Nevanlinna-Pick interpolation via graph spaces and Kreĭn-space geometry: a survey

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Dedicated to Bill Helton, a dear friend and collaborator

Abstract. The Grassmannian/Kreĭn-space approach to interpolation theory introduced in the 1980s gives a Kreĭn-space geometry approach to arriving at the resolvent matrix which parametrizes the set of solutions to a Nevanlinna-Pick interpolation or Nehari-Takagi best-approximation problem. We review the basics of this approach and then discuss recent extensions to multivariable settings which were not anticipated in the 1980s.

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1. Introduction

We take this opportunity to update the Grassmannian approach to matrix- and operator-valued Nevanlinna-Pick interpolation theory introduced in [24]. It was a privilege for the first-named current author to be a participant with Bill Helton in the development of all these operator-theory ideas and their connections with Kreĭn-space projective geometry and engineering applications (in particular, circuit theory and control). Particularly memorable was the eureka moment when Bill observed that our J-Beurling-Lax representer was the same as the Adamjan-Arov-Kreĭn "resolvent matrix" Θ parameterizing all solutions of a Nehari-Takagi problem. This gave us an alternative way of constructing and understanding the origin of such resolvent matrices, and provided a converse direction for Bill's earlier results on orbits of matrix-function linear-fractional maps [48].

The present paper is organized as follows. Following this Introduction, in Section 2 we review the Grassmannian approach to the basic bitangential Sarason interpolation problem, including an indication of how the simplest bitangential matrix Nevanlinna-Pick interpolation problem is included as a special case. We also highlight along the way where some additional insight has been gained over the years. In Section 3 we show how a reformulation of the problem as a bitangential operator-argument interpolation problem leads to a set of coordinates which leads to state-space realization formulas for the Beurling-Lax representer, i.e., the resolvent matrix providing the linear-fractional parametrization for solutions of the interpolation problem. The rational case of this construction essentially appears in the book [17] while the general operator-valued case is more recent (see [30]). The final Section 4 surveys extensions of the Grassmannian method to more general settings, with the main focus on the results from [43] where it is shown that the Grassmannian approach applies to left-tangential operator-argument interpolation problems for contractive multipliers on the Drury-Arveson space (commuting variables) and on the Fock space (noncommuting variables).

2. The Sarason bitangential interpolation problem via the Grassmannian approach

We formulate the *bitangential Sarason* (**BTS**) interpolation problem as follows. Given an input Hilbert space $\mathcal{U}_I$ and an output Hilbert space $\mathcal{U}_O$, we let $H^\infty_{\mathcal{L}(\mathcal{U}_I,\mathcal{U}_O)}$ denote the space of bounded holomorphic functions on the unit disk $\mathbb{D}$ with values in the space $\mathcal{L}(\mathcal{U}_I,\mathcal{U}_O)$ of bounded linear operators between $\mathcal{U}_I$ and $\mathcal{U}_O$. We let $\mathcal{S}(\mathcal{U}_I,\mathcal{U}_O)$ denote the *Schur class* consisting of the elements of $H^\infty_{\mathcal{L}(\mathcal{U}_I,\mathcal{U}_O)}$ with infinity norm over the unit disk at most 1. For a general coefficient Hilbert space $\mathcal{U}$, an element B of $H^\infty_{\mathcal{L}(\mathcal{U})}$ is said to be *two-sided inner* if the nontangential strong-limit boundary-values $B(\zeta)$ of B on the unit circle $\mathbb{T}$ are unitary operators on $\mathcal{U}$ for almost all $\zeta \in \mathbb{T}$. The *data set* for a bitangential Sarason interpolation problem $\mathfrak{D}_{BTS}$ consists of a triple (S_0, B_I, B_O) where S_0 is a function in $H^\infty_{\mathcal{L}(\mathcal{U}_I,\mathcal{U}_O)}$, and B_I and B_O are two-sided inner functions with values in $\mathcal{L}(\mathcal{U}_I)$ and $\mathcal{L}(\mathcal{U}_O)$ respectively. Then we formulate the bitangential Sarason interpolation problem as follows:

Problem BTS (Bitangential Sarason Interpolation Problem): Given a data set $\mathfrak{D}_{BTS} = (S_0, B_I, B_O)$ as above, find $S \in \mathcal{S}(\mathcal{U}_I,\mathcal{U}_O)$ so that the function $Q := B_I^{-1}(S - S_0)B_O^{-1}$ is in $H^\infty_{\mathcal{L}(\mathcal{U}_I,\mathcal{U}_O)}$.

By way of motivation, let us consider the special case where $\mathcal{U}_I = \mathbb{C}^{n_I}$ and $\mathcal{U}_O = \mathbb{C}^{n_O}$ are finite-dimensional and where for simplicity we assume that $\det B_I$ and $\det B_O$ are finite Blaschke products of respective degrees n_I and n_O. Let us also assume that all zeros of $\det B_I$ and of $\det B_O$ are simple (but possibly overlapping). Then it is not hard to see that the **BTS** interpolation problem is equivalent to a *bitangential Nevanlinna-Pick* (**BTNP**) interpolation problem which we now describe. We suppose that we are given nonzero row vectors $x_1, \dots, x_{n_I}$ of size $1 \times n_O$, row vectors $y_1, \dots, y_{n_O}$ of size $1 \times n_I$, distinct points $z_1, \dots, z_{n_O}$ in $\mathbb{D}$ (the zeros of $\det B_O$), together with nonzero column vectors $u_1, \dots, u_{n_I}$ of size $n_I \times 1$, column vectors $v_1, \dots, v_{n_I}$ of size $n_O \times 1$, and distinct points $w_1, \dots, w_{n_I}$ in

$\mathbb{D}$ (the zeros of $\det B_I$, possibly overlapping with the z_i's), together with complex numbers ρ_{ij} for any pair of indices i, j such that $z_i = w_j =: \xi_{ij}$. The bitangential Nevanlinna-Pick problem then is:

Problem BTNP (Bitangential Nevanlinna-Pick interpolation problem): *Given a data set $\mathfrak{D} = \mathfrak{D}_{BTNP}$ given by*

$$\mathfrak{D} = \{(x_i, y_i, z_i) \text{ for } i = 1, \dots, n_O, (u_j, v_j, w_j) \text{ for } j = 1, \dots, n_I, \xi_{ij} \text{ for } z_i = w_j\}$$

as described above, find a matrix Schur-class function $S \in \mathcal{S}(\mathbb{C}^{n_I}, \mathbb{C}^{n_O})$ so that S satisfies the collection of interpolation conditions:

$$\begin{aligned} &x_i S(z_i) = y_i \text{ for } i = 1, \dots, n_O, \\ &S(w_j) u_j = v_j \text{ for } j = 1, \dots, n_I, \text{ and} \\ &x_i S'(\xi_{ij}) u_j = \rho_{ij} \text{ for } i, j \text{ such that } z_i = w_j =: \xi_{ij}. \end{aligned} \tag{2.1}$$

We remark that it is Donald Sarason [70] who first made this connection between the operator-theoretic interpolation problem Problem **BTS** and the classical point-by-point interpolation problem Problem **BTNP** for the scalar case.

We now present the solution of **BTS** problem as originally presented in [24, 26]. In addition to the function spaces $H^\infty_{\mathcal{L}(\mathcal{U}_I, \mathcal{U}_O)}$ already introduced above, let us now introduce the spaces of vector-valued functions $L^2_{\mathcal{U}}$ (measurable $\mathcal{U}$-valued functions on $\mathbb{T}$ which are norm-square integrable) and its subspace $H^2_{\mathcal{U}}$ consisting of those $L^2_{\mathcal{U}}$-functions with vanishing Fourier coefficients of negative index; as is standard, we can equivalently view $H^2_{\mathcal{U}}$ as holomorphic $\mathcal{U}$-valued functions f on the unit disk $\mathbb{D}$ for which the 2-norm over circles of radius r centered at the origin are uniformly bounded as r increases to 1. The space $L^2_{\mathcal{U}}$ comes equipped with the bilateral shift operator M_z of multiplication by the coordinate functions z (on the unit circle):

$$M_z \colon f(z) \mapsto z f(z).$$

When restricted to $H^2_{\mathcal{U}}$, we get the unilateral shift (of multiplicity equal to $\dim \mathcal{U}$—not included in the notation M_z). For F a function in $H^\infty_{\mathcal{L}(\mathcal{U}_I, \mathcal{U}_O)}$, there is an associated multiplication operator

$$M_F \colon f(z) \mapsto F(z) f(z)$$

mapping $H^2_{\mathcal{U}_I}$ into $H^2_{\mathcal{U}_O}$ and intertwining the respective shift operators: $M_F M_z = M_z M_F$. More generally, we may consider M_F as an operator from $L^2_{\mathcal{U}_I}$ into $L^2_{\mathcal{U}_O}$ which intertwines the respective bilateral shift operators; in this setting we need not restrict F to $H^\infty_{\mathcal{L}(\mathcal{U}_I, \mathcal{U}_O)}$ but may allow $F \in L^\infty_{\mathcal{L}(\mathcal{U}_I, \mathcal{U}_O)}$. A key feature of this correspondence between functions and operators is the correspondence of norms: given $F \in H^\infty_{\mathcal{L}(\mathcal{U}_I, \mathcal{U}_O)}$, the operator norm of M_F is the same as the supremum norm (over the unit disk or over the the unit circle) of the function F:

$$\|M_F\|_{op} = \|F\|_\infty := \sup\{\|F(z)\| \colon z \in \mathbb{D}\} = \text{ess-sup}\{\|F(\zeta)\| \colon \zeta \in \mathbb{T}\}.$$

