma_i \in \{1, 2, \ldots, 2^{2q}\},\ 1 \leq i \leq j\}.$$

Let W_0 denote the set of the empty word $\emptyset$. We will write $|\gamma|$ for the length of the word γ. That is, for $\gamma = \gamma_1 \cdots \gamma_j$ with $\gamma_1, \ldots, \gamma_j \in \{1, 2, \ldots 2^{2q}\}$, we define $|\gamma| = j$. The length of the empty word is defined to be 0. We "compose" words in the usual way. That is, for $\gamma = \gamma_1 \cdots \gamma_j \in W_j$ and $u = u_1 \cdots u_k \in W_k$, $k \geq 1$, we define

$$\gamma u = \gamma_1 \cdots \gamma_j u_1 \cdots u_k \in W_{j+k}.$$

For every word γ, the word $\emptyset\gamma$ is defined to be γ itself.

Next we fix a $k_0 \in \mathbf{N}$. For each natural number $j \geq 1$, we define

$$k_j = k_0 + jq,$$

where, again, q is the natural number provided by Lemma 7.1. Thus $k_j + q = k_{j+1}$. We pick an arbitrary $\eta_\emptyset \in S$. That is, we have defined $\eta_\gamma \in S$ in the case where γ is the empty word $\emptyset$. Suppose that $j \geq 0$ and that we have defined $\eta_\gamma \in S$ for every word $\gamma \in W_j$. By Lemma 7.1, for each $\gamma \in W_j$ there are $\eta_{\gamma 1}, \eta_{\gamma 2}, \ldots, \eta_{\gamma 2^{2q}} \in B(\eta_\gamma, 2^{-k_j-N-1})$ such that

$$B(\eta_{\gamma i}, 2^{-k_{j+1}+M+1}) \cap B(\eta_{\gamma i'}, 2^{-k_{j+1}+M+1}) = \emptyset \tag{7.2}$$

for all $1 \le i < i' \le 2^{2q}$ and such that

$$B(\eta_{\gamma i}, 2^{-k_{j+1}+M+1}) \subset B(\eta_\gamma, 2^{-k_j-N-1}) \tag{7.3}$$

for every $1 \le i \le 2^{2q}$. This defines $\{\eta_u : u \in W_{j+1}\} \subset S$. Inductively, this defines $\eta_\gamma \in S$ for every word $\gamma \in \cup_{j=0}^{\infty} W_j$. Note that (7.2) and (7.3) together imply that

$$B(\eta_\gamma, 2^{-k_j+M+1}) \cap B(\eta_{\gamma'}, 2^{-k_j+M+1}) = \emptyset \quad \text{for all} \quad \gamma \ne \gamma' \quad \text{in} \quad W_j, \tag{7.4}$$

$j \ge 1$. We now define

$$w_\gamma = (1 - 2^{-2k_{|\gamma|}})^{1/2} \eta_\gamma \tag{7.5}$$

for every $\gamma \in \cup_{j=0}^{\infty} W_j$.

With w_γ given by (7.5), we are ready to define h_L. For each $L \in \mathbf{N}$, we define

$$h_L = \sum_{j=0}^{L-1} \sum_{\gamma \in W_j} m_{w_\gamma}^\kappa .$$

But before we can define f_L, we need to make an estimate.

Lemma 7.2. *For every $L \in \mathbf{N}$ we have*

$$\left| \sum_{u \in W_L} \langle h_L m_{w_u}^\kappa, m_{w_u}^\kappa \rangle \right| \ge (\alpha_0/2) 2^{-2k_0} L,$$

where α_0 is the same as in Lemma 6.1.

Proof. Consider any word γ with $|\gamma| < L$. Then $W_L = \{\gamma W_{L-|\gamma|}\} \cup \{W_L \backslash \gamma W_{L-|\gamma|}\}$, where $\gamma W_{L-|\gamma|} = \{\gamma v : v \in W_{L-|\gamma|}\}$. By (7.3) and an induction on $|v|$, we see that $\eta_{\gamma v} \in B(\eta_\gamma, 2^{-k_{|\gamma|}-N})$. That is,

$$d(\eta_{\gamma v}, \eta_\gamma) < 2^{-k_{|\gamma|}-N} = 2^{-N}(1 - |w_\gamma|^2)^{1/2}.$$

