ANALYTICAL ASPECTS OF THE DRURY-ARVESON SPACE

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Abstract. We give a brief review of the Drury-Arveson space and its associated operator theory. We then survey those recent results on that space where analytical methods play a predominant role. A number of open problems are also discussed.

1. Introduction

The space that is now simply denoted H_n^2, first appeared in [24, 40, 41]. In [41], Lubin used this space to produce the first example of a tuple of commuting subnormal operators that does not admit a joint normal extension. Drury's motivation in [24] was to find the correct multi-operator analogue of the von Neumann inequality for contractions.

But it was Arveson's seminal paper [5] published twenty years ago that really brought H_n^2 to the attention of the operator-theory community. Although it initially went under various appellations, the community seems to have now settled on the name "Drury-Arveson space" for H_n^2. Therefore we will use the term "Drury-Arveson space" in this survey, as we have always done in our previous writings.

Let $\mathbf{B}$ be the open unit ball in $\mathbf{C}^n$. Throughout the article, the complex dimension n is always assumed to be greater than or equal to 2. Recall that the Drury-Arveson space H_n^2 is the Hilbert space of analytic functions on $\mathbf{B}$ that has the function

$$K_z(\zeta) = \frac{1}{1 - \langle \zeta, z\rangle} \tag{1.1}$$

as its reproducing kernel [5]. Equivalently, H_n^2 can be described as the Hilbert space of analytic functions on $\mathbf{B}$ where the inner product is given by

$$\langle h, g\rangle = \sum_{\alpha \in \mathbf{Z}_+^n} \frac{\alpha!}{|\alpha|!} a_\alpha \overline{b_\alpha}$$

for

$$h(\zeta) = \sum_{\alpha \in \mathbf{Z}_+^n} a_\alpha \zeta^\alpha \quad \text{and} \quad g(\zeta) = \sum_{\alpha \in \mathbf{Z}_+^n} b_\alpha \zeta^\alpha.$$

Here and throughout, we follow the standard multi-index notation [46, page 3].

Arveson derived the space H_n^2 from the view point of dilation theory, and he was the first to recognize that H_n^2 is a reproducing-kernel Hilbert space with (1.1) as its reproducing kernel [5]. Based on this Arveson regarded H_n^2 as an generalization of the Hardy space. At the same

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time, Arveson recognized that H^2_n is fundamentally different from the traditional Hardy space and Bergman space in that it cannot be defined in terms of a measure.

Today, we view the Drury-Arveson space, the Hardy space and the Bergman space all as members of a continuum family of reproducing-kernel Hilbert spaces parametrized by their "weight", $t \in [-n, \infty)$.

However one views the Drury-Arveson space, it is a place where one can do a lot of exciting operator theory, function theory and analysis. In the last twenty years, the collective effort of the operator-theory community has produced a huge body of literature on H^2_n. In this article we will offer some of our own perspectives on the theory of Drury-Arveson space. We will also take a look at some of the recent developments. This review is not intended to be comprehensive in any way. On the contrary, we will limit the article to the aspects of the Drury-Arveson space that are most familiar to us.

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2. von Neumann inequality for row contractions

Both Drury and Arveson arrived at the space H^2_n by considering the proper generalization of the von Neumann inequality to commuting tuples $(A_1, \dots, A_n)$ of operators. After initial experimentations [44, 48] with $\|\cdot\|_\infty$ as the "right-hand side" of such an inequality, it was quickly realized that a different norm is needed, and that the condition that each A_i individually be a contraction is not enough. The proper setting for such a generalization is the *row contarctions.*

A commuting tuple of operators $(A_1, \dots, A_n)$ on a Hilberts space $\mathcal{H}$ is said to be a row contraction if the operator inequality

$$A_1A_1^* + \cdots + A_nA_n^* \leq 1$$

holds on $\mathcal{H}$. Equivalently, $(A_1, \dots, A_n)$ is a row contraction if and only if

$$\|A_1x_1 + \cdots + A_nx_n\|^2 \leq \|x_1\|^2 + \cdots + \|x_n\|^2$$

for all $x_1, \dots, x_n \in \mathcal{H}$. For such a tuple, the von Neumann inequality reads

$$\|p(A_1, \dots, A_n)\| \leq \|p\|_{\mathcal{M}} \tag{2.1}$$

for every $p \in \mathbf{C}[z_1, \dots, z_n]$ [5, 24]. Here, $\|p\|_{\mathcal{M}}$ is the *multiplier norm* of p, the norm of the operator of multiplication by the polynomial p on H^2_n. Both Drury and Arveson showed that in general $\|p\|_{\mathcal{M}}$ is not dominated by the supremum norm of p on $\mathbf{B}$.

Drury's proof of (2.1) in [24] is particularly simple and can be easily explained here. First of all, Drury observed that in order to prove (2.1), it suffices to consider commuting tuples $A = (A_1, \dots, A_n)$ on $\mathcal{H}$ satisfying the condition

$$A_1A_1^* + \cdots + A_nA_n^* \leq r \tag{2.2}$$

for some $0 < r < 1$. For such a tuple, one easily verifies that the combinatorial identity

$$\sum_{\alpha \in \mathbf{Z}^n_+} \frac{|\alpha|!}{\alpha!} A^\alpha (1 - A_1A_1^* - \cdots - A_nA_n^*) A^{*\alpha} = 1 \tag{2.3}$$

holds. Now define the operator $Z : \mathcal{H} \to H^2_n \otimes \mathcal{H}$ by the formula

$$(Zx)(\zeta) = \sum_{\alpha \in \mathbf{Z}^n_+} \frac{|\alpha|!}{\alpha!}(1 - A_1A_1^* - \cdots - A_nA_n^*)^{1/2}A^{*\alpha}x\zeta^\alpha, \quad x \in \mathcal{H}. \tag{2.4}$$

Then (2.3) ensures that Z is an *isometry*. It is straightforward to verify that

$$Zp(A_1^*, \dots, A_n^*) = (p(M^*_{\zeta_1}, \dots, M^*_{\zeta_n}) \otimes 1)Z \tag{2.5}$$

for every $p \in \mathbf{C}[z_1, \dots, z_n]$. Since Z is an isometry, this implies (2.1).

We would like to make two comments at this point. First, if one is aware of the combinatorial identity (2.3), one can go backwards to figure out the proper definition for H^2_n based on the requirement that the operator Z defined by (2.4) be an isometry. Second, (2.3) is what one usually calls a *resolution* of the identity operator. Such resolutions can be exploited in various ways, and we will come back to this point later.