Let us suppose that we are given a data set $\mathfrak{D}_{BTS} = (S_0, B_I, B_O)$ for a BTS problem as above. We introduce the space $\mathcal{K} = L^2_{\mathcal{U}_O} \oplus B_I^{-1} H^2_{\mathcal{U}_I}$ (elements of which will be written as column vectors $\left[\begin{smallmatrix} f \\ g \end{smallmatrix}\right]$ with $f \in L^2_{\mathcal{U}_O}$ and $g \in B_i^{-1} H^2_{\mathcal{U}_I}$). We use the signature matrix $J_{\mathcal{K}} := \left[\begin{smallmatrix} I_{\mathcal{U}_O} & 0 \\ 0 & -I_{\mathcal{U}_I} \end{smallmatrix}\right]$ to define a Kreĭn-space inner product on $\mathcal{K}$:

$$\left[\left[\begin{smallmatrix} f \\ B_I^{-1} g \end{smallmatrix}\right], \left[\begin{smallmatrix} f \\ B_I^{-1} g \end{smallmatrix}\right]\right]_{\mathcal{K}} := \left\langle J_{\mathcal{K}} \left[\begin{smallmatrix} f \\ B_I^{-1} g \end{smallmatrix}\right], \left[\begin{smallmatrix} f \\ B_I^{-1} g \end{smallmatrix}\right] \right\rangle_{L^2} = \|f\|^2_{L^2} - \|g\|^2_{H^2} \text{ for } \left[\begin{smallmatrix} f \\ B_I^{-1} g \end{smallmatrix}\right] \in \mathcal{K}.$$

We note that a *Kreĭn space* is simply a linear space $\mathcal{K}$ equipped with an indefinite inner product $[\cdot,\cdot]$ with respect to which $\mathcal{K}$ has an orthogonal decomposition $\mathcal{K} = \mathcal{K}_+ \oplus \mathcal{K}_-$ with $\mathcal{K}_+$ a Hilbert space in the $[\cdot,\cdot]$-inner product and $\mathcal{K}_-$ a Hilbert space in the $-[\cdot,\cdot]$-inner product; good references for more complete information are the books [8, 34]. We then consider the subspace $\mathcal{M}$ of $\mathcal{K}$ completely determined by the data set $\mathfrak{D}_{BTS} = (S_0, B_I, B_O)$:

$$\mathcal{M} := \mathcal{M}_{S_0,B_I,B_O} = \begin{bmatrix} B_O & S_0 B_I^{-1} \\ 0 & B_I^{-1} \end{bmatrix} \begin{bmatrix} H^2_{\mathcal{U}_O} \\ H^2_{\mathcal{U}_I} \end{bmatrix} \tag{2.2}$$

Then one checks that the function $S \in L^\infty_{\mathcal{L}(\mathcal{U}_I,\mathcal{U}_O)}$ is a solution of **BTS** problem if and only if its graph $\mathcal{G} := \left[\begin{smallmatrix} M_S \\ I \end{smallmatrix}\right] B_I^{-1} H^2_{\mathcal{U}_I}$ satisfies:

1. $\mathcal{G}$ *is a subspace of* $\mathcal{M}_{S_0,B_I,B_O}$ (and hence also is a subspace of $\mathcal{K}$),
2. $\mathcal{G}$ *is a negative subspace of* $\mathcal{K}$, i.e., $[g,g]_{\mathcal{K}} \le 0$ for all $g \in \mathcal{G}$, and, moreover, $\mathcal{G}$ is maximal with respect to this property: if $\mathcal{N}$ is another subspace of $\mathcal{K}$ with $\mathcal{G} \subset \mathcal{N}$, then $\mathcal{G} = \mathcal{N}$, and
3. $\mathcal{G}$ *is shift-invariant*, i.e., whenever $g \in \mathcal{G}$ then the vector function $\widetilde{g}$ given by $\widetilde{g}(z) = zg(z)$ is also in $\mathcal{G}$.

Let us verify each of these conditions in turn:

(1): If $S = S_0 + B_O Q B_I$ where $Q \in H^\infty_{\mathcal{L}(\mathcal{U}_I,\mathcal{U}_O)}$, then

$$\begin{aligned} \begin{bmatrix} M_S \\ I \end{bmatrix} B_I^{-1} H^2_{\mathcal{U}_I} &= \begin{bmatrix} M_{S_0} + M_{B_O} M_Q M_{B_I} \\ I \end{bmatrix} B_I^{-1} H^2_{\mathcal{U}_I} \\ &\subset \begin{bmatrix} B_O \cdot M_Q \\ 0 \end{bmatrix} H^2_{\mathcal{U}_I} + \begin{bmatrix} S_0 B_I^{-1} \\ B_I^{-1} \end{bmatrix} H^2_{\mathcal{U}_I} \ (\text{since } M_Q \colon H^2_{\mathcal{U}_I} \to H^2_{\mathcal{U}_O}) \\ &= \mathcal{M}_{S_0,B_I,B_O}. \end{aligned}$$

(2): If $S \in \mathcal{S}(\mathcal{U}_I, \mathcal{U}_O)$, then by the remarks above it follows that $\|M_S\|_{op} \le 1$. This is enough to imply that $\mathcal{G}$ is $\mathcal{K}$-maximal negative.

(3): Due to the intertwining properties of M_z mentioned above, we have

$$M_z \begin{bmatrix} M_S \\ I \end{bmatrix} M_{B_I}^{-1} H^2_{\mathcal{U}_I} = \begin{bmatrix} M_S \\ I \end{bmatrix} M_{B_I}^{-1} M_z H^2_{\mathcal{U}_I} \subset \begin{bmatrix} M_S \\ I \end{bmatrix} B_I^{-1} H^2_{\mathcal{U}_I}$$

from which we see that $\mathcal{G}$ is invariant under M_z.

Conversely, one can show that if $\mathcal{G}$ is *any* subspace of $\mathcal{K}$ satisfying conditions (1), (2), (3) above, then $\mathcal{G}$ has the form $\mathcal{G} = \left[\begin{smallmatrix} M_S \\ I \end{smallmatrix}\right] B_I^{-1} H^2_{\mathcal{U}_I}$ with S a solution of the BTS problem. Indeed, condition (2) forces $\mathcal{G}$ to be the graph space $\mathcal{G} = \left[\begin{smallmatrix} X \\ I \end{smallmatrix}\right] B_I^{-1} H^2_{\mathcal{U}_I}$ for a contraction operator $X \colon B_I^{-1} H^2_{\mathcal{U}_I} \to L^2_{\mathcal{U}_O}$. Condition (3) then forces X to be a multiplier $X = M_S$ for some $S \in L^\infty_{\mathcal{U}_I,\mathcal{U}_O}$ and $\|X\| \le 1$ implies that $\|S\|_\infty \le 1$. Finally, condition (1) then forces S to be of the form $S = S_0 + B_O K B_I$ with $K \in H^\infty_{\mathcal{L}(\mathcal{U}_I,\mathcal{U}_O)}$ from which we see that S is solution of the BTS problem.

Elementary Kreĭn-space geometry implies that, if there exists a $\mathcal{G}$ satisfying conditions (1) and (2), then necessarily the orthogonal complement of $\mathcal{M}$ inside $\mathcal{K}$ with respect to the indefinite Kreĭn-space inner product must be a positive subspace:

$$\mathcal{P} := \mathcal{P}_{S,B_I,B_O} = \mathcal{K} \ominus_J \mathcal{M}_{S,B_I,B_O} \text{ is a positive subspace,} \tag{2.3}$$

i.e., $[p,p]_\mathcal{K} \ge 0$ for all $p \in \mathcal{P}$.

We conclude that *the subspace* $\mathcal{P} := \mathcal{P}_{S_0,B_I,B_O}$ *being a positive subspace is a necessary condition for the existence of solutions to the* **BTS** *Problem.* More explicitly one can work out that positivity of $\mathcal{P}$ in (2.3) is equivalent to contractivity of the Sarason model operator:

$$\|T_{S_0,B_I,B_O}\| \le 1 \text{ where } T_{S_0,B_I,B_O} = P_{L^2_{\mathcal{U}_O} \ominus B_O H^2_{\mathcal{U}_O}} M_{S_0}|_{B_I^{-1} H^2_{\mathcal{U}_I}}. \tag{2.4}$$

In terms of the **BTNP** formulation, condition (2.3) translates to positive semidefiniteness of the associated Pick matrix $\Lambda_{\mathfrak{D}_{BTNP}}$:

$$\Lambda_{\mathfrak{D}_{BTNP}} := \begin{bmatrix} \Lambda_I & (\Lambda_{OI})^* \\ \Lambda_{OI} & \Lambda_O \end{bmatrix} \ge 0 \tag{2.5}$$

where

$$\Lambda_I = \left[\frac{u_i^* u_j - v_i v_j^*}{1 - \overline{w}_i w_j}\right], \quad [\Lambda_{OI}]_{ij} = \begin{cases} \frac{x_i v_j - y_i u_j}{w_j - z_i} & \text{for } z_i \ne w_j, \\ \rho_{ij} & \text{for } z_i = w_j \end{cases}, \ \Lambda_O = \left[\frac{x_i x_j^* - y_i y_j^*}{1 - z_i \overline{z}_j}\right].$$

To prove sufficiency of any of the three equivalent conditions (2.3), (2.4), (2.5), we must be able to show that solutions of the BTS problem exist when $\mathcal{P}$ is a positive subspace. Let us therefore suppose that the subspace $\mathcal{P} := \mathcal{P}_{S,B_I,B_O}$ is a positive subspace of $\mathcal{K}$. Then *any* subspace $\mathcal{G}$ contained in $\mathcal{M}_{S_0,B_I,B_O}$ which is maximal as a negative subspace of $\mathcal{M}_{S_0,B_I,B_O}$ is also maximal as a negative subspace of $\mathcal{K}$ (i.e., $\mathcal{M}_{S_0,B_I,B_O}$*-maximal negative implies* $\mathcal{K}$*-maximal negative*) and hence $\mathcal{G}$ satisfies conditions (1) and (2). The rub is to find such a $\mathcal{G}$ which also satisfies the shift-invariance condition (3).