Moreover,

$$1 - |w_{\gamma v}|^2 = 2^{-2k_{|\gamma v|}} = 2^{-2k_{|\gamma|}-2|v|q} = 2^{-2|v|q}(1 - |w_\gamma|^2) \le 2^{-2N}(1 - |w_\gamma|^2)$$

for every $v \in W_{L-|\gamma|}$. Thus it follows from Lemma 6.4 that

$$\sum_{v \in W_{L-|\gamma|}} \langle m_{w_\gamma}^\kappa m_{w_{\gamma v}}^\kappa, m_{w_{\gamma v}}^\kappa \rangle = \sum_{v \in W_{L-|\gamma|}} p(w_{\gamma v}, w_\gamma) + \sum_{v \in W_{L-|\gamma|}} c(w_{\gamma v}, w_\gamma), \tag{7.6}$$

where for every $v \in W_{L-|\gamma|}$, $p(w_{\gamma v}, w_\gamma)$ is a positive number satisfying the inequality

$$p(w_{\gamma v}, w_\gamma) \ge \alpha_0(1 - |w_{\gamma v}|^2) = \alpha_0 2^{-2k_{|\gamma v|}} = \alpha_0 2^{-2k_{|\gamma|}} 2^{-2(L-|\gamma|)q} \tag{7.7}$$

and

$$|c(w_{\gamma v}, w_\gamma)| \le (\alpha_0/4)(1-|w_{\gamma v}|^2) = (\alpha_0/4)2^{-2k_{|\gamma|}}2^{-2(L-|\gamma|)q}. \tag{7.8}$$

On the other hand, if $u \in W_L \backslash \gamma W_{L-|\gamma|}$, then $u = \gamma' v'$, where $\gamma' \in W_{|\gamma|}\backslash\{\gamma\}$ and $v' \in W_{L-|\gamma|}$. By (7.3), we have $\eta_u \in B(\eta_{\gamma'}, 2^{-|\gamma|-N})$. By (7.4), this means

$$\eta_u \notin B(\eta_\gamma, 2^{-k_{|\gamma|}+M+1}) \quad \text{for every} \quad u \in W_L\backslash\gamma W_{L-|\gamma|}.$$

Hence it follows from Lemma 6.8 that

$$\sum_{u\in W_L\backslash\gamma W_{L-|\gamma|}} |\langle m^\kappa_{w_\gamma} m^\kappa_{w_u}, m^\kappa_{w_u}\rangle| \le (\alpha_0/4)\mathcal{N}(H_L, 2^{-k_{|\gamma|}})2^{-2k_0-2Lq}, \tag{7.9}$$

where $H_L = \{\eta_u : u \in W_L\}$. We need to estimate $\mathcal{N}(H_L, 2^{-k_{|\gamma|}})$. If $x \in S$ is such that $B(x, 2^{-k_{|\gamma|}}) \cap B(\eta_a, 2^{-k_{|\gamma|}-N}) \ne \emptyset$ for some $a \in W_{|\gamma|}$, then by (7.4) we have $B(x, 2^{-k_{|\gamma|}}) \cap B(\eta_b, 2^{-k_{|\gamma|}-N}) = \emptyset$ for every $b \in W_{|\gamma|}\backslash\{a\}$. That is, if $B(x, 2^{-k_{|\gamma|}}) \cap B(\eta_a, 2^{-k_{|\gamma|}-N}) \ne \emptyset$, then $H_L \cap B(x, 2^{-k_{|\gamma|}}) \subset \{\eta_{av} : v \in W_{L-|\gamma|}\}$, which implies

$$\operatorname{card}(H_L \cap B(x, 2^{-k_{|\gamma|}})) \le 2^{2q(L-|\gamma|)}.$$

On the other hand, if $x \in S$ is such that $B(x, 2^{-k_{|\gamma|}}) \cap B(\eta_a, 2^{-|\gamma|-N}) = \emptyset$ for every $a \in W_{|\gamma|}$, then obviously $\operatorname{card}(H_L \cap B(x, 2^{-k_{|\gamma|}})) = 0$. Thus we conclude that

$$\mathcal{N}(H_L, 2^{-k_{|\gamma|}}) \le 2^{2q(L-|\gamma|)}. \tag{7.10}$$

Substituting this in (7.9), we obtain

$$\sum_{u\in W_L\backslash\gamma W_{L-|\gamma|}} |\langle m^\kappa_{w_\gamma} m^\kappa_{w_u}, m^\kappa_{w_u}\rangle| \le (\alpha_0/4)2^{-2k_0-2|\gamma|q}. \tag{7.11}$$

For convenience, let us write $G_L = \cup_{j=0}^{L-1} W_j$. Recalling (7.6), we have

$$\begin{aligned}
\sum_{u\in W_L} \langle h_L m^\kappa_{w_u}, m^\kappa_{w_u}\rangle &= \sum_{\gamma\in G_L}\sum_{u\in W_L} \langle m^\kappa_{w_\gamma} m^\kappa_{w_u}, m^\kappa_{w_u}\rangle \\
&= \sum_{\gamma\in G_L}\sum_{v\in W_{L-|\gamma|}} \langle m^\kappa_{w_\gamma} m^\kappa_{w_{\gamma v}}, m^\kappa_{w_{\gamma v}}\rangle + \sum_{\gamma\in G_L}\sum_{u\in W_L\backslash\gamma W_{L-|\gamma|}} \langle m^\kappa_{w_\gamma} m^\kappa_{w_u}, m^\kappa_{w_u}\rangle \\
&= \sum_{\gamma\in G_L}\sum_{v\in W_{L-|\gamma|}} p(w_{\gamma v}, w_\gamma) + T_1 + T_2,
\end{aligned}$$

where

$$T_1 = \sum_{\gamma\in G_L}\sum_{v\in W_{L-|\gamma|}} c(w_{\gamma v}, w_\gamma) \quad \text{and}$$

$$T_2 = \sum_{\gamma\in G_L}\sum_{u\in W_L\backslash\gamma W_{L-|\gamma|}} \langle m^\kappa_{w_\gamma} m^\kappa_{w_u}, m^\kappa_{w_u}\rangle.$$