By Drury's combinatorial argument, for a commuting row contraction $(A_1, \dots, A_n) = A$, the resolution of identity (2.3) holds whenever

$$\lim_{k\to\infty} \sum_{|\alpha|=k} \frac{|\alpha|!}{\alpha!}A^\alpha A^{*\alpha} = 0 \tag{2.6}$$

in the strong operator topology. Obviously, (2.6) is a much weaker condition than (2.2). Thus (2.3) holds for many more commuting row contractions than those satisfying condition (2.2). A commuting row contraction satisfying (2.6) is said to be *pure* [5, 6].

3. The multipliers

A fundamental contribution that Arveson made in [5] was that he took the idea of $\|\cdot\|_{\mathcal{M}}$ one step further to introduce general *multipliers* for H^2_n, which turn out to be a constant source of fascination for the Drury-Arveson space community today. A function $f \in H^2_n$ is said to be a multiplier of the Drury-Arveson space if $fh \in H^2_n$ for every $h \in H^2_n$ [5]. We will write $\mathcal{M}$ for the collection of the multipliers of H^2_n. If $f \in \mathcal{M}$, then the multiplication operator M_f is bounded on H^2_n [5], and the multiplier norm $\|f\|_{\mathcal{M}}$ is defined to be the operator norm $\|M_f\|$ on H^2_n.

There are many more questions about the multipliers than there are answers. We begin with the few things that we do know about $\mathcal{M}$. The most significant piece of knowledge about $\mathcal{M}$ is the corona theorem due to Costea, Sawyer and Wick.

Theorem 3.1. [16] *For $g_1, \dots, g_k \in \mathcal{M}$, if there is a $c > 0$ such that*

$$|g_1(\zeta)| + \cdots + |g_k(\zeta)| \ge c$$

for every $\zeta \in \mathbf{B}$, then there exist $f_1, \dots, f_k \in \mathcal{M}$ such that

$$f_1g_1 + \cdots + f_kg_k = 1.$$

This is so far the only success story with regard to the corona theorem in the multi-variable setting. A special case of the corona theorem is the one-function corona theorem:

Corollary 3.2. *For $f \in \mathcal{M}$, if there is a $c > 0$ such that $|f(\zeta)| \ge c$ for every $\zeta \in \mathbf{B}$, then $1/f \in \mathcal{M}$.*

While the proof of the full version of the corona theorem involves difficult analysis [16] and is very long, the one-function corona theorem can be directly proved with a short, soft and elementary argument [30]. An essential ingredient in the direct proof of Corollary 3.2 in [30] was the von Neumann inequality (2.1). But it is worth mentioning that other than the use of (2.1), the rest of the proof in [30] was completely self-contained.

Since the publication of [30], there has been a proof of Corollary 3.2 in [45] that claims to be "significantly shorter". But the proof in [45] relies heavily on facts established in [1, 12, 43]. To begin with, the proof in [45] required the fact that $\mathcal{M}$ is contained in the collection $\mathcal{C}H_n^2$ defined in that paper, and for this inclusion [45] simply cited [43]. Thus if one follows the argument in [45], then much of the work for proving Corollary 3.2 was actually done in [1, 12, 43]. This differs significantly from our purpose in [30], namely to give a proof of Corollary 3.2 that does not require references.

From Corollary 3.2 we immediately obtain

Corollary 3.3. *Let $f \in \mathcal{M}$. Then the spectrum of the multiplication operator M_f on H_n^2 is contained in the closure of $\{f(z) : z \in \mathbf{B}\}$.*

In the case $f \in \mathbf{C}[z_1, \dots, z_n]$, this fact was already known to Arveson in [5], where he proved it using a Banach-algebra argument. The converse of Corollary 3.3, namely for $f \in \mathcal{M}$ the spectrum of M_f contains $\{f(z) : z \in \mathbf{B}\}$, is trivial. In fact, using the reproducing kernel, for every $f \in \mathcal{M}$ we have $M_f^* K_z = \overline{f(z)} K_z$, $z \in \mathbf{B}$. Thus for each $z \in \mathbf{B}$, $\overline{f(z)}$ is an eigenvalue for M_f^*.

Nowadays, one commonly views the Drury-Arveson space H_n^2 as a Hilbert module over the polynomial ring $\mathbf{C}[z_1, \dots, z_n]$. In fact, one often refers to H_n^2 as *the Drury-Arveson module*, in various contexts. But where there are modules, there are submodules. A *submodule* of the Drury-Arveson module is a closed linear subspace $\mathcal{S}$ of H_n^2 that is invariant under the multiplication by $\mathbf{C}[z_1, \dots, z_n]$. In other words, a submodule is what one would otherwise call an invariant subspace of H_n^2. The next proposition sets H_n^2 apart from other reproducing-kernel Hilbert spaces on $\mathbf{B}$.

Proposition 3.4. [6] *Let $\mathcal{S}$ be a submodule of the Drury-Arveson module. If $\mathcal{S} \neq \{0\}$, then $\mathcal{S} \cap \mathcal{M} \neq \{0\}$.*

One obvious consequence of Proposition 3.4 is the following:

Corollary 3.5. *Let $\mathcal{S}_1$, $\mathcal{S}_2$ be submodules of the Drury-Arveson module. If $\mathcal{S}_1 \neq \{0\}$ and $\mathcal{S}_2 \neq \{0\}$, then $\mathcal{S}_1 \cap \mathcal{S}_2 \neq \{0\}$.*

In the jargon of invariant-subspace theory, Corollary 3.5 says that there are no non-trivial invariant subspaces of H_n^2 that are *disjoint*. If $n = 1$, then this is a consequence of Beurling's theorem. So one might interpret Corollary 3.5 as saying that this one particular aspect of Beurling's theorem is retained by H_n^2 for all $n \geq 2$.

This is definitely not the case for some of the other reproducing-kernel Hilbert spaces. For example, one can easily construct invariant subspaces N_1, N_2 of the Bergman space $L_a^2(\mathbf{B}, dv)$ such that $N_1 \cap N_2 = \{0\}$ while $N_1 \neq \{0\}$ and $N_2 \neq \{0\}$. In particular, this implies that $N_1 \cap H^\infty(\mathbf{B}) = \{0\}$ and $N_2 \cap H^\infty(\mathbf{B}) = \{0\}$.