It is at this point that we make a leap of faith and assume what is called in [9] the *Beurling-Lax Axiom: there exists a (bounded) J-unitary function $\Theta(z)$ so that*

$$\mathcal{M}_{S_0,B_I,B_O} = \Theta \cdot H^2_\mathcal{U} \tag{2.6}$$

for some appropriate Kreĭn space $\mathcal{U}$. Thus we assume that $\mathcal{U}$ has a Kreĭn-space inner product induced by a fundamental decomposition $\mathcal{U} = \mathcal{U}_+ \oplus \mathcal{U}_-$ with $\mathcal{U}_+$ a Hilbert space and $\mathcal{U}_-$ an anti-Hilbert space. More concretely, we simply take $\mathcal{U}_+$ and $\mathcal{U}_-$ to be Hilbert spaces and the Kreĭn-space inner product on $\mathcal{U} = \mathcal{U}_+ \oplus \mathcal{U}_-$ is given by

$$\left[\left[\begin{smallmatrix} u_+ \\ u_- \end{smallmatrix}\right], \left[\begin{smallmatrix} u_+ \\ u_- \end{smallmatrix}\right]\right]_\mathcal{U} = \|u_+\|^2_{\mathcal{U}_+} - \|u_-\|^2_{\mathcal{U}_-}.$$

The J-unitary property of Θ means that the values $\Theta(\zeta)$ of Θ are J-unitary for almost all ζ in the unit circle $\mathbb{T}$ (as a map between Kreĭn coefficient spaces $\mathcal{U}$ and $\mathcal{U}_O \oplus \mathcal{U}_I$ with the inner product induced by $J_K = \left[\begin{smallmatrix} I_{\mathcal{U}_O} & 0 \\ 0 & -I_{\mathcal{U}_I} \end{smallmatrix}\right]$). It then follows that without loss of generality we may take $\mathcal{U}_+ = \mathcal{U}_O$ and $\mathcal{U}_- = \mathcal{U}_I$. The crucial point is that then the operator M_Θ of multiplication by Θ *is a Kreĭn-space isomorphism* between $H^2_{\mathcal{U}}$ $(\mathcal{U} = \mathcal{U}_O \oplus \mathcal{U}_I)$ and $\mathcal{M}_{S_0,B_I,B_O}$, i.e., M_Θ maps $H^2_{\mathcal{U}}$ one-to-one and onto $\mathcal{M}_{S_0,B_I,B_O}$ and preserves the respective Kreĭn-space inner products:

$$[\Theta u, \Theta u]_{\mathcal{K}} = [u, u]_{\mathcal{U}},$$

and simultaneously *intertwines the respective shift operators*:

$$M_\Theta M_z = M_z M_\Theta.$$

It turns out that if condition (2.3) holds, then any such J-unitary representer Θ for $\mathcal{M}$ is actually J*-inner*, i.e., Θ has meromorphic pseudocontinuation to the unit disk $\mathbb{D}$ such that the values $\Theta(z)$ are J-contractive at all points of analyticity z inside the unit disk:

$$J - \Theta(z)^* J \Theta(z) \geq 0 \text{ for } z \in \mathbb{D}, \quad \Theta \text{ analytic at } z.$$

Under the assumption that we have such a representation (2.6) for $\mathcal{M}_{S_0,B_I,B_O}$, we can complete the solution of the **BTS** problem (under the assumption that the subspace $\mathcal{P}_{S_0,B_I,B_O}$ is a positive subspace) as follows. Since $M_\Theta \colon H^2_{\mathcal{U}} \to \mathcal{M}_{S_0,B_I,B_O}$ is a Kreĭn-space isomorphism, all the Kreĭn-space geometry is preserved. Thus a subspace $\mathcal{N}$ of $H^2_{\mathcal{U}}$ is maximal negative as a subspace of $H^2_{\mathcal{U}}$ if and only if its image $M_\Theta \mathcal{N} = \Theta \cdot \mathcal{N}$ is maximal negative as a subspace of $\mathcal{M}_{S_0,B_I,B_O}$. Moreover, since $M_z M_\Theta = M_\Theta M_z$, we see that $\mathcal{N}$ is shift-invariant in $H^2_{\mathcal{U}}$ if and only if its image $\Theta \cdot \mathcal{N}$ is a shift-invariant subspace of $\mathcal{M}_{S_0,B_I,B_O}$. From the observations made above, under the assumption that the subspace $\mathcal{P}_{S_0,B_I,B_O}$ is positive, getting a subspace $\mathcal{G}$ to satisfy conditions (1) and (2) in the Grassmannian reduction of the **BTS** problem is the same as getting $\mathcal{G} \subset \mathcal{M}_{S_0,B_I,B_O}$ to be maximal negative as a subspace of $\mathcal{M}_{S_0,B_I,B_O}$. We conclude that $\mathcal{G}$ meets all three conditions (1), (2), (3) in the Grassmannian reduction of the **BTS** problem if and only if $\mathcal{G} = \Theta \cdot \mathcal{N}$ where $\mathcal{N}$ is maximal negative as a subspace of $H^2_{\mathcal{U}}$ and is shift invariant. But these subspaces are easy: they are just subspaces of the form $\mathcal{N} = \left[\begin{smallmatrix} M_G \\ I \end{smallmatrix}\right] H^2_{\mathcal{U}_I}$ where G is in the Schur class $\mathcal{S}(\mathcal{U}_I, \mathcal{U}_O)$. We conclude that S solves the **BTS** problem if and only the graph $\mathcal{G}_S = \left[\begin{smallmatrix} M_S \\ I \end{smallmatrix}\right] B_I^{-1} H^2_{\mathcal{U}_I}$ satisfies

$$\begin{aligned} \begin{bmatrix} M_S \\ I \end{bmatrix} B_I^{-1} \cdot H^2_{\mathcal{U}_I} &= \Theta \cdot \begin{bmatrix} M_G \\ I \end{bmatrix} H^2_{\mathcal{U}_I} \\ &= \begin{bmatrix} \Theta_{11} G + \Theta_{12} \\ \Theta_{21} G + \Theta_{22} \end{bmatrix} \cdot H^2_{\mathcal{U}_I}. \end{aligned} \tag{2.7}$$

Next note that the operator $M_\Theta \left[\begin{smallmatrix} M_G \\ I \end{smallmatrix}\right]$, as the composition of injective maps, is injective as an operator acting on $H^2_{\mathcal{U}_O}$. We claim that the bottom component $M_{\Theta_{21} G + \Theta_{22}}$ is already injective. Indeed, if $(\Theta_{21} G + \Theta_{22}) h = 0$ for some nonzero

$h \in H^2_{\mathcal{U}_I}$, then $\left[\begin{smallmatrix} (\Theta_{11}G+\Theta_{12})h \\ 0 \end{smallmatrix}\right]$ would be a strictly positive element of the negative subspace $\Theta\left[\begin{smallmatrix} G \\ I \end{smallmatrix}\right] \cdot H^2_{\mathcal{U}_I}$, a contradiction. Thus $M_{\Theta_{21}G+\Theta_{22}}$ must be injective as claimed. From the identity of bottom components in (2.7), we see that multiplication by $\Theta_{21}G + \Theta_{22}$ maps $H^2_{\mathcal{U}_I}$ onto $B_I^{-1}H^2_I$. We conclude that the function $K := B_I(\Theta_{21}G+\Theta_{22})$ and its inverse are in $H^\infty_{\mathcal{L}(\mathcal{U}_I)}$. Then we may rewrite (2.7) as

$$\begin{aligned} \begin{bmatrix} S \\ I \end{bmatrix} B_I^{-1} \cdot H^2_{\mathcal{U}_I} &= \begin{bmatrix} (\Theta_{11}G+\Theta_{12})(\Theta_{21}G+\Theta_{22})^{-1} \\ I \end{bmatrix} B_I^{-1}K \cdot H^2_{\mathcal{U}_I} \\ &= \begin{bmatrix} (\Theta_{11}G+\Theta_{12})(\Theta_{21}G+\Theta_{22})^{-1} \\ I \end{bmatrix} B_I^{-1} \cdot H^2_{\mathcal{U}_I}. \end{aligned}$$

Thus for each $h \in B_I^{-1}H^2_{\mathcal{U}_I}$ there is an element h' of $B_I^{-1}H^2_{\mathcal{U}_I}$ such that

$$\begin{bmatrix} S \\ I \end{bmatrix} h = \begin{bmatrix} (\Theta_{11}G+\Theta_{12})(\Theta_{21}G+\Theta_{22})^{-1} \\ I \end{bmatrix} h'.$$

Equality of the bottom components forces $h = h'$ and then equality of the top components for all h leads to the linear-fractional parametrization for the set of solutions of the **BTS** problem: *S solves the* **BTS** *Problem if and only if S has the form*

$$S = (\Theta_{11}G+\Theta_{12})(\Theta_{21}G+\Theta_{22})^{-1} \tag{2.8}$$

for a uniquely determined $G \in \mathcal{S}(\mathcal{U}_I, \mathcal{U}_O)$. In this way we arrive at the linear-fractional parametrization of the set of all solutions appearing in the work of Nevanlinna [66] for the classical Nevanlinna-Pick interpolation problem and in the work of Adamjan-Arov-Kreĭn [1] in the context of the Nehari-Takagi problem.

Remark 2.1. We note that the derivation of the linear-fractional parametrization (2.8) used essentially only coordinate-free Kreĭn-space geometry. It is also possible to arrive at this parametrization without any appeal to Kreĭn-space geometry via working directly with properties of J-inner functions: see e.g. [17] where a winding number argument plays a key role, and [39] for an alternative reproducing-kernel method.

All the success of the preceding paragraphs is predicated on the validity of the so-called Beurling-Lax Axiom (2.6). Validity of the Beurling-Lax Axiom requires at a minimum that the subspace $\mathcal{M}_{S_0,B_I,B_O}$ be a Kreĭn space in the indefinite inner product inherited from $\mathcal{K}$. Unlike the Hilbert space case, this is not automatic (see e.g. [8, Section 1.7]). We say that the subspace $\mathcal{M}$ of the Kreĭn space $\mathcal{K}$ is *regular* if it is the case that $\mathcal{M}$ is itself a Kreĭn space with inner product inherited from $\mathcal{K}$; an equivalent condition is that $\mathcal{K}$ decomposes as an orthogonal (in the Kreĭn-space inner product) direct sum $\mathcal{K} = \mathcal{M} \oplus \mathcal{M}^{[\perp]}$ (where $\mathcal{M}^{[\perp]}$ is the orthogonal complement of $\mathcal{M}$ inside $\mathcal{K}$ with respect to the Kreĭn-space inner product). For the Nevanlinna-Pick problem involving only finitely many interpolation conditions, regularity of $\mathcal{M}$ is automatic under the condition that the solution of the interpolation problem is not unique (completely indeterminate in the language of some authors). Nevertheless, even when the subspace $\mathcal{M} =$

$\mathcal{M}_{S_0,B_I,B_O}$ is regular in $\mathcal{K} = \begin{bmatrix} L^2_{\mathcal{U}_O} \\ B_I^{-1} H^2_{\mathcal{U}_I} \end{bmatrix}$, it can happen that only a weakened version of the Beurling-Lax Axiom holds. The following is one of the main results from [24] (see [56] for extensions to shift-invariant subspaces contractively included in $H^2_{\mathcal{U}}$).