Since $p(w_{\gamma v}, w_\gamma)$ is a positive number for every pair of $\gamma \in G_L$ and $v \in W_{L-|\gamma|}$, we have

$$\left|\sum_{u\in W_L} \langle h_L m_{w_u}^\kappa, m_{w_u}^\kappa\rangle\right| \geq \sum_{\gamma\in G_L} \sum_{v\in W_{L-|\gamma|}} p(w_{\gamma v}, w_\gamma) - |T_1| - |T_2|.$$

Note that $\mathrm{card}(W_j) = 2^{2jq}$ for every $j \geq 0$. Applying (7.7), we now have

$$\begin{aligned}\left|\sum_{u\in W_L} \langle h_L m_{w_u}^\kappa, m_{w_u}^\kappa\rangle\right| &\geq \alpha_0 \sum_{\gamma\in G_L} 2^{-2k_{|\gamma|}} - |T_1| - |T_2| \\ &= \alpha_0 \sum_{j=0}^{L-1} \mathrm{card}(W_j) 2^{-2k_j} - |T_1| - |T_2| \\ &= \alpha_0 2^{-2k_0} L - |T_1| - |T_2|. \qquad (7.12)\end{aligned}$$

On the other hand, by (7.8) we have

$$|T_1| \leq (\alpha_0/4) \sum_{j=0}^{L-1} \mathrm{card}(W_j)\mathrm{card}(W_{L-j}) 2^{-2k_j} 2^{-2(L-j)q} = (\alpha_0/4) 2^{-2k_0} L. \qquad (7.13)$$

Similarly, by (7.11) we have

$$|T_2| \leq (\alpha_0/4) \sum_{j=0}^{L-1} \mathrm{card}(W_j) 2^{-2k_0-2jq} = (\alpha_0/4) 2^{-2k_0} L. \qquad (7.14)$$

Substituting (7.13) and (7.14) in (7.12), the lemma follows. □

To define f_L, let $\nu : W_L \to \mathbf{Z}$ be an injective map. For each τ in the unit circle $\mathbf{T}$, we define the function

$$\varphi_{L,\tau} = \sum_{u\in W_L} \tau^{\nu(u)} m_{w_u}^\kappa.$$

Since $\nu(u) - \nu(u') \in \mathbf{Z}\backslash\{0\}$ for every pair of $u \neq u'$ in W_L, we have

$$\begin{aligned}\int \langle \varphi_{L,\tau} h_L, \varphi_{L,\tau}\rangle dm(\tau) &= \sum_{u,u'\in W_L} \langle h_L m_{w_u}^\kappa, m_{w_{u'}}^\kappa\rangle \int \tau^{\nu(u)-\nu(u')} dm(\tau) \\ &= \sum_{u\in W_L} \langle h_L m_{w_u}^\kappa, m_{w_u}^\kappa\rangle.\end{aligned}$$

Thus it follows from Lemma 7.2 that

$$\int |\langle \varphi_{L,\tau} h_L, \varphi_{L,\tau}\rangle| dm(\tau) \geq (\alpha_0/2) 2^{-2k_0} L.$$

In particular, this means that there exists a $\tau_L \in \mathbf{T}$ such that

$$|\langle \varphi_{L,\tau_L} h_L, \varphi_{L,\tau_L} \rangle| \geq (\alpha_0/2)2^{-2k_0} L.$$

With this $\tau_L \in \mathbf{T}$, we define $f_L = \varphi_{L,\tau_L}$. The above inequality tells us that (3.4) holds for the constant $\delta = (\alpha_0/2)2^{-2k_0}$.

Next we prove (3.3) and (3.2). By (7.4) and (7.5), for every $L \in \mathbf{N}$ the set

$$\bigcup_{j=0}^{L-1} \{w_\gamma : \gamma \in W_j\}$$

is a quasi-lattice in $\mathbf{B}$. Recall that $\kappa = 2n+2$, therefore $\kappa - 1 > 2n - 1$. Also recall (5.1). Hence it follows from Corollary 5.5 that

$$\begin{aligned} \|h_L\| &= \left\| \sum_{j=0}^{L-1} \sum_{\gamma \in W_j} m^\kappa_{w_\gamma} \right\| = \left\| \sum_{j=0}^{L-1} \sum_{\gamma \in W_j} (1 - |w_\gamma|^2)^{1/2} \psi_{w_\gamma, \kappa-1} \right\| \\ &\leq C_{5.5}(\kappa - 1) \left(\sum_{j=0}^{L-1} \sum_{\gamma \in W_j} (1 - |w_\gamma|^2) \right)^{1/2} = C_{5.5}(\kappa-1)(2^{-2k_0} L)^{1/2}. \end{aligned}$$