Incidentally, Proposition 3.4 can also be proved using Drury's ideas. In fact, define

$$\mathcal{Z}_{\mathcal{S},i} = M_{\zeta_i}|\mathcal{S}, \quad i = 1, \dots, n. \tag{3.1}$$

It is easy to verify that $(\mathcal{Z}_{\mathcal{S},1}, \dots, \mathcal{Z}_{\mathcal{S},n})$ is a row contraction on $\mathcal{S}$. Moreover, it is easy to show that (2.6) holds for the tuple $(A_1, \dots, A_n) = (\mathcal{Z}_{\mathcal{S},1}, \dots, \mathcal{Z}_{\mathcal{S},n})$, consequently so does the resolution of identity (2.3). Then, taking adjoints on both sides of (2.5), we find that

$$p(\mathcal{Z}_{\mathcal{S},1}, \dots, \mathcal{Z}_{\mathcal{S},n})Z^* = Z^*(p(M_{\zeta_1}, \dots, M_{\zeta_n}) \otimes 1), \tag{3.2}$$

$p \in \mathbf{C}[z_1, \dots, z_n]$. For each $y \in \mathcal{S}$, set $\varphi_y = Z^*(1 \otimes y) \in \mathcal{S}$. The above gives us

$$\|p\varphi_y\|_{H^2_n} = \|p\varphi_y\|_{\mathcal{S}} \le \|p \otimes y\|_{H^2_n \otimes \mathcal{S}} = \|p\|_{H^2_n}\|y\|_{\mathcal{S}}$$

for every $p \in \mathbf{C}[z_1, \dots, z_n]$. This means that $\varphi_y \in \mathcal{M}$. Finally, since the linear span of all $p\varphi_y = Z^*(p \otimes y)$ is dense in $\mathcal{S}$, the condition $\mathcal{S} \neq \{0\}$ implies that $\varphi_y \neq 0$ for some $y \in \mathcal{S}$.

The argument above has the virtue that it is based on (2.5), the same thing that establishes the von Neumann inequality for commuting row contractions. This connection clearly makes Proposition 3.4 a "close cousin" of the von Neumann inequality (2.1).

The above argument can be pushed even further. Let $\{u_j : j \in J\}$ be an orthonormal basis in $\mathcal{S}$. For each $j \in J$, write $\varphi_j = Z^*(1 \otimes u_j)$, which we now know is a multiplier for H^2_n. Let $\{f_j : j \in J\} \subset H^2_n$ be such that $\sum_{j \in J} \|f_j\|^2 < \infty$. Then it follows from (3.2) that

$$Z^* \sum_{j \in J} f_j \otimes u_j = \sum_{j \in J} f_j\varphi_j = \sum_{j \in J} M_{\varphi_j} f_j.$$

In other words, there is a unitary operator U such that Z^*U is the row operator

$$[M_{\varphi_1}, M_{\varphi_2}, \dots] : H^2_n \oplus H^2_n \oplus \cdots \to \mathcal{S}.$$

Since $Z^*Z = 1$ on $\mathcal{S}$, this leads to the representation

$$P_{\mathcal{S}} = \sum_{j \in J} M_{\varphi_j} M^*_{\varphi_j}$$

for the orthogonal projection $P_{\mathcal{S}} : H^2_n \to \mathcal{S}$, which first appeared in Arveson's paper [6].

Theorem 3.6. [27] *Let $\mathcal{S}$ be a submodule of the Drury-Arveson module H^2_n and define the corresponding defect operator*

$$D_{\mathcal{S}} = [\mathcal{Z}^*_{\mathcal{S},1}, \mathcal{Z}_{\mathcal{S},1}] + \cdots + [\mathcal{Z}^*_{\mathcal{S},n}, \mathcal{Z}_{\mathcal{S},n}].$$

Suppose that $\mathcal{S} \neq \{0\}$. Then there is an $\epsilon = \epsilon(\mathcal{S}) > 0$ such that

$$s_1(D_{\mathcal{S}}) + \cdots + s_k(D_{\mathcal{S}}) \ge \epsilon k^{(n-1)/n}$$

for every $k \in \mathbf{N}$. Consequently, $D_{\mathcal{S}}$ does not belong to the Schatten class $\mathcal{C}_n$.

Analogue of Theorem 3.6 also holds for submodules of the Hardy module [27]. But, thanks to Proposition 3.4, the proof of Theorem 3.6 is much easier than the proof of its analogue on the Hardy space. This is a good example of applications of multipliers. Incidentally, the analogue of Theorem 3.6 is not known to be true or false in the Bergman-space case, although one would certainly expect this analogue to be true. In fact, the possible Bergman-space analogue of Theorem 3.6 seems to be a very hard problem.

One reason the Bergman space and the Hardy space attract attention is that there are Toeplitz operators defined there, and every aspect of these operators is thoroughly studied. For the Drury-Arveson space H_n^2, $n \geq 2$, there is no L^2 space associated with it. Thus the only kind of "Toeplitz operators" on H_n^2 are the multipliers and their adjoints. Nevertheless, these operators retain certain familiar properties of Toeplitz operators. One noticeable such property is the essential commutativity. Arveson showed in [5] that the commutators $[M_{\zeta_i}^*, M_{\zeta_j}]$ on H_n^2 all belong to the Schatten class $\mathcal{C}_p$ for $p > n$. Such essential commutativity was later generalized to include multipliers:

Theorem 3.7. [28] *Let $f \in \mathcal{M}$ and $j \in \{1, \dots, n\}$. Then on H_n^2, the commutator $[M_f^*, M_{\zeta_j}]$ belongs to the Schatten class $\mathcal{C}_p$ for $p > 2n$. Moreover, for each $2n < p < \infty$, there is a C such that*

$$\|[M_f^*, M_{\zeta_j}]\|_p \leq C\|f\|_{\mathcal{M}}$$

for all $f \in \mathcal{M}$ and $j \in \{1, \dots, n\}$.

An enduring challenge in the theory of the Drury-Arveson space, since its very inception, has been the quest for a *good* characterization of the membership in $\mathcal{M}$. In other words, here we ask a very instinctive question, what does a general $f \in \mathcal{M}$ look like?

Let $k \in \mathbf{N}$ be such that $2k \geq n$. Then given any $f \in H_n^2 \cap H^\infty(\mathbf{B})$, one can define the measure $d\mu_f$ on $\mathbf{B}$ by the formula

$$d\mu_f(\zeta) = |(R^k f)(\zeta)|^2 (1 - |\zeta|^2)^{2k-n} dv(\zeta), \tag{3.3}$$

where dv is the normalized volume measure on $\mathbf{B}$ and R denotes the radial derivative $z_1\partial_1 + \cdots + z_n\partial_n$. Ortega and Fàbrega showed that f is a multiplier of the Drury-Arveson space if and only if $d\mu_f$ is an H_n^2-Carleson measure [43]. That is, $f \in \mathcal{M}$ if and only if there is a C such that

$$\int |h(\zeta)|^2 d\mu_f(\zeta) \leq C\|h\|^2$$

for every $h \in H_n^2$. In [4], Arcozzi, Rochberg and Sawyer gave a characterization for all the H_n^2-Carleson measures on $\mathbf{B}$.