Theorem 2.2. *Suppose that $\mathcal{M}$ is a regular subspace of $L^2_{\mathcal{U}}$ (where $L^2_{\mathcal{U}}$ is considered to be a Kreĭn space in the indefinite inner product induced by the Kreĭn-space inner product on the space of constants $\mathcal{U} = \mathcal{U}_O \oplus \mathcal{U}_I$). Then there exists a multiplier Θ with values in $\mathcal{L}(\mathcal{U})$ such that*

1. $M_{\Theta^{\pm 1}} : \mathcal{U} \to L^2_{\mathcal{U}}$,
2. $\Theta(\zeta)^* J \Theta(\zeta) = J$ *for almost all* $\zeta \in \mathbb{T}$ *(where* $J = \begin{bmatrix} I_{\mathcal{U}_O} & 0 \\ 0 & -I_{\mathcal{U}_I} \end{bmatrix}$*),*
3. *the densely defined operator* $M_\Theta P_{H^2_{\mathcal{U}}} M_{\Theta^{-1}} = M_\Theta P_{H^2_{\mathcal{U}}} J M_{\Theta^*} J$ *extends to define a bounded J-orthogonal projection operator on* $L^2_{\mathcal{U}}$*, and*
4. *the space $\mathcal{M}$ is equal to the closure of $\Theta \cdot (H^2_{\mathcal{U}})_0$, where $(H^2_{\mathcal{U}})_0$ is the space of analytic trigonometric polynomials $p(\zeta) = \sum_{k=0}^n u_k \zeta^k$ with coefficients u_k in $\mathcal{U}$ $(n = 0, 1, 2, \dots)$.*

Conversely, whenever Θ is a multiplier satisfying conditions (1), (2), and (3) and the subspace $\mathcal{M}$ is defined via (4), then $\mathcal{M}$ is a regular subspace of $L^2_{\mathcal{U}}$ (with J-orthogonal projection onto $\mathcal{M}$ along $\mathcal{M}^{[\perp]}$ given by the bounded extension of $M_{\Theta^{-1}} P_{H^2_{\mathcal{U}}} M_\Theta$ onto all of $L^2_{\mathcal{U}}$).

This illustrates a general phenomenon in the Kreĭn-space setting in contrast with the Hilbert-space setting: there is no reason why unitary operators need be bounded. The moral of the story is: the Beurling-Lax Axiom does hold in case $\mathcal{M}_{S_0,B_I,B_O}$ is a regular subspace of $\mathcal{K}$, but only with in general densely defined and unbounded Beurling-Lax representer Θ. This technical detail in turn complicates the Kreĭn-space geometry argument given above leading to the existence and parametrization of the set of all solutions of the BTS Problem under the necessary condition (2.3) that the subspace $\mathcal{P}_{\mathcal{S}_0,B_I,B_O}$ be a positive subspace. This point was handled in [24] (and revisited in [29]) via an approximation argument using the fact that bounded functions are dense in any shift-invariant subspace of H^2.

Here we use an idea from [43] based on ingredient from the approach of Dym [39] to obtain a smoother derivation of the linear-fractional parametrization even for the case where the Beurling-Lax representer may be unbounded. The following lemma proves to be helpful.

Lemma 2.3. *(See Lemma 2.3.1 in* [43]*.) Let $\mathcal{K}$ be a Kreĭn space and let $\mathcal{M}$ be a regular subspace of $\mathcal{K}$ such that $\mathcal{M}^{[\perp]}$ is a positive subspace. If $\mathcal{G}$ is a maximal negative subspace of $\mathcal{K}$, then the following are equivalent:*

1. $\mathcal{G} \subset \mathcal{M}$.
2. $P_{\mathcal{M}} \mathcal{G}^{[\perp]}$ *is a positive subspace, where* $P_{\mathcal{M}}$ *is the J-orthogonal projection of $\mathcal{K}$ onto $\mathcal{M}$.*

Now we suppose that $\mathcal{P}_{S_0,B_I,B_O}$ is a positive subspace (as is necessary for solutions to the **BTS** problem to exist) and that $S \in \mathcal{S}(\mathcal{U}_I, \mathcal{U}_O)$ is a solution. Thus $\mathcal{G} = \left[\begin{smallmatrix} S \\ I \end{smallmatrix}\right] B_I^{-1} \cdot H^2_{\mathcal{U}_I}$ is maximal negative and contained in $\mathcal{M}_{S_0,B_I,B_O}$. According to the lemma, this means that $P_{\mathcal{M}}\mathcal{G}^{[\perp]}$ is a positive subspace. By the result of Theorem 2.2 we know that $P_{\mathcal{M}} = M_\Theta J P_{H^2_{\mathcal{U}}} M_{\Theta^*} J$ (formally unbounded but having bounded extension to the whole space). Also, an elementary computation gives

$$\mathcal{G}^{[\perp]} = \begin{bmatrix} I \\ P_{B_I^{-1}H^2_{\mathcal{U}_I}} M_{S^*} \end{bmatrix} L^2_{\mathcal{U}_O}.$$

Thus the condition (2) in Lemma 2.3 becomes

$$\left\langle JM_\Theta J P_{H^2} M_{\Theta^*} J \begin{bmatrix} I \\ P_{B_I^{-1}H^2_{\mathcal{U}_I}} M_{S^*} \end{bmatrix} f, \begin{bmatrix} I \\ P_{B_I^{-1}H^2_{\mathcal{U}_I}} M_{S^*} \end{bmatrix} f \right\rangle_{L^2 \oplus B_I^{-1} H^2} \geq 0 \tag{2.9}$$

for all $f \in L^2_{\mathcal{U}_O}$. Since the range of M_Θ is contained in $\mathcal{M}$ which in turn is contained in $\begin{bmatrix} L^2_{\mathcal{U}_O} \\ B_I^{-1} H^2_{\mathcal{U}_I} \end{bmatrix}$, we see that the projection $P_{B_I^{-1}H^2_{\mathcal{U}_I}}$ in (2.9) is removable. We can then rewrite (2.9) as

$$\left\langle \begin{bmatrix} I & -M_S \end{bmatrix} M_\Theta J P_{H^2} M_{\Theta^*} \begin{bmatrix} I \\ -M_{S^*} \end{bmatrix} f, f \right\rangle \geq 0.$$

Restricting to an appropriate dense domain and writing F in place of M_F for multiplication operators for simplicity, we arrive at the operator inequality

$$\begin{aligned} 0 &\leq \begin{bmatrix} \Theta_{11} - S\Theta_{21} & \Theta_{12} - S\Theta_{22} \end{bmatrix} P_{H^2} \begin{bmatrix} \Theta_{11}^* - \Theta_{21}^* S^* \\ \Theta_{12}^* - \Theta_{22}^* S^* \end{bmatrix} \\ &= (\Theta_{11} - S\Theta_{21}) P_{H^2} (\Theta_{11} - S\Theta_{21})^* - (\Theta_{12} - S\Theta_{22}) P_{H^2} (\Theta_{12} - S\Theta_{22})^*. \end{aligned} \tag{2.10}$$

It is a consequence of the Commutant Lifting Theorem (in this form actually a version of the Leech Theorem—see [68]) that (2.10) implies that there is a Schur-class function written as $-G \in \mathcal{S}(\mathcal{U}_I, \mathcal{U}_O)$ so that

$$\Theta_{12} - S\Theta_{22} = (\Theta_{11} - S\Theta_{21})(-G).$$

It is now a straightforward matter to solve for S in terms of G to arrive at

$$S = (\Theta_{11} G + \Theta_{12})(\Theta_{21} G + \Theta_{22})^{-1}. \tag{2.11}$$

Conversely the steps are reversible: for any Schur-class function $G \in \mathcal{S}(\mathcal{U}_I, \mathcal{U}_O)$, the formula (2.11) leads to a solution S of the **BTS** problem. In this way we arrive at the linear-fractional parametrization (2.8) for the set of all solutions of the **BTS** problem even in the case where $\mathcal{M}$ is regular but its Beurling-Lax representer Θ is not bounded.

Remark 2.4. The case where $\mathcal{M}$ is regular is only a particular instance of the so-called "completely indeterminate case" where solutions of the BTS Problem exist having norm strictly less than 1. In this case there is still a linear-fractional parametrization of the set of all solutions of the form (2.8) even though the associated interpolation subspace $\mathcal{M}_{S_0,B_I,B_O}$ is not a regular subspace of $\mathcal{K}$; see [5].