Hence (3.3) holds for the constant $C_{\mathrm{c}} = C_{5.5}(\kappa - 1)2^{-k_0}$. Similarly, since the set $\{w_u : u \in W_L\}$ is a quasi-lattice, Corollary 5.5 also gives us

$$\begin{aligned} \|f_L\| &= \left\| \sum_{u \in W_L} \tau_L^{\nu(u)} m^\kappa_{w_u} \right\| = \left\| \sum_{u \in W_L} \tau_L^{\nu(u)} (1 - |w_u|^2)^{1/2} \psi_{w_u, \kappa-1} \right\| \\ &\leq C_{5.5}(\kappa - 1) \left(\sum_{u \in W_L} (1 - |w_u|^2) \right)^{1/2} = C_{5.5}(\kappa - 1)2^{-k_0}. \end{aligned}$$

This proves (3.2).

The proof of (3.5) is easy. Indeed for every pair of $z \in \mathbf{B}$ and $h \in H^2_n$ we have

$$\langle M^*_{m^\kappa_z} 1, h \rangle = \langle 1, m^\kappa_z h \rangle = \overline{m^\kappa_z(0) h(0)} = (1 - |z|^2)^\kappa \langle 1, h \rangle.$$

Thus $M^*_{m^\kappa_z} 1 = (1 - |z|^2)^\kappa$. Consequently, if we write $a_L = M^*_{f_L} 1$, then $a_L \in \mathbf{C}$ and

$$|a_L| \leq \sum_{u \in W_L} (1 - |w_u|^2)^\kappa = 2^{2Lq} \cdot 2^{-2k_L \kappa} \leq 2^{-2k_L(\kappa - 1)} \tag{7.15}$$

(recall that $k_L = k_0 + Lq$). On the other hand, since $\|M^*_{m^\kappa_z}\| \leq 2^\kappa$ for $z \in \mathbf{B}$, we have

$$\|M^*_{f_L} h_L\| \leq 2^\kappa \cdot \mathrm{card}(W_L) \cdot \|h_L\| \leq 2^\kappa \cdot 2^{2Lq} \cdot C_{\mathrm{c}} L^{1/2}, \tag{7.16}$$

where for the second $\le$ we use (3.3). Combining (7.15), (7.16) and the Cauchy-Schwarz inequality, we obtain

$$|\langle M_{f_L} M_{f_L}^* h_L, 1\rangle| = |\langle M_{f_L}^* h_L, M_{f_L}^* 1\rangle| \le 2^\kappa C_{\mathrm{c}} L^{1/2} 2^{-2k_L(\kappa-2)}.$$

Since $\kappa - 2 = 2n$ and $k_L = k_0 + Lq$, this proves (3.5).

Thus only (3.1) remains to be proved.

8. More almost orthogonality

The proof of (3.1) will be based on a number of estimates of vector norms in H_n^2. We need some general lemmas.

Lemma 8.1. [11,Corollary 2.2] *For all $k \in \mathbf{Z}_+$ and $z \in \mathbf{B}$, the operator $M_{m_z^k}$ is subnormal on H_n^2.*

Our next lemma illustrates how to use m_w, $w \in \mathbf{B}$, as "partition" functions in H_n^2.

Lemma 8.2. *There exists a constant $C_{8.2}$ such that the following estimate holds: Suppose that $0 \le r < 1$ and that X is a subset of S satisfying the condition*

$$B(\xi, \sqrt{1-r^2}) \cap B(\xi', \sqrt{1-r^2}) = \emptyset. \tag{8.1}$$

for all $\xi \ne \xi'$ in X. Then for every set of vectors $\{h_\xi : \xi \in X\}$ in H_n^2 we have

$$\left\| \sum_{\xi \in X} m_{r\xi}^{\kappa-2} h_\xi \right\| \le C_{8.2} \left(\sum_{\xi \in X} \|h_\xi\|^2 \right)^{1/2}.$$

Proof. We need the following elementary fact from operator theory: If T is a hyponormal operator on a Hilbert space $\mathcal{H}$, then

$$\|T^* v\| \le \|Tv\| \quad \text{for every} \quad v \in \mathcal{H}. \tag{8.2}$$

Next, note that by (4.2), there exists a constant C such that for all r and X satisfying the condition in the lemma and for all $\eta \in S$ and $k \in \mathbf{Z}_+$, we have

$$\mathrm{card}\left\{\xi \in X : d(\xi, \eta) < 2^k \sqrt{1-r^2}\right\} \le C 2^{2nk}. \tag{8.3}$$