For a given Borel measure on $\mathbf{B}$, the conditions in [4, Theorem 34] are not the easiest to verify. More to the point, [4, Theorem 34] deals with all Borel measures on $\mathbf{B}$, not just the class of measures $d\mu_f$ of the form (3.3). Thus it is natural to ask, is there a simpler, or a more direct, characterization of the membership $f \in \mathcal{M}$?

Since the Drury-Arveson space is a reproducing-kernel Hilbert space, it is natural to turn to the reproducing kernel for possible answers. Recall that the normalized reproducing kernel for H_n^2 is given by the formula

$$k_z(\zeta) = \frac{(1 - |z|^2)^{1/2}}{1 - \langle \zeta, z \rangle},$$

$z, \zeta \in \mathbf{B}$. For any $f \in H_n^2$, its Berezin transform

$$\langle f k_z, k_z \rangle$$

is none other than $f(z)$ itself. Given what is known about H_n^2 (see the related discussions in Section 4 below), one does not expect the boundedness of Berezin transform on $\mathbf{B}$ to be enough to guarantee the membership $f \in \mathcal{M}$. But what about something stronger than the Berezin transform of f? For example, anyone who gives any thought about multipliers is likely to come up with the natural question, for $f \in H_n^2$, does the condition

$$\sup_{|z|<1} \|fk_z\| < \infty$$

imply the membership $f \in \mathcal{M}$? Conditions of this type are now called "reproducing-kernel thesis" [42] and are among the first things that one would check when it comes to boundedness. One can rephrase the question thus: is it enough to determine the boundedness of M_f by testing it on the special subset $\{k_z : |z| < 1\}$ of the unit ball in H_n^2?

It is very tempting to think that the answer to the above question might be affirmative, and that was what we thought for quite a while. After all, an affirmative answer would provide a very simple characterization of the membership $f \in \mathcal{M}$. But that would be too simplistic, as it turns out. The answer is just the opposite:

Theorem 3.8. [32] *There exists an $f \in H_n^2$ satisfying the conditions $f \notin \mathcal{M}$ and*

$$\sup_{|z|<1} \|fk_z\| < \infty.$$

Thus, unfortunately, we are back where we started, looking for a non-trivial characterization of the membership $f \in \mathcal{M}$ that is reasonably simple and straightforward.

4. A family of reproducing-kernel Hilbert spaces

For each real number $-n \le t < \infty$, consider the kernel

$$\frac{1}{(1 - \langle \zeta, z\rangle)^{n+1+t}}, \quad \zeta, z \in \mathbf{B}.$$

Let $\mathcal{H}^{(t)}$ be the corresponding reproducing kernel Hilbert space of analytic functions on $\mathbf{B}$. Alternately, one can describe $\mathcal{H}^{(t)}$ as the completion of $\mathbf{C}[z_1, \dots, z_n]$ with respect to the norm $\|\cdot\|_t$ arising from the inner product $\langle \cdot, \cdot \rangle_t$ defined according to the following rules: $\langle z^\alpha, z^\beta \rangle_t = 0$ whenever $\alpha \ne \beta$,

$$\langle z^\alpha, z^\alpha \rangle_t = \frac{\alpha!}{\prod_{j=1}^{|\alpha|}(n + t + j)}$$

if $\alpha \in \mathbf{Z}_+^n \backslash \{0\}$, and $\langle 1, 1\rangle_t = 1$. Clearly, we have

$$\mathcal{H}^{(s)} \subset \mathcal{H}^{(t)}$$

for all $-n \le s < t < \infty$.

Obviously, $\mathcal{H}^{(0)}$ is the Bergman space $L_a^2(\mathbf{B}, dv)$, $\mathcal{H}^{(-1)}$ is the Hardy space $H^2(S)$, and $\mathcal{H}^{(-n)}$ is none other than the Drury-Arveson space H_n^2. Moreover, for each $-1 < t < \infty$, $\mathcal{H}^{(t)}$ is a weighted Bergman space. One can view the Bergman space $\mathcal{H}^{(0)} = L_a^2(\mathbf{B}, dv)$ as a benchmark, against which the other spaces in the family should be compared.

This suggests that for each $-n \le t < \infty$, we should think of t as the weight of the space $\mathcal{H}^{(t)}$. In particular, the Drury-Arveson space H_n^2 has weight $-n$. This approach tells us that the Drury-Arveson space is but one member of a continuum family of reproducing kernel Hilbert spaces. Thus, for example, if one obtains a result on one space of the family, then one should take it as a hint for things to come: does its analogue also hold for other spaces in the family?

There is no simple answer to this general question, and often different techniques are required for different spaces.

Using the radial derivative $R = z_1\partial_1 + \cdots + z_n\partial_n$, the norm $\|\cdot\|_t$ can be expressed in an equivalent form. Indeed for a given $-n \le t < \infty$, let m be any non-negative integer such that $2m + t > -1$. Then it is easy to verify that

$$\|f\|_t^2 \approx |f(0)|^2 + \int |(R^m f)(\zeta)|^2 (1 - |\zeta|^2)^{2m+t} dv(\zeta)$$

for $f \in \mathcal{H}^{(t)}$. For computations and estimates, the right-hand side is often more convenient.

For each $-n \le t < \infty$, consider $(M_{\zeta_1}^{(t)}, \ldots, M_{\zeta_n}^{(t)})$, the tuple of multiplication by the coordinate functions on $\mathcal{H}^{(t)}$. It was Lubin who first proved that, when $n \ge 2$, the tuple $(M_{\zeta_1}^{(-n)}, \ldots, M_{\zeta_n}^{(-n)})$ is not jointly subnormal [41]. Arveson arrived at the same conclusion in [5] by showing that there is some $p \in \mathbf{C}[z_1, \ldots, z_n]$ with the property that the operator of multiplication by p on H_n^2 is not hyponormal. Later, Arazy and Zhang showed that for each $-n < t < -1$, the tuple $(M_{\zeta_1}^{(t)}, \ldots, M_{\zeta_n}^{(t)})$ on $\mathcal{H}^{(t)}$ is also not jointly subnormal [3]. This means that when $-n < t < -1$, $\mathcal{H}^{(t)}$ has more in common with the Drury-Arveson space than with the Hardy space or the Bergman space.

In practical terms, the lack of joint subnormality for $(M_{\zeta_1}^{(t)}, \ldots, M_{\zeta_n}^{(t)})$ means that it is more difficult to do estimates on $\mathcal{H}^{(t)}$, and analytical results are hard to come by. This problem is particularly acute with H_n^2, $n \ge 2$. For example, the proof of Theorem 3.8 in [32] practically represents all the analytical techniques we have on H_n^2 at the moment.