3. State-space realization of the J-Beurling-Lax representer

Various authors ([40, 17]), perhaps beginning with Nudelman [67]) have noticed that the detailed interpolation conditions (2.1) can be written more compactly in aggregate form as

$$\frac{1}{2\pi i}\int_{\mathbb{T}}(zI-Z)^{-1}XS(z)\,\mathtt{d}z = Y,$$
$$\frac{1}{2\pi i}\int_{\mathbb{T}}S(z)U(zI-A)^{-1}\,\mathtt{d}z = V, \tag{3.1}$$
$$\frac{1}{2\pi i}\int_{\mathbb{T}}(zI-Z)^{-1}XS(z)U(zI-A)^{-1}\,\mathtt{d}z = \Gamma, \tag{3.2}$$

where the collection of seven matrices $\mathfrak{D}_{BTOA} = (U,V,A,Z,X,Y,\Gamma)$ (the label **BTOA** refers to the *bitangential operator-argument* interpolation problem which is described below) is given by

$$X = \begin{bmatrix} x_1 \\ \vdots \\ x_{n_O} \end{bmatrix}, \quad Y = \begin{bmatrix} y_1 \\ \vdots \\ y_{n_O} \end{bmatrix}, \quad Z = \begin{bmatrix} z_1 & & \\ & \ddots & \\ & & z_{n_O} \end{bmatrix},$$
$$U = \begin{bmatrix} u_1 & \cdots & u_{n_I} \end{bmatrix}, \quad V = \begin{bmatrix} v_1 & \cdots & v_{n_I} \end{bmatrix}, \quad A = \begin{bmatrix} w_1 & & \\ & \ddots & \\ & & w_{n_O} \end{bmatrix},$$
$$[\Gamma]_{ij} = \begin{cases} \frac{x_i v_j - y_i u_j}{w_j - z_i} & \text{if } w_j \neq z_i, \\ \rho_{ij} & \text{if } w_j = z_i \end{cases} \quad \text{for } 1 \le i \le n_O,\ 1 \le j \le n_I. \tag{3.3}$$

The interpolation conditions expressed in this form (3.2) make sense even if the matrices A and Z, while maintaining spectrum inside the unit disk, have more general Jordan canonical forms (i.e., are not diagonalizable); in this way we get a compact way of expressing higher order bitangential interpolation conditions. By expanding the resolvent operators inside the contour integrals in Laurent series, it is not hard to see that we can rewrite the interpolation/moment conditions in (3.2) in the form

$$P_{H^{2\perp}_{\mathcal{U}_O}} M_S \widehat{\mathcal{O}}^b_{U,A} = \widehat{\mathcal{O}}^b_{V,A}, \quad \widehat{\mathcal{C}}^b_{Z,X} M_S|_{H^2_{\mathcal{U}_I}} = \widehat{\mathcal{C}}^b_{Z,Y}, \quad \widehat{\mathcal{C}}^b_{Z,X} P_{H^2_{\mathcal{U}_O}} M_S \widehat{\mathcal{O}}^b_{U,A} = \Gamma \tag{3.4}$$

where $\widehat{\mathcal{O}}^b_{U,A} \colon \mathbb{C}^{n_I} \to H^{2\perp}_{\mathcal{U}_I}$ and $\widehat{\mathcal{O}}^b_{V,A} \colon \mathbb{C}^{n_I} \to H^{2\perp}_{\mathcal{U}_O}$ are the *backward-time observation operators* given by

$$\widehat{\mathcal{O}}^b_{U,A} \colon x \mapsto U(zI-A)^{-1}x = \sum_{n=1}^{\infty}(UA^{n-1}x)z^{-n},$$
$$\widehat{\mathcal{O}}^b_{V,A} \colon x \mapsto V(zI-A)^{-1}x = \sum_{n=1}^{\infty}(VA^{n-1}x)z^{-n},$$

and where $\widehat{\mathcal{C}}^b_{Z,X} \colon H^2_{\mathcal{U}_I} \to \mathbb{C}^{n_O}$, $\widehat{\mathcal{C}}^b_{Z,Y} \colon H^2_{\mathcal{U}_I} \to \mathbb{C}^{n_O}$ are the *backward-time control operators* given by

$$\widehat{\mathcal{C}}^b_{Z,X} \colon f(z) = \sum_{n=0}^{\infty} f_n z^n \mapsto \sum_{n=0}^{\infty} Z^n X f_n, \quad \widehat{\mathcal{C}}^b_{Z,Y} \colon g(z) = \sum_{n=0}^{\infty} g_n z^n \mapsto \sum_{n=0}^{\infty} Z^n Y g_n.$$

The terminology is suggested from the following connections with linear systems. Given a discrete-time state-output linear system running in backwards time with specified initial condition at time $n=0$

$$\begin{array}{rcl} x(n) & = & Ax(n+1) \\ y(n) & = & Cx(n+1) \end{array}, \quad x(0) = x_0, \tag{3.5}$$

the resulting output string $\{y(n)\}_{n=-1,-2,\dots}$ is given by

$$y(-n) = CA^{n-1}x_0 \text{ for n=1,2, } \dots.$$

It is natural to let $\mathcal{O}^b_{C,A}$ denote the *time-domain backward-time observation operator* given by

$$\mathcal{O}^b_{C,A} \colon x \mapsto \{y(n)\}_{n=-1,-2,\dots} = \{CA^{-n-1}x\}_{n=-1,-2,\dots}.$$

Upon taking Z-transform $\{y(n)\} \mapsto \widehat{y}(z) = \sum_{n\in\mathbb{Z}} y(n)z^n$, we arrive at the *frequency-domain backward-time observation operator* $\widehat{\mathcal{O}}^b_{C,A}$ given by

$$\widehat{\mathcal{O}}^b_{C,A} \colon x \mapsto \widehat{y}(z) = \sum_{n=1}^{\infty}(CA^{n-1}x)z^{-n} = C(zI-A)^{-1}x.$$

In these computations we assumed that the matrix A has spectrum inside the disk; we conclude that $C(zI-A)^{-1}x \in H^{2\perp}_{\mathcal{U}_O}$ when viewed as a function on the circle; note that $C(zI-A)^{-1}x$ is rational with all poles inside the disk and vanishes at infinity.

Similarly, given a discrete-time input-state linear system running in backwards time

$$x(n) = Zx(n+1) + Xu(n+1) \tag{3.6}$$

where we assume that $x(n) = 0$ for $n \geq N$ and $u(n) = 0$ for all $n > N$ for some large N, solving the recursion successively for $x(N-1)$, $x(N-2)$, $\dots$, $x(0)$ leads to the formula

$$x(0) = \sum_{k=0}^{\infty} Z^k X u(k) = \begin{bmatrix} X & ZX & Z^2X & \cdots \end{bmatrix} \begin{bmatrix} u(0) \\ u(1) \\ u(2) \\ \vdots \end{bmatrix}.$$

As Z by assumption has spectrum inside the unit disk, the matrix

$$\begin{bmatrix} X & ZX & Z^2X & \cdots \end{bmatrix},$$

initially defined only on input strings having finite support, extends to the space of all $\mathcal{U}_I$-valued ℓ^2 input-strings $\ell^2_{\mathcal{U}_I}$. It is natural to define the *frequency-domain backward-time control operator* $\mathcal{C}^b_{Z,X}$ by

$$\mathcal{C}^b_{Z,X} \colon \{u(n)\}_{n \geq 0} \mapsto \begin{bmatrix} X & ZX & Z^2X & \cdots \end{bmatrix} \begin{bmatrix} u(0) \\ u(1) \\ u(2) \\ \vdots \end{bmatrix}.$$

Application of the inverse Z-transform to $\{u(n)\}_{n=0,1,2,\dots}$ then leads us to the *frequency-domain backward-time control operator* $\widehat{\mathcal{C}}^b_{Z,X} \colon H^2_{\mathcal{U}_I} \to \mathbb{C}^{n_O}$ given by

$$\widehat{\mathcal{C}}^b_{Z,X} \colon u(z) = \sum_{n=0}^{\infty} u(n) z^n \mapsto \sum_{n=0}^{\infty} Z^n X u(n).$$

The next step is to observe that conditions (3.4) make sense even if the data set $\mathfrak{D}_{BTOA}$ does not consist of matrices. Instead, we now view X, Y, Z, U, V, A, Γ as operators

$$\begin{aligned} &X \colon \mathcal{U}_O \to \mathcal{X}_L, \quad Y \colon \mathcal{U}_I \to \mathcal{X}_L, \quad Z \colon \mathcal{X}_L \to \mathcal{X}_L, \\ &U \colon \mathcal{X}_R \to \mathcal{U}_I, \quad V \colon \mathcal{X}_R \to \mathcal{U}_O, \quad A \colon \mathcal{X}_R \to \mathcal{X}_R, \quad \Gamma \colon \mathcal{X}_R \to \mathcal{X}_L. \end{aligned} \tag{3.7}$$

Note that when the septet $(X, Y, Z, U, V, A, \Gamma)$ is of the form as in (3.3), then the Sylvester equation

$$\Gamma A - Z\Gamma = XV - YU. \tag{3.8}$$

is satisfied. To avoid degeneracies, it is natural to impose some additional controllability and observability assumptions. The full set of admissibility requirements is as follows.

Definition 3.1. *Given a septet of operators* $\mathfrak{D}_{BTOA} := (X, Y, Z, U, V, A, \Gamma)$ *as in* (3.7)*, we say that* $\mathfrak{D}_{BTOA}$ *is* admissible *if the following conditions are satisfied:*

1. (X, Z) *is a* stable exactly controllable input pair*, i.e.,* $\widehat{\mathcal{C}}^b_{Z,X}$ *defines a bounded operator from* $H^2_{\mathcal{U}_I}$ *into* $\mathcal{X}_L$ *with range equal to the whole space* $\mathcal{X}_L$.
2. (U, A) *is a* stable exactly observable output pair*, i.e.,* $\widehat{\mathcal{O}}^b_{U,A}$ *maps the state space* $\mathcal{X}_R$ *into* $H^{2\perp}_{\mathcal{U}_I}$ *and is bounded below:*

$$\|\widehat{\mathcal{O}}^b_{U,A} x\|^2_{H^{2\perp}_{\mathcal{U}_I}} \geq \delta \|x\|^2_{\mathcal{X}_R} \textit{ for some } \delta > 0.$$

3. *The operator* Γ *is a solution of the Sylvester equation* (3.8)*.*

We can now formulate the promised *bitangential operator-argument interpolation problem.*

Problem BTOA (Bitangential Operator Argument Interpolation Problem): *Given an admissible operator-argument interpolation data set* $\mathfrak{D}_{BTOA}$ *as described in Definition* 3.1*, find a function* S *in the Schur class* $\mathcal{S}(\mathcal{U}_I, \mathcal{U}_O)$ *which satisfies the interpolation conditions* (3.4)*.*

It can be shown that there is a bijection between **BTS** data sets $\mathfrak{D}_{BTS} = \{S_0, B_I, B_O\}$ and admissible **BTOA** data sets $\mathfrak{D}_{BTOA}$ (3.7) so that the corresponding interpolation problems **BTS** and **BTOA** have exactly the same set of solutions. For the rational matrix-valued case, details can be found in [17] (see Theorem 16.9.3 there); the result for the general case can be worked out using these ideas and the results from [30].