For each pair of $\xi \ne \xi'$ in X, we have

$$|\langle m_{r\xi}^{\kappa-2} h_\xi, m_{r\xi'}^{\kappa-2} h_{\xi'}\rangle| = |\langle M^*_{m_{r\xi'}^{\kappa-2}} m_{r\xi}^{\kappa-2} h_\xi, h_{\xi'}\rangle| \le \|M^*_{m_{r\xi'}^{\kappa-2}} m_{r\xi}^{\kappa-2} h_\xi\| \|h_{\xi'}\|.$$

Applying Lemma 8.1 and inequality (8.2), we have

$$|\langle m_{r\xi}^{\kappa-2} h_\xi, m_{r\xi'}^{\kappa-2} h_{\xi'}\rangle| \le \|m_{r\xi'}^{\kappa-2} m_{r\xi}^{\kappa-2} h_\xi\| \|h_{\xi'}\| \le \|m_{r\xi'}^{\kappa-2} m_{r\xi}^{\kappa-2}\|_{\mathcal{M}} \|h_\xi\| \|h_{\xi'}\|. \tag{8.4}$$

Recall that $\kappa - 2 = 2n$. By Lemma 6.5, we have

$$\begin{aligned}\|m_{r\xi'}^{\kappa-2}m_{r\xi}^{\kappa-2}\|_{\mathcal{M}} = \|(m_{r\xi'}m_{r\xi})^{\kappa-2}\|_{\mathcal{M}} &\le \|m_{r\xi'}m_{r\xi}\|_{\mathcal{M}}^{\kappa-2} \\ &\le 192^{2n}\left(\frac{1-r^2}{|1-\langle\xi,\xi'\rangle|}\right)^{2n} = 192^{2n}\left(\frac{\sqrt{1-r^2}}{d(\xi,\xi')}\right)^{4n}.\end{aligned}$$

Substituting this in (8.4), we obtain

$$|\langle m_{r\xi}^{\kappa-2}h_\xi, m_{r\xi'}^{\kappa-2}h_{\xi'}\rangle| \le 192^{2n}\left(\frac{\sqrt{1-r^2}}{d(\xi,\xi')}\right)^{4n}\|h_\xi\|\|h_{\xi'}\|. \tag{8.5}$$

For each $k \in \mathbf{N}$, define

$$E^{(k)} = \left\{(\xi,\xi') \in X \times X : 2^{k-1}\sqrt{1-r^2} \le d(\xi,\xi') < 2^k\sqrt{1-r^2}\right\}.$$

Then by (8.1) we have

$$\begin{aligned}\left\|\sum_{\xi\in X} m_{r\xi}^{\kappa-2}h_\xi\right\|^2 &= \sum_{\xi,\xi'\in X}\langle m_{r\xi}^{\kappa-2}h_\xi, m_{r\xi'}^{\kappa-2}h_{\xi'}\rangle \\ &= \sum_{\xi\in X}\|m_{r\xi}^{\kappa-2}h_\xi\|^2 + \sum_{k=1}^{\infty}\sum_{(\xi,\xi')\in E^{(k)}}\langle m_{r\xi}^{\kappa-2}h_\xi, m_{r\xi'}^{\kappa-2}h_{\xi'}\rangle. \end{aligned}\tag{8.6}$$

By (8.3) and Lemma 5.3, for each $k \in \mathbf{N}$ there is a partition

$$E^{(k)} = E_1^{(k)} \cup \cdots \cup E_{2m(k)}^{(k)}$$

with $m(k) \le C2^{2nk}$ such that for each $1 \le j \le 2m(k)$, $E_j^{(k)}$ has the property that for $(\xi,\xi'),(x,x') \in E_j^{(k)}$, the condition $(\xi,\xi') \ne (x,x')$ implies both $\xi \ne x$ and $\xi' \ne x'$. In other words, both projections $(\xi,\xi') \mapsto \xi$ and $(\xi,\xi') \mapsto \xi'$ are injective on $E_j^{(k)}$. By (8.5) and this injectivity, we have

$$\begin{aligned}\sum_{(\xi,\xi')\in E_j^{(k)}}|\langle m_{r\xi}^{\kappa-2}h_\xi, m_{r\xi'}^{\kappa-2}h_{\xi'}\rangle| &\le 192^{2n}\sum_{(\xi,\xi')\in E_j^{(k)}}\left(\frac{\sqrt{1-r^2}}{d(\xi,\xi')}\right)^{4n}\|h_\xi\|\|h_{\xi'}\| \\ &\le 192^{2n}2^{-4n(k-1)}\sum_{(\xi,\xi')\in E_j^{(k)}}\|h_\xi\|\|h_{\xi'}\| \le 192^{2n}2^{4n}2^{-4nk}\sum_{\xi\in X}\|h_\xi\|^2.\end{aligned}$$