One can even define $\mathcal{H}^{(t)}$ and $(M_{\zeta_1}^{(t)}, \ldots, M_{\zeta_n}^{(t)})$ in the range $-n - 1 < t < -n$. But when $t < -n$, the tuple $(M_{\zeta_1}^{(t)}, \ldots, M_{\zeta_n}^{(t)})$ is no longer a row contraction. That is why we only consider the weight range $-n \le t < \infty$.

Arveson was the first to notice that, when $n \ge 2$, H_n^2 does not contain $H^\infty(\mathbf{B})$, the collection of bounded analytic function on $\mathbf{B}$. In [5], he explicitly constructed an $f \in H^\infty(\mathbf{B})$ that does not belong to H_n^2. This construction was based on the function

$$\theta(\zeta_1, \ldots, \zeta_n) = \zeta_1 \cdots \zeta_n$$

on $\mathbf{B}$. Arveson observed that

$$\|\theta^k\|_\infty = \frac{1}{n^{kn/2}} \quad \text{while} \quad \|\theta^k\|_{H_n^2} = \left(\frac{(k!)^n}{(nk)!}\right)^{1/2} \approx \frac{k^{(n-1)/4}}{n^{kn/2}}.$$

Once this is seen, for $n \ge 2$ it is easy to come up with coefficients $a_0, a_1, \ldots a_k, \ldots$ such that $f = \sum_{k=0}^\infty a_k \theta^k$ is in $H^\infty(\mathbf{B})$ but not in H_n^2. In fact, one can even require f to be continuous on the closure of the unit ball $\mathbf{B}$.

It should be mentioned that examples of $f \in H^\infty(\mathbf{B})$, $f \notin H_n^2$ actually existed in plain sight. From the last chapter of Rudin's famous book [46] we know that when $n \ge 2$, if u is a non-constant inner function on $\mathbf{B}$, then $|\nabla u|$ is not square-integrable with respect to the volume measure on $\mathbf{B}$. Using the spaces introduced in this section, we can rephrase this result as saying that if u is a non-constant inner function on $\mathbf{B}$, then $u \notin \mathcal{H}^{(-2)}$. In particular, $\mathcal{H}^{(-n)} = H_n^2$ does not contain any non-constant inner function.

But when Rudin's book was published in 1980, it was not yet known whether non-constant inner functions existed in the case $n \geq 2$. In fact, Rudin offered the gradient result in his book as evidence against the existence of non-constant inner functions. But shortly thereafter, non-constant inner functions were successfully constructed by Løw [39] and Aleksandrov [2]. Amazingly, H_n^2 somehow misses all these functions!

We know that the Hardy space $H^2(S) = \mathcal{H}^{(-1)}$ contains all the inner functions, whereas $\mathcal{H}^{(-2)}$ contains none, other than the constants. This comparison raises an interesting open question: what about the weights $-2 < t < -1$? That is, if $-2 < t < -1$, does the space $\mathcal{H}^{(t)}$ contain any non-constant inner function? To us, this question is extremely interesting. But unfortunately we have no clue to offer, one way or the other.

5. Essential normality

If $\mathcal{S}$ is a submodule of H_n^2 as defined in Section 3, then $\mathcal{Q} = H_n^2 \ominus \mathcal{S}$ is a quotient module of the Drury-Arveson module. The term "quotient module" is justified by the fact that for $f, g \in \mathbf{C}[z_1, \dots, z_n]$ and $h \in H_n^2$, we have

$$P_{\mathcal{Q}} fgh = P_{\mathcal{Q}} f P_{\mathcal{Q}} gh,$$

where $P_{\mathcal{Q}} : H_n^2 \to \mathcal{Q}$ is the orthogonal projection. For a submodule $\mathcal{S}$, recall that the module operators $\mathcal{Z}_{\mathcal{S},1}, \dots, \mathcal{Z}_{\mathcal{S},n}$ are given by (3.1). For the corresponding quotient module $\mathcal{Q}$, we also have the module operators

$$\mathcal{Z}_{\mathcal{Q},i} = P_{\mathcal{Q}} M_{\zeta_i} | \mathcal{Q},$$

$i = 1, \dots, n$. Suppose that $\mathcal{F}$ is either a submodule or a quotient module. Then for $1 \leq p < \infty$, the module $\mathcal{F}$ is said to be *p-essentially normal* if the commutators

$$[\mathcal{Z}_{\mathcal{F},i}^*, \mathcal{Z}_{\mathcal{F},j}], \quad i, j \in \{1, \dots, n\},$$

all belong to the Schatten class $\mathcal{C}_p$. In this regard, we have the famous

Arveson Conjecture. [8, 9] Every graded submodule $\mathcal{G}$ of $H_n^2 \otimes \mathbf{C}^m$ is p-essentially normal for every $p > n$.

Here, the term "graded" means that $\mathcal{G}$ admits an orthogonal decomposition in terms of homogeneous polynomials. Arveson's original thinking was that the study of the module operators on such a $\mathcal{G}$ is really an operator-theoretic version of algebraic geometry, which has broad implications. There has been a lot of work on the Arveson Conjecture [8, 18, 26, 34, 35, 36, 37, 38, 47], and the best results to date are due to Guo and K. Wang [36].

Douglas made a more refined conjecture in [19] that also covers more modules. Together these essential-normality problems are now called the Arveson-Douglas Conjecture, which is quite all encompassing. We state a somewhat specialized version of it below:

Arveson-Douglas Conjecture. Suppose that I is an ideal in $\mathbf{C}[z_1, \dots, z_n]$, and let V_I denote the zero variety of I. Then the quotient module

$$\mathcal{Q} = \{h \in H_n^2 : h \perp I\}$$

is p-essentially normal for all $p > \dim_{\mathbf{C}} V_I$.

This is a more refined conjecture in the respect that it asserts $p > \dim_{\mathbf{C}} V_I$, which is more than just $p > n$. In practice, though, it is a serious challenge to reach the lower limit $p > \dim_{\mathbf{C}} V_I$, and this is true even on the Bergman space. However, as is shown in [49], reaching the lower limit $p > \dim_{\mathbf{C}} V_I$ leads to real applications.

If one only considers graded submodules, the work is inherently algebraic in nature. In 2011, Douglas and K. Wang made a breakthrough in which analysis became predominant:

Theorem 5.1. [22] *For every $q \in \mathbf{C}[z_1, \dots, z_n]$, the submodule $[q]$ of the Bergman module is p-essentially normal for every $p > n$.*

What is remarkable about this result is that it is *unconditional* in the sense that it makes no assumptions about the polynomial q. This sets a very high standard for all the essential-normality results to come. In [29], we were able to show that the analogue of Theorem 5.1 holds for the Hardy module $H^2(S)$.