Let us now suppose that $\mathfrak{D}_{BTS} = \{S_0, B_I, B_O\}$ and $\mathfrak{D}_{BTOA}$ (3.7) are equivalent in this sense. Then the subspace $\mathcal{M}_{S_0,B_I,B_O}$ (2.2) is the subspace of $L^2_{\mathcal{U}_O} \oplus B_I^{-1} H^2_{\mathcal{U}_I}$ spanned by the graph spaces $\left[\begin{smallmatrix} S \\ I \end{smallmatrix}\right] B_I^{-1} \cdot H^2_{\mathcal{U}_I}$ of solutions S of the interpolation problem BTS. Hence this same subspace can be expressed as the span $\mathcal{M}_{BTOA}$ of the graph spaces of all solutions S of the interpolation problem BTOA. One can work out that $\mathcal{M}_{BTOA}$ can be expressed directly in terms of the data set $\mathfrak{D}_{BTOA}$ as:

$$\mathcal{M}_{BTOA} = \left\{ \widehat{\mathcal{O}}^b_{\left[\begin{smallmatrix} V \\ U \end{smallmatrix}\right], A} x + \begin{bmatrix} f_+ \\ f_- \end{bmatrix} : x \in \mathcal{X}_R, \begin{bmatrix} f_+ \\ f_- \end{bmatrix} \in H^2_{\mathcal{U}_O \oplus \mathcal{U}_I} \right.$$
$$\left. \text{such that } \widehat{\mathcal{C}}^b_{Z, \left[\begin{smallmatrix} X & -Y \end{smallmatrix}\right]} \begin{bmatrix} f_+ \\ f_- \end{bmatrix} = \Gamma x \right\}. \tag{3.9}$$

Remark 3.2. For the representation of general shift-invariant subspaces in terms of null-pole data developed in [30], the coupling operator Γ in general is only a closed (possibly unbounded) operator with dense domain in $\mathcal{X}_R$. In the context of the **BTOA** interpolation problem as we have here, from the last of the interpolation conditions (3.4) we see that Γ is bounded whenever the **BTOA** interpolation problem has solutions. Therefore for the discussion here we may avoid the complications of unbounded Γ and always assume that Γ is bounded.

By the analysis of the previous section, we see that parametrization of the set of all solutions of the **BTOA** interpolation problem follows from a J-Beurling-Lax representation for the subspace $\mathcal{M}_{BTOA}$ (3.9) as in Theorem 2.2. Toward this end we have the following result; we do not go into details here but the main ingredients can be found [30] (see Corollary 6.4, Theorem 6.5 and Theorem 7.1 there).

Theorem 3.3. *Let* $\mathfrak{D}_{BTOA}$ *be an admissible bitangential operator-argument interpolation data set as in Definition* 3.1 *and let* $\mathcal{M}_{BTOA}$ *be the associated shift-invariant subspace as in* (3.9)*. Then:*

1. *$\mathcal{M}_{BTOA}$ is regular as a subspace of the Kreĭn space $L^2_{\mathcal{U}_O} \oplus B_I^{-1} \cdot H^2_{\mathcal{U}_I}$, or equivalently, as a subspace of the Kreĭn space $L^2_{\mathcal{U}_O} \oplus L^2_{\mathcal{U}_I}$ (both with the indefinite inner product induced by $J = \left[\begin{smallmatrix} I_{\mathcal{U}_O} & 0 \\ 0 & -I_{\mathcal{U}_I} \end{smallmatrix}\right]$) if and only if the operator*

$$\Lambda_{BTOA} := \begin{bmatrix} -(\widehat{\mathcal{O}}^b_{\left[\begin{smallmatrix} V \\ U \end{smallmatrix}\right], A})^* J \widehat{\mathcal{O}}^b_{\left[\begin{smallmatrix} V \\ U \end{smallmatrix}\right], A} & \Gamma^* \\ \Gamma & \widehat{\mathcal{C}}^b_{Z, [\,X \ -Y\,]} J (\widehat{\mathcal{C}}^b_{Z, [\,X \ -Y\,]})^* \end{bmatrix} : \begin{bmatrix} \mathcal{X}_R \\ \mathcal{X}_L \end{bmatrix} \to \begin{bmatrix} \mathcal{X}_R \\ \mathcal{X}_L \end{bmatrix} \tag{3.10}$$

is invertible.

2. *The subspace*

$$\mathcal{P}_{BTOA} := \mathcal{K} \ominus_J \mathcal{M}_{BTOA} \text{ where } \mathcal{K} = \begin{bmatrix} \widehat{\mathcal{O}}^b_{V,A} & 0 \\ 0 & \widehat{\mathcal{O}}^b_{U,A} \end{bmatrix} \mathcal{X}_R \oplus \begin{bmatrix} H^2_{\mathcal{U}_O} \\ H^2_{\mathcal{U}_I} \end{bmatrix}$$

is a positive subspace if and only the the **BTOA** *Pick matrix Λ_{BTOA} as in* (3.10) *is positive semidefinite.*

3. *Assume that Λ_{BTOA} is invertible. Then a Beurling-Lax representer Θ for $\mathcal{M}_{BTOA}$ has bidichotomous realization*

$$\Theta(z) = \begin{bmatrix} V \\ U \end{bmatrix} (zI - A)^{-1} \mathbf{B}_- + \mathbf{D} + z \begin{bmatrix} X^* \\ Y^* \end{bmatrix} (I - zZ^*)^{-1} \mathbf{B}_+. \tag{3.11}$$

where the operators appearing in (3.11) *not specified in the data set $\mathfrak{D}_{BTOA}$, namely $\mathbf{B}_-$, $\mathbf{B}_+$, and $\mathbf{D}$, are constructed so that the operator*

$$\begin{bmatrix} \mathbf{B}_- \\ \mathbf{B}_+ \\ \mathbf{D} \end{bmatrix} : \begin{bmatrix} \mathcal{U}_O \\ \mathcal{U}_I \end{bmatrix} \to \begin{bmatrix} \mathcal{X}_R \\ \mathcal{X}_L \\ \left[\begin{smallmatrix} \mathcal{U}_O \\ \mathcal{U}_I \end{smallmatrix}\right] \end{bmatrix}$$

is a J-unitary isomorphism from $\left[\begin{smallmatrix} \mathcal{U}_O \\ \mathcal{U}_I \end{smallmatrix}\right]$ onto $Ker\, \Psi \subset \mathcal{X}_R \oplus \mathcal{X}_L \oplus \left[\begin{smallmatrix} \mathcal{U}_O \\ \mathcal{U}_I \end{smallmatrix}\right]$, where

$$\Psi = \begin{bmatrix} \Gamma & -Z\widehat{\mathcal{C}}^b_{Z, [\,X \ -Y\,]} J (\widehat{\mathcal{C}}^b_{Z, [\,X \ -Y\,]})^* & [\,-X \ Y\,] \\ -A^* (\widehat{\mathcal{O}}^b_{\left[\begin{smallmatrix} V \\ U \end{smallmatrix}\right], A})^* J \widehat{\mathcal{O}}^b_{\left[\begin{smallmatrix} V \\ U \end{smallmatrix}\right]} & -\Gamma^* & [\,-V^* \ U^*\,] \end{bmatrix},$$

and where $\mathcal{X}_R \oplus \mathcal{X}_L \oplus \left[\begin{smallmatrix} \mathcal{U}_O \\ \mathcal{U}_I \end{smallmatrix}\right]$ carries the indefinite inner product induced by the selfadjoint operator

$$\mathcal{J} := \begin{bmatrix} (\mathcal{O}^b_{\left[\begin{smallmatrix} V \\ U \end{smallmatrix}\right], A})^* J \widehat{\mathcal{O}}^b_{\left[\begin{smallmatrix} V \\ U \end{smallmatrix}\right], A} & 0 & 0 \\ 0 & \widehat{\mathcal{C}}^b_{Z, [\,X \ -Y\,]} J (\widehat{\mathcal{C}}^b_{Z, [\,X \ -Y\,]})^* & 0 \\ 0 & 0 & J \end{bmatrix}.$$

In case Λ_{BTOA} in (3.4) *is also positive definite, then Θ parametrizes all solutions of the* **BTOA** *interpolation problem via the formula* (2.8) *with free parameter $G \in \mathcal{S}(\mathcal{U}_I, \mathcal{U}_O)$.*

Remark 3.4. The idea for the derivation of the formula (3.11) for the Beurling-Lax representer Θ for the subspace $\mathcal{M}_{BTOA}$ in Theorem 3.3 goes back to [24]: Θ, when viewed as an operator from the Kreĭn space of constant functions $\mathcal{U}_O \oplus \mathcal{U}_I$

into the Kreĭn space of functions $L^2_{\mathcal{U}_O} \oplus L^2_{\mathcal{U}_I}$ is a Kreĭn-space isomorphism from $\mathcal{U}_O \oplus \mathcal{U}_I$ to the wandering subspace $\mathcal{L} := M_z(\mathcal{M}_{BTOA})^{[\perp]} \cap \mathcal{M}_{BTOA}$. Similar state-space realizations hold for affine Beurling-Lax representations (or the *Beurling-Lax Theorem for the Lie group* $GL(n,\mathbb{C})$ in the terminology of [25]). Here one is given a pair of subspaces $(\mathcal{M}, \mathcal{M}^\times)$ such that $\mathcal{M}$ is forward shift invariant, $\mathcal{M}^\times$ is backward shift invariant, $\mathcal{M}$ and $\mathcal{M}^\times$ form a direct-sum decomposition for $\mathcal{L}^2_{\mathcal{U}}$, and one seeks an invertible operator function Θ on the circle $\mathbb{T}$ so that $\mathcal{M} = \Theta \cdot H^2_{\mathcal{U}}$ and $\mathcal{M}^\times = \Theta \cdot H^{2\perp}_{\mathcal{U}}$. State-space implementations for the Beurling-Lax representer Θ where $\mathcal{M}$ and $\mathcal{M}^\times$ are assumed to have representations of the form (3.9) are worked out in [30] (see also [17, Theorem 5.5.2] and [16] for the rational matrix-valued case).