Since $m(k) \le C2^{2nk}$, for each $k \in \mathbf{N}$ we now have

$$\begin{aligned}\sum_{(\xi,\xi')\in E^{(k)}}|\langle m_{r\xi}^{\kappa-2}h_\xi, m_{r\xi'}^{\kappa-2}h_{\xi'}\rangle| &= \sum_{j=1}^{2m(k)}\sum_{(\xi,\xi')\in E_j^{(k)}}|\langle m_{r\xi}^{\kappa-2}h_\xi, m_{r\xi'}^{\kappa-2}h_{\xi'}\rangle| \\ &\le 2C2^{2nk}192^{2n}2^{4n}2^{-4nk}\sum_{\xi\in X}\|h_\xi\|^2 = C_1 2^{-2nk}\sum_{\xi\in X}\|h_\xi\|^2,\end{aligned}$$

where $C_1 = 2C192^{2n}2^{4n}$. Combining this with (8.6), we obtain

$$\left\| \sum_{\xi \in X} m_{r\xi}^{\kappa-2} h_\xi \right\|^2 \le 2^{4n} \sum_{\xi \in X} \|h_\xi\|^2 + C_1 \sum_{k=1}^{\infty} 2^{-2nk} \sum_{\xi \in X} \|h_\xi\|^2.$$

Hence the lemma holds for the constant $C_{8.2} = (2^{4n} + C_1 \sum_{k=1}^{\infty} 2^{-2nk})^{1/2}$. $\square$

Lemma 8.3. *There is a constant $C_{8.3}$ such that for all $z, w \in \mathbf{B}$ satisfying the condition $|w| \le |z|$, we have*

$$\|m_w k_z\| \le C_{8.3} \frac{1 - |w|^2}{|1 - \langle z, w \rangle|}.$$

Proof. Let $z \in \mathbf{B}$ and let φ_z be the corresponding Möbius transform (see (4.3)). Recall that the formula

$$(U_z h)(\zeta) = h(\varphi_z(\zeta)) k_z(\zeta)$$

defines a unitary operator on H_n^2. Suppose that $w \in \mathbf{B}$ and that $|w| \le |z|$. It suffices to consider the case where $w \ne z$. Since $\varphi_z \circ \varphi_z = \mathrm{id}$, we have $\|m_w k_z\| = \|U_z(m_w \circ \varphi_z)\| = \|m_w \circ \varphi_z\|$. Therefore it suffices to estimate $\|m_w \circ \varphi_z\|$. The rest of the proof is similar to the proof of Lemma 2.4 in [11].

Set $\lambda = \varphi_z(w)$. Then $w = \varphi_z(\lambda)$. By [17,Theorem 2.2.2], we have

$$m_w(\varphi_z(\zeta)) = \frac{1 - |w|^2}{1 - \langle \varphi_z(\zeta), \varphi_z(\lambda) \rangle} = (1 - |w|^2) \frac{(1 - \langle \zeta, z \rangle)(1 - \langle z, \lambda \rangle)}{(1 - |z|^2)(1 - \langle \zeta, \lambda \rangle)}.$$

On the other hand, the same theorem gives us

$$1 - \langle z, \lambda \rangle = 1 - \langle \varphi_z(0), \varphi_z(w) \rangle = \frac{1 - |z|^2}{1 - \langle z, w \rangle}. \tag{8.7}$$

Therefore

$$m_w(\varphi_z(\zeta)) = \frac{1 - |w|^2}{1 - \langle z, w \rangle} \cdot \frac{1 - \langle \zeta, z \rangle}{1 - \langle \zeta, \lambda \rangle}.$$

Consequently, it suffices to consider the function $\psi(\zeta) = (1 - \langle \zeta, z \rangle)/(1 - \langle \zeta, \lambda \rangle)$.

Set $s = |\lambda|$. Let $U : \mathbf{C}^n \to \mathbf{C}^n$ be a unitary transformation such that

$$U^* \lambda = (s, 0, 0, \dots, 0) \quad \text{and}$$
$$U^* z = (\bar{a}, \bar{b}, 0, \dots, 0),$$

where $\bar{a} = \langle z, \lambda/s \rangle$ and $|b|^2 = |z|^2 - |\langle z, \lambda/s \rangle|^2$. Again by [17,Theorem 2.2.2], we have

$$1 - s^2 = 1 - |\varphi_z(w)|^2 = \frac{(1 - |z|^2)(1 - |w|^2)}{|1 - \langle w, z \rangle|^2}. \tag{8.8}$$

Since $1 - sa = 1 - \langle \lambda, z \rangle$ and $|z| \geq |w|$, it follows from (8.7) and (8.8) that

$$|1-a| \leq 2|1-sa| = 2\frac{1-|z|^2}{|1-\langle w, z\rangle|} \leq 2\sqrt{1-s^2}. \tag{8.9}$$

Also,

$$|b|^2 \leq 1 - |\langle z, \lambda \rangle|^2 \leq 2|1 - \langle z, \lambda \rangle| \leq 2\sqrt{1-s^2}.$$