As it turns out, for essential-normality problems on the unit ball, there is a very simple parameter that measures both progress and the level of difficulty: it is the weight t introduced in Section 3. Theorem 5.1 covers the weight $t = 0$, whereas our Hardy-space analogue [29] covers $t = -1$. It is the value of t that actually determines the difficulty of the problem: the more negative the value of t, the harder it is to solve the corresponding essential-normality problem. For the Drury-Arveson space analogue of Theorem 5.1, the level of difficulty is set at $t = -n$. The following is the best that we can do at the moment:

Theorem 5.2. [33] *Let q be an arbitrary polynomial in $\mathbf{C}[z_1, \dots, z_n]$. Then for every $-3 < t < \infty$, the submodule $[q]^{(t)}$ of $\mathcal{H}^{(t)}$ is p-essentially normal for every $p > n$.*

Specializing this to the weight $t = -2$ gives us the only unconditional essential normality that we have at the moment in a Drury-Arveson space case:

Corollary 5.3. *For every $q \in \mathbf{C}[z_1, z_2]$, the submodule $[q]$ of the two-variable Drury-Arveson module H^2_2 is p-essentially normal for every $p > 2$.*

For three variables, we must deal with the weight $t = -3$, which represents substantially more difficulty. At the moment, we only have partial results for the weight $t = -3$ [33], and even that requires non-trivial work. Although we are not able to solve it at the moment, the stumbling block for the case $t = -3$ can be stated as a very explicit estimate:

Problem 5.4. Denote $R = z_1\partial_1 + \cdots + z_n\partial_n$, the radial derivative on $\mathbf{B}$. Given an arbitrary $q \in \mathbf{C}[z_1, \dots, z_n]$, does there exist a constant $C = C(q)$ such that

$$\int |(Rq)(\zeta)f(\zeta)|^2 dv(\zeta) \le C \int |R(qf)(\zeta)|^2 dv(\zeta)$$

for every $f \in \mathbf{C}[z_1, \dots, z_n]$ satisfying the condition $f(0) = 0$?

For weights $t < -3$, the main difficulty can also be stated in terms of explicit estimates of this type [33, Definition 1.7]. In fact, we think that the value of these essential-normality problems lies precisely in the fact that they embody such non-trivial analysis.

6. Expanding on Drury's idea

A reasonable way to interpret the von Neumann inequality (2.1) is to say that the tuple $(M_{\zeta_1}, \dots, M_{\zeta_n})$ on H_n^2 "dominates" every other row contraction. In other words, the row contraction $(M_{\zeta_1}, \dots, M_{\zeta_n})$ on H_n^2 is the "master" among all row contractions. This interpretation of (2.1) inspires us to consider the following question. Suppose that we have two row contractions, $(A_1, \dots, A_n)$ and $(B_1, \dots, B_n)$. It seems fair to say that $(B_1, \dots, B_n)$ dominates $(A_1, \dots, A_n)$ if the inequality

$$\|p(A_1, \dots, A_n)\| \le \|p(B_1, \dots, B_n)\|$$

holds for every polynomial $p \in \mathbf{C}[z_1, \dots, z_n]$. Or, perhaps one can relax this condition somewhat: if there is a constant $0 < C < \infty$ such that

$$\|p(A_1, \dots, A_n)\| \le C\|p(B_1, \dots, B_n)\|$$

for every polynomial $p \in \mathbf{C}[z_1, \dots, z_n]$, one might still say that the tuple $(B_1, \dots, B_n)$ dominates the tuple $(A_1, \dots, A_n)$.

The main point is this: we can also ask the rather restricted question whether a given tuple $(B_1, \dots, B_n)$ dominates (whatever the word means) a particular $(A_1, \dots, A_n)$, not just the question whether it dominates a general class of $(A_1, \dots, A_n)$'s. In other words, the tuple $(B_1, \dots, B_n)$ may not be as dominating as the tuple $(M_{\zeta_1}, \dots, M_{\zeta_n})$ on H_n^2, but does it dominate a particular $(A_1, \dots, A_n)$ nonetheless?

The first hint of a possible hierarchical structure among commuting tuples comes from the fact that the Drury-Arveson space H_n^2 is really "the head" of the family of reproducing-kernel Hilbert spaces $\{\mathcal{H}^{(t)} : -n \le t < \infty\}$ introduced in Section 4. An obvious question is, what about the "lesser" tuples $(M_{\zeta_1}^{(t)}, \dots, M_{\zeta_n}^{(t)})$, $-n < t < \infty$. What do they dominate?

In other words, here we are asking whether there is some sort of hierarchy, albeit partial, among commuting tuples of operators. Obviously, such a general question represents a monumental undertaking, one that perhaps requires the efforts of many researchers over many years.

But one thing encouraging is that there are plenty of interesting examples of such a hierarchy. One way to construct such examples is to consider reproducing-kernel Hilbert spaces $\mathcal{H}^{(v)}$ that are even more general than the $\mathcal{H}^{(t)}$ introduced in Section 4.

Suppose that $v = \{v_\alpha : \alpha \in \mathbf{Z}_+^n\}$ is a set of positive numbers satisfying the condition

$$\sum_{\alpha \in \mathbf{Z}_+^n} v_\alpha |w^\alpha|^2 < \infty$$

for every $w \in \mathbf{B}$. We define an inner product $\langle \cdot, \cdot \rangle_v$ on $\mathbf{C}[z_1, \dots, z_n]$ according to the following rules: $\langle z^\alpha, z^\beta \rangle_v = 0$ whenever $\alpha \ne \beta$, and

$$\langle z^\alpha, z^\alpha \rangle_v = 1/v_\alpha$$

for $\alpha \in \mathbf{Z}_+^n$. Let $\|\cdot\|_v$ be the norm induced by the inner product $\langle \cdot, \cdot \rangle_v$, and let $\mathcal{H}^{(v)}$ be the Hilbert space obtained as the completion of $\mathbf{C}[z_1, \dots, z_n]$ with respect to $\|\cdot\|_v$. Let $(M_{\zeta_1}^{(v)}, \dots, M_{\zeta_n}^{(v)})$ be the tuple of multiplication by the coordinate functions on $\mathcal{H}^{(v)}$. Each $\mathcal{H}^{(v)}$ has its own collection of multipliers, and for each multiplier f we write $M_f^{(v)}$ for the operator of multiplication by f on $\mathcal{H}^{(v)}$.