Remark 3.5. When we consider the result of Theorem 3.3 for the case of matricial data, arguably the solution is not as explicit as one would like; one must find a J-orthonormal basis for a certain finite-dimensional regular subspace of $L^2_{\mathcal{U}_O \oplus \mathcal{U}_I}$. One can explain this as follows. In this general setting, no assumptions are made on the locations of the poles (i.e., the spectrum of A inside the unit disk and the reflection of the spectrum of Z to outside the disk) and zeros (i.e., the spectrum of Z and the reflection of the spectrum of A to outside the disk) in the extended complex plane; hence there is no global chart with respect to which one can set up coordinates. This issue can be resolved in several ways. For example, one could specify a point ζ_0 on the unit circle at which no interpolation conditions are specified, and demand that $\Theta(\zeta_0)$ be some given J-unitary matrix (e.g., $I_{\mathcal{U}_O \oplus \mathcal{U}_I}$ (see e.g. Theorem 7.5.2 in [17]); however in the case of infinite-dimensional data it is possible for Z and A to have spectrum including the whole unit circle thereby making this approach infeasible. Alternatively, one might assume that both A and Z are invertible (no interpolation conditions at the point 0) in which case Theorem 7.1.7 in [17] gives a more explicit formula for Θ. A difficulty for numerical implementation of the formulas is the challenge of inverting the Pick matrix in these formulas; this difficulty was later addressed by adapting the use of fast recursive algorithms for the inversion of structured matrices by Olshevsky and his collaborators (see e.g. [59, 60]).

3.1. A special case: left tangential operator-argument interpolation

We now discuss the special case of Theorem 3.3 where there are only left tangential interpolation conditions present (so the right-side state space $\mathcal{X}_R = \{0\}$ is trivial). In this case the bitangential operator-argument interpolation data set $\mathfrak{D}_{BTOA}$ consisting of seven operators collapses to a left tangential operator-argument data set $\mathfrak{D}_{LTOA}$ consisting of only three operators

$$\mathfrak{D}_{LTOA} = \{X, Y, Z\} \text{ where } X : \mathcal{U}_O \to \mathcal{X}_L, \quad Y : \mathcal{U}_I \to \mathcal{X}_L, \quad Z : \mathcal{X}_L \to \mathcal{X}_L,$$

the interpolation problem collapses to just the second of the conditions (3.4) which can be written also in more succinct left-tangential operator-argument form

$$(\widehat{XS})^{\wedge L}(Z) := \sum_{n=0}^{\infty} Z^n X S_n = Y \tag{3.12}$$

(here S_n $(n = 0, 1, 2, \dots)$ are the Taylor coefficients of S: $S(z) = \sum_{n=0}^{\infty} S_n z^n$ for $z \in \mathbb{D}$). The shift-invariant subspace $\mathcal{M}_{BTOA}$ collapses to the left-tangential version

$$\begin{aligned}\mathcal{M}_{LTOA} &= \left\{ \begin{bmatrix} f_+ \\ f_- \end{bmatrix} : \widehat{\mathcal{C}}^b_{Z, [\, X \ -Y \,]} \begin{bmatrix} f_+ \\ f_- \end{bmatrix} = 0 \right\} \\ &= \operatorname{Ker} \widehat{\mathcal{C}}^b_{Z, [\, X \ -Y \,]} \subset H^2_{\mathcal{U}_O \oplus \mathcal{U}_I}, \end{aligned} \tag{3.13}$$

and where the solution criterion $\Lambda_{BTOA} \geq 0$ collapses to

$$\Lambda_{LTOA} := \widehat{\mathcal{C}}^b_{Z, [\, X \ -Y \,]} J (\widehat{\mathcal{C}}^b_{Z, [\, X \ -Y \,]})^* \geq 0.$$

In the regular case (which we now assume), we have in addition that Λ_{LTOA} is invertible. It follows that $H^2_{\mathcal{U}_O \oplus \mathcal{U}_I} \ominus_J \mathcal{M}_{LTOA}$ is given by

$$\begin{aligned} H^2_{\mathcal{U}_O \oplus \mathcal{U}_I} \ominus_J \mathcal{M}_{LTOA} &= \operatorname{Ran} \left(\widehat{\mathcal{C}}^b_{Z, [\, X \ -Y \,]} \right)^* J \\ &= \operatorname{Ran} \widehat{\mathcal{O}}^f_{\left[\begin{smallmatrix} X^* \\ Y^* \end{smallmatrix}\right], Z^*} := \left\{ \begin{bmatrix} X^* \\ Y^* \end{bmatrix} (I - zZ^*)^{-1} x \colon x \in \mathcal{X}_L \right\}. \end{aligned}$$

To simplify the notation let us introduce the quantities

$$C = \begin{bmatrix} X^* \\ Y^* \end{bmatrix}, \quad A = Z^* \tag{3.14}$$

so that we may write $\widehat{\mathcal{O}}^f_{C,A}$ rather than the more cumbersome $\widehat{\mathcal{O}}^f_{\left[\begin{smallmatrix} X^* \\ Y^* \end{smallmatrix}\right], Z^*}$ and write simply $\mathcal{M}$ for $\mathcal{M}_{LTOA}$ and $\mathcal{M}^{[\perp]}$ for $H^2_{\mathcal{U}_O \oplus \mathcal{U}_I} \ominus_J \mathcal{M}_{LTOA}$. Then the regularity of $\mathcal{M}$ and the positivity of Λ_{LTOA} can be expressed as

$$\Lambda_{LTOA} = (\widehat{\mathcal{O}}^f_{C,A})^* J \widehat{\mathcal{O}}^f_{C,A} > 0.$$

If we impose the positive-definite inner product induced by Λ_{LTOA} on $\mathcal{X}_L$, then the map

$$\iota \colon x \mapsto \mathcal{O}^f_{C,A} \tag{3.15}$$

is a Kreĭn-space isomorphism between $\mathcal{X}_L$ and $\mathcal{M}^{[\perp]}$. If we set

$$K(z, w) = C(I - zA)^{-1} \Lambda^{-1} (I - \overline{w} A^*)^{-1} C^* \tag{3.16}$$

(with $\Lambda = \Lambda_{LTOA}$), then one can use the J-unitary property of the map ι (3.15) to compute, for $f(z) = (\widehat{\mathcal{O}}^f_{C,A} x)(z) = C(I - zA)^{-1} x$, $w \in \mathbb{D}$ and $u \in \mathcal{U}_O \oplus \mathcal{U}_I$,

$$\begin{aligned} \langle Jf, \, K(\cdot, w) u \rangle_{H^2_{\mathcal{U}_O \oplus \mathcal{U}_I}} &= \langle \Lambda x, \Lambda^{-1} (I - \overline{w} A^*)^{-1} C^* u \rangle_{\mathcal{X}_L} \\ &= \langle C(I - wA)^{-1} x, u \rangle_{\mathcal{U}_O \oplus \mathcal{U}_I} \\ &= \langle f(w), u \rangle_{\mathcal{U}_O \oplus \mathcal{U}_I} \end{aligned}$$

from which we see that $K(z,w)$ is the *reproducing kernel* for the space $\mathcal{M}^{[\perp]}$. On the other hand, if we construct $\left[\begin{smallmatrix}\mathbf{B}\\ \mathbf{D}\end{smallmatrix}\right]$ so that

$$\begin{bmatrix} A & \mathbf{B} \\ C & \mathbf{D} \end{bmatrix} \begin{bmatrix} \Lambda^{-1} & 0 \\ 0 & J \end{bmatrix} \begin{bmatrix} A^* & C^* \\ \mathbf{B}^* & \mathbf{D}^* \end{bmatrix} = \begin{bmatrix} \Lambda^{-1} & 0 \\ 0 & J \end{bmatrix}, \tag{3.17}$$

and set

$$\Theta(z) = \mathbf{D} + zC(I - zA)^{-1}\mathbf{B},$$

then Θ is J-inner with associated kernel $K_\Theta(z,w)$ satisfying

$$K_\Theta(z,w) := \frac{J - \Theta(z)J\Theta(w)^*}{1 - z\overline{w}} = C(I - zA)^{-1}\Lambda^{-1}(I - \overline{w}A^*)^{-1}C^* = K(z,w)$$

where $K(z,w)$ is as in (3.16). From this it is possible to show that the closure of $\Theta \cdot (H^2_{\mathcal{U}})_0$ is exactly $(\mathcal{M}^{[\perp]})^{[\perp]} = \mathcal{M}$, i.e., the J-Beurling-Lax representer for $\mathcal{M}$ can be constructed in this way. To make the construction of $\left[\begin{smallmatrix}\mathbf{B}\\ \mathbf{D}\end{smallmatrix}\right]$, note that solving (3.17) for $\mathbf{B}$ and $\mathbf{D}$ amounts to solving the J-Cholesky factorization problem

$$\begin{bmatrix} \mathbf{B} \\ \mathbf{D} \end{bmatrix} J \begin{bmatrix} \mathbf{B}^* & \mathbf{D}^* \end{bmatrix} = \begin{bmatrix} \Lambda^{-1} & 0 \\ 0 & J \end{bmatrix} - \begin{bmatrix} A \\ C \end{bmatrix} \Lambda^{-1} \begin{bmatrix} A^* & C^* \end{bmatrix}. \tag{3.18}$$

An amusing exercise is to check that this recipe is equivalent to that in Theorem 3.3 when specialized to the case where $\mathcal{X}_R = \{0\}$.

4. Extensions and generalizations of the Grassmannian method

The CBMS monograph [49] and the survey article [9] mention various adaptations of the Grassmannian method to other sorts of interpolation and extension problems. We also mention the Grassmannian version of the abstract band method (including the Tagaki version where one seeks a solution in a generalized Schur class (the kernel $K_S(z,w) = [I - S(z)S(w)^*]/(1 - z\overline{w})$ is required to have a most some number κ of negative squares rather than to be a positive kernel)) worked out in [53]. Also the Grassmannian approach certainly influenced the theory of time-varying interpolation developed in [18, 19, 20]. Moreover, one can argue that the Grassmannian approach to interpolation, in particular the point of view espoused in [27], foreshadowed the behavioral formulation and solution of the H^∞-control problem (see [58, 71]). Here we discuss some more recent extensions of the Grassmannian method to several variable contexts.