Since $\|\psi\| = \|\psi \circ U\|$, it suffices to estimate the latter. We have

$$\psi(U\zeta) = \frac{1 - \langle U\zeta, z\rangle}{1 - \langle U\zeta, \lambda\rangle} = \frac{1 - a\zeta_1 - b\zeta_2}{1 - s\zeta_1} = \psi_1(\zeta) + \psi_2(\zeta) - \psi_3(\zeta),$$

where

$$\psi_1(\zeta) = \frac{1-a}{1-s\zeta_1}, \quad \psi_2(\zeta) = a\frac{1-\zeta_1}{1-s\zeta_1} \quad \text{and} \quad \psi_3(\zeta) = \frac{b\zeta_2}{1-s\zeta_1}.$$

Obviously, $\|\psi_1\| = |1-a|/\sqrt{1-s^2}$. Therefore, by (8.9), we have $\|\psi_1\| \leq 2$. Also, (4.1) gives us $\|\psi_2\| \leq 2$. On the other hand, since $\|\zeta^\alpha\|^2 = \alpha!/|\alpha|!$, $\alpha \in \mathbf{Z}_+^n$, we have

$$\|\psi_3\|^2 = |b|^2 \sum_{k=0}^{\infty} s^{2k} \|\zeta_1^k \zeta_2\|^2 = |b|^2 \sum_{k=0}^{\infty} \frac{s^{2k}}{k+1} \leq 2\sqrt{1-s^2}\left(1 + \log\frac{1}{1-s^2}\right).$$

Thus if we set

$$C = 2 \sup_{0 \leq t < 1} \sqrt{1-t^2}\left(1 + \log \frac{1}{1-t^2}\right),$$

which is finite, then we have $\|\psi_3\| \leq C^{1/2}$. Therefore the lemma holds for the constant $C_{8.3} = 4 + C^{1/2}$. □

With the above preparation, we are now ready to prove (3.1). Recall from Section 7 that $f_L = \varphi_{L,\tau_L}$ for a suitably chosen $\tau_L \in \mathbf{T}$. Obviously, we need to estimate $\|f_L k_z\|$ for all $L \in \mathbf{N}$ and $z \in \mathbf{B}$. By (7.4) and (7.5), we can apply Lemma 8.2 to obtain

$$\|f_L k_z\| = \left\| \sum_{u \in W_L} m_{w_u}^{\kappa - 2} \cdot \tau_L^{\nu(u)} m_{w_u}^2 k_z \right\| \leq C_{8.2} \left(\sum_{u \in W_L} \|m_{w_u}^2 k_z\|^2 \right)^{1/2}. \tag{8.10}$$

We divide the estimate of $\|f_L k_z\|$ into three cases, according to the value of $|z|$.

(1) Suppose that $\sqrt{1-|z|^2} \geq 2^{-k_0}$. This is the trivial case, for in this case we obvious have $\|f_L k_z\| = (1-|z|^2)^{-1/2}\|m_z f_L\| \leq 2(1-|z|^2)^{-1/2}\|f_L\|$. Thus by (3.2) we have

$$\|f_L k_z\| \leq 2^{k_0+1} C_{\mathrm{b}} \tag{8.11}$$

in the case $\sqrt{1-|z|^2} \geq 2^{-k_0}$.

(2) Suppose that $2^{-k_j} \le \sqrt{1-|z|^2} < 2^{-k_{j-1}}$ for some $1 \le j \le L$. In this case we have $|z| \le |w_u|$ for every $u \in W_L$. There is a $\xi \in S$ such that $z = |z|\xi$. With this ξ we set

$$\begin{aligned} X_0 &= B(\xi, 2^{-k_{j-1}}) \quad \text{and} \\ X_i &= B(\xi, 2^{-k_{j-1}+iq}) \backslash B(\xi, 2^{-k_{j-1}+(i-1)q}) \end{aligned}$$

for every $i \in \mathbf{N}$. For each $i \ge 0$, let $W_{L,i} = \{u \in W_L : \eta_u \in X_i\}$. By (8.10), we have

$$\|f_L k_z\| \le C_{8.2} \left(\sum_{i=0}^{\infty} g_i \right)^{1/2}, \tag{8.12}$$

where

$$g_i = \sum_{u \in W_{L,i}} \|m_{w_u}^2 k_z\|^2,$$

$i \ge 0$. Next we estimate each g_i. Note that

$$\|m_{w_u}^2 k_z\| = (1-|z|^2)^{-1/2} \|m_{w_u}^2 m_z\|.$$

For $u \in W_{L,0}$, we use the estimate $\|m_{w_u}^2 m_z\| \le 4\|m_{w_u}\| = 4\sqrt{1-|w_u|^2} = 4 \cdot 2^{-k_L}$. Consequently

$$\|m_{w_u}^2 k_z\| \le 4 \cdot 2^{k_j} 2^{-k_L} = 4 \cdot 2^{-(L-j)q}$$

for every $u \in W_{L,0}$. Hence $g_0 \le (4 \cdot 2^{-(L-j)q})^2 \mathrm{card}(W_{L,0})$. By (7.10) we have $\mathrm{card}(W_{L,0}) \le 2^{2q(L-j+1)}$. Therefore