For each $j \in \{1, \dots, n\}$, let ϵ_j denote the element in $\mathbf{Z}_+^n$ whose j-th component is 1 and whose other components are 0. If $\alpha \in \mathbf{Z}_+^n$ and if the j-th component of α is not 0, then, of course, $\alpha - \epsilon_j \in \mathbf{Z}_+^n$. Given a set of positive numbers $v = \{v_\alpha : \alpha \in \mathbf{Z}_+^n\}$, for $\alpha \in \mathbf{Z}_+^n$ we define $v_{\alpha-\epsilon_j} = 0$ if the j-th component of α is 0. Suppose that $v = \{v_\alpha : \alpha \in \mathbf{Z}_+^n\}$ satisfies the condition

$$\sum_{j=1}^{n} \frac{v_{\alpha-\epsilon_j}}{v_\alpha} \le 1 \quad \text{for every} \quad \alpha \in \mathbf{Z}_+^n.$$

Then it is easy to verify that $(M_{\zeta_1}^{(v)}, \dots, M_{\zeta_n}^{(v)})$ is a row contraction on $\mathcal{H}^{(v)}$.

In our view, identity (2.3) is the heart and soul of the theory of Drury-Arveson space. Therefore one way to uncover a possible hierarchy described above is to tinker with (2.3) and see what happens. For example, we can try to replace the coefficients $|\alpha|!/\alpha!$ in (2.3) with general v_α. If $|\alpha|!/\alpha!$ is replaced by v_α, then obviously the defect operator

$$D = 1 - A_1A_1^* - \cdots - A_nA_n^*$$

in (2.3) also needs to be replaced accordingly. But what replaces D?

Theorem 6.1. [31] *Let $A = (A_1, \dots, A_n)$ be a commuting row contraction on a Hilbert space H. Suppose that there is a positive operator W on H such that the sum*

$$Y = \sum_{\alpha \in \mathbf{Z}_+^n} v_\alpha A^\alpha W A^{*\alpha}$$

converges in the weak operator topology. Furthermore, suppose that the sum Y satisfies the operator inequality $c \le Y \le C$ on H for some scalars $0 < c \le C < \infty$. Then the inequality

$$\|f(A)\| \le (C/c)\|M_f^{(v)}\|$$

holds for every multiplier f of the space $\mathcal{H}^{(v)}$.

Let B be a bounded operator on a Hilbert space H. Then its *essential norm* is

$$\|B\|_{\mathcal{Q}} = \inf\{\|B + K\| : K \in \mathcal{K}(H)\},$$

where $\mathcal{K}(H)$ is the collection of compact operators on H.

Theorem 6.2. [31] *Let $A = (A_1, \dots, A_n)$ be a commuting row contraction on a separable Hilbert space H. Suppose that there is a positive, compact operator W on H such that the sum*

$$Y = \sum_{\alpha \in \mathbf{Z}_+^n} v_\alpha A^\alpha W A^{*\alpha}$$

converges in the weak operator topology. Furthermore, suppose that the operator Y has the following two properties:

(a) *There are scalars $0 < c \le C < \infty$ such that the operator inequality $c \le Y \le C$ holds on H;*

(b) *$Y = 1 + K$, where K is a compact operator on H.*

Then the inequality

$$\|f(A)\|_{\mathcal{Q}} \le \|M_f^{(v)}\|_{\mathcal{Q}}$$

holds for every multiplier f of the space $\mathcal{H}^{(v)}$.

Families of non-trivial (albeit quite technical) examples of A, v, W were given in [31]. In particular, if $-n < s < t < \infty$, then the tuple $(M^{(s)}_{\zeta_1}, \dots, M^{(s)}_{\zeta_n})$ dominates the tuple $(M^{(t)}_{\zeta_1}, \dots, M^{(t)}_{\zeta_n})$.

7. Closure of the polynomials

Let $\mathcal{A}$ be the closure of $\mathbf{C}[z_1, \dots, z_n]$ in $\mathcal{M}$ with respect to the multiplier norm. It is easy to understand that $\mathcal{A}$ is a special set of multipliers. Obviously, $\mathcal{A}$ is contained in $A(\mathbf{B})$, the ball algebra. Thus all multipliers in $\mathcal{A}$ are continuous on $\overline{\mathbf{B}}$, but not all continuous multipliers are in $\mathcal{A}$, although this latter statement is not completely trivial. Arveson showed in [5] that the maximal ideal space of $\mathcal{A}$ is homeomorphic to $\overline{\mathbf{B}}$.

In [13], Clouâtre and Davidson identified the first and second dual of $\mathcal{A}$: there is a commutative von Neumann algebra W such that

$$\mathcal{A}^* \simeq \mathcal{M}_* \oplus_1 W_* \quad \text{and} \quad \mathcal{A}^{**} \simeq \mathcal{M} \oplus_\infty W.$$

They established analogues of several classical results concerning the dual space of the ball algebra. These developments are deeply intertwined with the problem of peak interpolation for multipliers. It is also worth mentioning that these results shed light on the nature of the extreme points of the unit ball of $\mathcal{A}^*$.

Theorem 7.1. [13] *Let $f \in \mathcal{A}$ with $\|f\|_\infty < \|f\|_{\mathcal{M}} = 1$. The set*

$$\mathcal{F} = \{\Psi \in \mathcal{M}_* : \|\Psi\|_{\mathcal{A}^*} = 1 = \Psi(f)\}$$

has extreme points, which are also extreme points of the closed unit ball of $\mathcal{M}_*$.

Building on the work in [13], Clouâtre and Davidson further gave a complete characterization of absolutely continuous commuting row contractions in measure theoretic terms [14]. They also showed that completely non-unitary row contractions are necessarily absolutely continuous, which is a direct analogue of the single-operator case.

References

[1] J. Agler and J. McCarthy, Pick interpolation and Hilbert function spaces, Graduate Studies in Mathematics **44**, American Mathematical Society, Providence, RI, 2002.
[2] A. Aleksandrov, The existence of inner functions in a ball, (Russian) Mat. Sb. (N.S.) **118(160)** (1982), 147-163, 287; (English) Math. USSR Sbornik 46 (1983), 143-159.
[3] J. Arazy and G. Zhang, Homogeneous multiplication operators on bounded symmetric domains, J. Funct. Anal. **202** (2003), 44-66.
[4] N. Arcozzi, R. Rochberg and E. Sawyer, Carleson Measures for the Drury-Arveson Hardy space and other Besov-Sobolev spaces on complex Balls, Adv. Math. **218** (2008), 1107-1180.
[5] W. Arveson, Subalgebras of C^*-algebras. III. Multivariable operator theory, Acta Math. **181** (1998), 159-228.
[6] W. Arveson, The curvature invariant of a Hilbert module over $\mathbf{C}[z_1, \dots, z_d]$, J. Reine Angew. Math. **522** (2000), 173-236.
[7] W. Arveson, The Dirac operator of a commuting d-tuple, J. Funct. Anal. **189** (2002), 53-79.
[8] W. Arveson, p-summable commutators in dimension d, J. Operator Theory **54** (2005), 101-117.
[9] W. Arveson, Myhill Lectures, SUNY Buffalo, April 2006.
[10] W. Arveson, Quotients of standard Hilbert modules, Trans. Amer. Math. Soc. **359** (2007), 6027-6055.
[11] J. Ball, T. Trent and V. Vinnikov, Interpolation and commutant lifting for multipliers on reproducing kernel Hilbert spaces, Op. Th. Adv. and App. **122** (2001), 89-138.

[12] C. Cascante, J. Fàbrega, and J. Ortega, On weighted Toeplitz, big Hankel operators and Carleson measures, Integral Equations Operator Theory **66** (2010), 495-528.

[13] R. Clouâtre and K. Davidson, Duality, convexity and peak interpolation in the Drury-Arveson space, Adv. Math. **295** (2016), 90-149.

[14] R. Clouâtre and K. Davidson, Absolute continuity for commuting row contractions, J. Funct. Anal. **271** (2016), 620-641.

[15] R. Clouâtre and K. Davidson, Ideals in a multiplier algebra on the ball, Trans. Amer. Math. Soc. **370** (2018), 1509-1527.

[16] S. Costea, E. Sawyer and B. Wick, The corona theorem for the Drury-Arveson Hardy space and other holomorphic Besov-Sobolev spaces on the unit ball in $\mathbf{C}^n$, Anal. PDE **4** (2011), 499-550.

[17] K. Davidson, The mathematical legacy of William Arveson, J. Operator Theory **68** (2012), 307-334.

[18] R. Douglas, Essentially reductive Hilbert modules, J. Operator Theory **55** (2006), 117-133.

[19] R. Douglas, A new kind of index theorem, Analysis, geometry and topology of elliptic operators, 369-382, World Sci. Publ., Hackensack, NJ, 2006.

[20] R. Douglas, K. Guo and Y. Wang, On the p-essential normality of principal submodules of the Bergman module on strongly pseudoconvex domains, preprint 2017, arXiv:1708.04949

[21] R. Douglas, X. Tang and G. Yu, An analytic Grothendieck Riemann Roch theorem, Adv. Math. **294** (2016), 307-331.

[22] R. Douglas and K. Wang, Essential normality of cyclic submodule generated by any polynomial, J. Funct. Anal. **261** (2011), 3155-3180.

[23] R. Douglas and Y. Wang, Geometric Arveson-Douglas conjecture and holomorphic extension, Indiana Univ. Math. J. **66** (2017), 1499-1535.

[24] S. Drury, A generalization of von Neumann's inequality to the complex ball, Proc. Amer. Math. Soc. **68** (1978), 300-304.

[25] M. Engliš and J. Eschmeier, Geometric Arveson-Douglas conjecture, Adv. Math. **274** (2015), 606-630.

[26] J. Eschmeier, Essential normality of homogeneous submodules, Integr. Equ. Oper. Theory **69** (2011), 171-182.

[27] Q. Fang and J. Xia, Defect operators associated with submodules of the Hardy module, Indiana Univ. Math. J. **60** (2011), 729-749.

[28] Q. Fang and J. Xia, Commutators and localization on the Drury-Arveson space, J. Funct. Anal. **260** (2011), 639-673.

[29] Q. Fang and J. Xia, Essential normality of polynomial-generated submodules: Hardy space and beyond, J. Funct. Anal. **265** (2013), 2991-3008.

[30] Q. Fang and J. Xia, Corrigendum to "Multipliers and essential norm on the Drury-Arveson space", Proc. Amer. Math. Soc. **141** (2013), 363-368.

[31] Q. Fang and J. Xia, A hierarchy of von Neumann inequalities?, J. Operator Theory **72** (2014), 219-239.

[32] Q. Fang and J. Xia, On the problem of characterizing multipliers for the Drury-Arveson space, Indiana Univ. Math. J. **64** (2015), 663-696.

[33] Q. Fang and J. Xia, On the essential normality of principal submodules of the Drury-Arveson module, to appear in Indiana Univ. Math. J.

[34] K. Guo, Defect operators for submodules of H_d^2, J. Reine Angew. Math. **573** (2004), 181-209.

[35] K. Guo and K. Wang, Essentially normal Hilbert modules and K-homology. II. Quasi-homogeneous Hilbert modules over the two dimensional unit ball, J. Ramanujan Math. Soc., **22** (2007), 3, 259-281.

[36] K. Guo and K. Wang, Essentially normal Hilbert modules and K-homology, Math. Ann. **340** (2008), 907-934.

[37] K. Guo and Ch. Zhao, p-essential normality of quasi-homogeneous Drury-Arveson submodules, J. Lond. Math. Soc. (2) **87** (2013), 899-916.

[38] M. Kennedy and O. Shalit, Essential normality and the decomposability of algebraic varieties, New York J. Math. **18** (2012), 877-890.

[39] E. Løw, A construction of inner functions on the unit ball in $\mathbf{C}^p$, Invent. Math. **67** (1982), 223-229.

[40] A. Lubin, Models for commuting contractions, Michigan Math. J. **23** (1976), 161-165.

[41] A. Lubin, Weighted shifts and products of subnormal operators, Indiana Univ. Math. J. **26** (1977), 839-845.

[42] M. Mitkovski and B. Wick, A reproducing kernel thesis for operators on Bergman-type function spaces, J. Funct. Anal. **267** (2014), 2028-2055.

[43] J. Ortega and J. Fàbrega, Pointwise multipliers and decomposition theorems in analytic Besov spaces, Math. Z. **235** (2000), 53-81.

[44] S. Parrott, Unitary dilations for commuting contractions, Pacific J. Math. **34** (1970), 481-490.

[45] S. Richter and J. Sunkes, Hankel operators, invariant subspaces, and cyclic vectors in the Drury-Arveson space, Proc. Amer. Math. Soc. **144** (2016), 2575-2586.

[46] W. Rudin, Function theory in the unit ball of $\mathbf{C}^n$, Springer-Verlag, New York, 1980.

[47] O. Shalit, Stable polynomial division and essential normality of graded Hilbert modules, J. Lond. Math. Soc. (2) **83** (2011), 273-289.

[48] N. Varopoulos, On an inequality of von Neumann and an application of the metric theory of tensor products to operators theory, J. Funct. Anal. **16** (1974), 83-100.

[49] Y. Wang and J. Xia, Essential normality for quotient modules and complex dimensions, preprint 2018.

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