$$g_0 \le 2^{2q+4}. \tag{8.13}$$

For $u \in W_{L,i}$ where $i \ge 1$, since $|z| \le |w_u|$, we use the inequality

$$\|m_{w_u} k_z\|_{\mathcal{M}} = (1-|z|^2)^{-1/2} \|m_{w_u} m_z\|_{\mathcal{M}} \le 96 \frac{\sqrt{1-|z|^2}}{|1-(|z|/|w_u|)\langle w_u, z\rangle|},$$

which follows from Lemma 6.5. Thus if $u \in W_{L,i}$, $i \ge 1$, then

$$\|m_{w_u} k_z\|_{\mathcal{M}} \le 192 \frac{\sqrt{1-|z|^2}}{d^2(\xi, \eta_u)} \le \frac{192}{2^{2(i-1)q}} \cdot 2^{k_{j-1}}.$$

Therefore for such a u we have

$$\|m_{w_u}^2 k_z\| \le \|m_{w_u} k_z\|_{\mathcal{M}} \|m_{w_u}\| \le \frac{192}{2^{2(i-1)q}} \cdot 2^{k_{j-1}} \cdot 2^{-k_L} \le \frac{192}{2^{2(i-1)q}} \cdot 2^{-(L-j)q}.$$

Consequently

$$g_i \le \left(\frac{192}{2^{2(i-1)q}} \cdot 2^{-(L-j)q} \right)^2 \mathrm{card}(W_{L,i}).$$

By (7.10), we have $\operatorname{card}(W_{L,i}) \le 2^{2q(L-j+1+i)}$. Substituting this in the above, we obtain

$$g_i \le \frac{(2^{3q}192)^2}{2^{2iq}}$$

for every $i \ge 1$. Combining this with (8.12) and (8.13), we find that

$$\|f_L k_z\| \le C_{8.2}\left(2^{2q+4} + (2^{3q}192)^2 \sum_{i=1}^{\infty} \frac{1}{2^{2iq}}\right)^{1/2} \tag{8.14}$$

in the case $2^{-k_j} \le \sqrt{1-|z|^2} < 2^{-k_{j-1}}$ for some $1 \le j \le L$.

(3) Suppose that $\sqrt{1-|z|^2} < 2^{-k_L}$. In this case we have $|z| > |w_u|$ for every $u \in W_L$. Again, there is a $\xi \in S$ such that $z = |z|\xi$, and we set

$$\begin{aligned} Y_0 &= B(\xi, 2^{-k_L}) \quad \text{and} \\ Y_i &= B(\xi, 2^{-k_L+iq}) \backslash B(\xi, 2^{-k_L+(i-1)q}) \end{aligned}$$

for every $i \in \mathbf{N}$. Let $\tilde{W}_{L,i} = \{u \in W_L : \eta_u \in Y_i\}$, $i \ge 0$. By (8.10), we have

$$\|f_L k_z\| \le C_{8.2}\left(\sum_{i=0}^{\infty} \tilde{g}_i\right)^{1/2}, \tag{8.15}$$

where

$$\tilde{g}_i = \sum_{u \in \tilde{W}_{L,i}} \|m_{w_u}^2 k_z\|^2 \le 4 \sum_{u \in \tilde{W}_{L,i}} \|m_{w_u} k_z\|^2,$$

$i \ge 0$. By (7.4), we have $\operatorname{card}(\tilde{W}_{L,0}) \le 1$. Therefore

$$\tilde{g}_0 \le 4 \cdot 4 = 16. \tag{8.16}$$

Suppose that $i \ge 1$. If $u \in \tilde{W}_{L,i}$, then by Lemma 8.3 we have

$$\|m_{w_u} k_z\| \le C_{8.3} \frac{1-|w_u|^2}{|1-\langle w_u, z\rangle|} \le 2C_{8.3} \frac{2^{-2k_L}}{d^2(\eta_u, \xi)} \le \frac{2^{2q+1}C_{8.3}}{2^{2iq}}.$$

By (7.10), we have $\operatorname{card}(\tilde{W}_{L,i}) \le 2^{2iq}$. Thus for each $i \ge 1$ we have

$$\tilde{g}_i \le 4 \cdot \left(\frac{2^{2q+1}C_{8.3}}{2^{2iq}}\right)^2 \cdot 2^{2iq} = \frac{(2^{2q+2}C_{8.3})^2}{2^{2iq}}.$$

Combining this with (8.15) and (8.16), we conclude that

$$\|f_L k_z\| \le C_{8.2}\left(16 + (2^{2q+2}C_{8.3})^2 \sum_{i=1}^{\infty} \frac{1}{2^{2iq}}\right)^{1/2} \tag{8.17}$$

in the case $\sqrt{1-|z|^2} < 2^{-k_L}$.

Finally, (3.1) follows from the combination of (8.11), (8.14) and (8.17). This completes the proofs of Theorems 2.2 and 2.3, and concludes our paper.

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