4.1. Interpolation problems for multipliers on the Drury-Arveson space

A multivariable generalization of the Szegö kernel much studied of late is the positive kernel

$$k_d(\boldsymbol{\lambda}, \boldsymbol{\zeta}) = \frac{1}{1 - \langle \boldsymbol{\lambda}, \boldsymbol{\zeta} \rangle}$$

on $\mathbb{B}^d \times \mathbb{B}^d$, where $\mathbb{B}^d = \left\{ \boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_d) \in \mathbb{C}^d \colon \langle \boldsymbol{\lambda}, \boldsymbol{\lambda} \rangle < 1 \right\}$ is the unit ball of the d-dimensional Euclidean space $\mathbb{C}^d$ and $\langle \boldsymbol{\lambda}, \boldsymbol{\zeta} \rangle$ is the standard inner product in $\mathbb{C}^d$. The reproducing kernel Hilbert space $H(k_d)$ associated with k_d is called the Drury-Arveson space (also denoted as H^2_d) and acts as a natural multivariable analogue of the Hardy space H^2 of the unit disk. The many references on this topic include [38, 6, 7, 2, 31, 42, 46, 57].

For $\mathcal{Y}$ an auxiliary Hilbert space, we consider the tensor product Hilbert space $H_{\mathcal{Y}}(k_d) := H(k_d) \otimes \mathcal{Y}$ whose elements can be viewed as $\mathcal{Y}$-valued functions in $H(k_d)$. Then $H_{\mathcal{Y}}(k_d)$ has the following characterization:

$$H_{\mathcal{Y}}(k_d) = \left\{ f(\boldsymbol{\lambda}) = \sum_{\mathbf{n} \in \mathbb{Z}^d_+} f_{\mathbf{n}} \boldsymbol{\lambda}^{\mathbf{n}} : \|f\|^2 = \sum_{\mathbf{n} \in \mathbb{Z}^d_+} \frac{\mathbf{n}!}{|\mathbf{n}|!} \cdot \|f_{\mathbf{n}}\|^2_{\mathcal{Y}} < \infty \right\}. \tag{4.1}$$

Here and in what follows, we use standard multivariable notations: for multi-integers $\mathbf{n} = (n_1, \dots, n_d) \in \mathbb{Z}^d_+$ and points $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_d) \in \mathbb{C}^d$ we set

$$|\mathbf{n}| = n_1 + n_2 + \dots + n_d, \qquad \mathbf{n}! = n_1! n_2! \dots n_d!, \qquad \boldsymbol{\lambda}^{\mathbf{n}} = \lambda_1^{n_1} \lambda_2^{n_2} \dots \lambda_d^{n_d}. \tag{4.2}$$

For coefficient Hilbert spaces $\mathcal{U}$ and $\mathcal{Y}$, the operator-valued Drury-Arveson Schur-multiplier class $\mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ is defined to be the space of functions S holomorphic on the unit ball $\mathbb{B}^d$ with values in the space of operators $\mathcal{L}(\mathcal{U}, \mathcal{Y})$ such that the multiplication operator

$$M_S \colon f(\boldsymbol{\lambda}) \to S(\boldsymbol{\lambda}) \cdot f(\boldsymbol{\lambda})$$

maps $H_{\mathcal{U}}(k_d)$ contractively into $H_{\mathcal{Y}}(k_d)$, or equivalently, the associated multivariable de Branges-Rovnyak kernel

$$K_S(\boldsymbol{\lambda}, \boldsymbol{\zeta}) := \frac{I - S(\boldsymbol{\lambda}) S(\boldsymbol{\zeta})^*}{1 - \langle \boldsymbol{\lambda}, \boldsymbol{\zeta} \rangle} \tag{4.3}$$

should be a positive kernel.

Let $\mathbf{A} = (A_1, \dots, A_d)$ be a commutative d-tuple of bounded, linear operators on the Hilbert space $\mathcal{X}$. If $C \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, then the pair $(C, \mathbf{A})$ is said to be *output-stable* if the associated observation operator

$$\widehat{\mathcal{O}}_{C, \mathbf{A}} \colon x \mapsto C(I - \lambda_1 A_1 - \dots - \lambda_d A_d)^{-1} x$$

maps $\mathcal{X}$ into $H_{\mathcal{Y}}(k_d)$, or equivalently (by the closed graph theorem), the observation operator is bounded. Just as in the single-variable case (see (3.5)), there is a system-theoretic interpretation for this operator, but now in the context of multidimensional systems (see [12] for details). We can then pose the Drury-Arveson space version of the left-tangential operator-argument interpolation (**LTOA**) problem formulated in Subsection 3.1 by replacing the unit disk $\mathbb{D}$ by the unit ball $\mathbb{B}^d$ and the Schur class $\mathcal{S}(\mathcal{U}, \mathcal{Y})$ by the Drury-Arveson Schur-multiplier class $\mathcal{S}_d(\mathcal{U}, \mathcal{Y})$.

Problem LTOA (Left Tangential Operator Argument Interpolation Problem): *Let $\mathcal{U}_I, \mathcal{U}_O$ and $\mathcal{X}$ be Hilbert spaces and suppose that we are given the data set $(\mathbf{Z}, X, Y)$ with $\mathbf{Z} = (Z_1, \cdots, Z_d) \in \mathcal{L}(\mathcal{X}, \oplus_1^d \mathcal{X})$, $X \in \mathcal{L}(\mathcal{U}_O, \mathcal{X})$, $Y \in \mathcal{L}(\mathcal{U}_I, \mathcal{X})$ such that $(\mathbf{Z}, X)$ is an input stable pair, or, $(X^*, \mathbf{Z}^*)$ is an output stable pair. Find $S \in \mathcal{S}_d(\mathcal{U}_I, \mathcal{U}_O)$ such that*

$$\left(\widehat{\mathcal{O}}_{X^*, \mathbf{Z}^*} \right)^* M_S = \left(\widehat{\mathcal{O}}_{Y^*, \mathbf{Z}^*} \right)^*, \tag{4.4}$$

or equivalently,

$$(\widehat{XS})^{\wedge L}(\mathbf{Z}) = Y, \tag{4.5}$$

where the multivariable left tangential operator-argument point-evaluation is given by

$$(\widehat{XS})^{\wedge L}(\mathbf{Z}) = \sum_{n\in\mathbb{Z}_+^d} \mathbf{Z}^n X S_n.$$

Here $S(z) = \sum_{n\in\mathbb{Z}_+^d} z^n$ is the multivariable Taylor series for S and we use the commutative multivariable notation

$$\mathbf{Z}^n = Z_1^{n_1}\cdots Z_d^{n_d} \text{ for } n = (n_1,\ldots,n_d) \in \mathbb{Z}_+^d.$$

We note that this and related interpolation problems were studied in [11] by using techniques from reproducing kernel Hilbert spaces, Schur-complements and isometric extensions borrowed from the work of [39, 55, 54]; here we show how the problem can be handled via the Grassmannian approach.

As a motivation for this formalism, we consider a simple example: take $\mathcal{U}_I = \mathcal{U}_O = \mathbb{C}$, $X = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$, $Y = \begin{bmatrix} w_1 \\ w_1 \\ \vdots \\ w_N \end{bmatrix}$, $\mathbf{Z} = (Z_1,\cdots,Z_d)$ with $Z_j = \begin{bmatrix} \lambda_j^{(1)} & & & \\ & \lambda_j^{(2)} & & \\ & & \ddots & \\ & & & \lambda_j^{(N)} \end{bmatrix}$, where $j = 1,\cdots,d$ and where $\lambda^{(i)} = (\lambda_1^{(i)},\ldots,\lambda_d^{(i)}) \in \mathbb{B}^d$ for $i = 1,\cdots,N$. Then the **LTOA** problem collapses to Nevanlinna-Pick-type interpolation problem for Drury-Arveson space multipliers, as studied in [64, 4, 37, 31]: *for given points* $\lambda^{(1)},\ldots,\lambda^{(N)}$ *in the ball* $\mathbb{B}^d$ *and given complex numbers* $w_1,\ldots,w_N$*, find* $S\in\mathcal{S}_d$ *so that*

$$S(\lambda^{(i)}) = w_i \qquad \text{for} \quad i = 1,\cdots,N.$$

We transform the problem to projective coordinates (following the Grassmannian approach) as follows. We identify the Drury-Arveson-space multiplier $S\in\mathcal{S}_d$ with its graph to convert the nonhomogeneous interpolation conditions to homogeneous interpolation conditions for the associated subspaces (i.e., we projectivize the problem). Then one checks that the function $S\in\mathcal{S}_d$ is a solution of the LTOA problem if and only if its graph $\mathcal{G} := \left[\begin{smallmatrix} M_S \\ I \end{smallmatrix}\right] H(k_d) \subset \left[\begin{smallmatrix} H(k_d) \\ H(k_d) \end{smallmatrix}\right]$ satisfies:

1. $\mathcal{G}$ *is a subspace of* $\mathcal{M} = \{f \in \left[\begin{smallmatrix} H(k_d) \\ H(k_d) \end{smallmatrix}\right] : \left[\begin{smallmatrix} 1 & -w_i \end{smallmatrix}\right] f(\lambda^i) = 0$ *for* $i = 1,\cdots,N\}$ (and hence also is a subspace of the Kreĭn space $\mathcal{K} = \left[\begin{smallmatrix} H(k_d) \\ H(k_d) \end{smallmatrix}\right]$ with $J = \left[\begin{smallmatrix} I_{H(k_d)} & \\ & -I_{H(k_d)} \end{smallmatrix}\right]$),
2. $\mathcal{G}$ *is maximal negative in* $\mathcal{K}$ and
3. $\mathcal{G}$ *is* M_{λ_k} *invariant for* $k = 1,\cdots,d$.

Conversely, if $\mathcal{G}$ as a subspace of $\left[\begin{smallmatrix} H(k_d) \\ H(k_d) \end{smallmatrix}\right]$ satisfies (1), (2), (3), then $\mathcal{G}$ is in the form of $\begin{bmatrix} M_S \\ I \end{bmatrix} H_{\mathcal{U}_I}(k_d)$ for a solution S of the interpolation problem. Thus the LTOA interpolation problem translates to the problem of finding subspaces $\mathcal{G}$ of $\left[\begin{smallmatrix} H(k_d) \\ H(k_d) \end{smallmatrix}\right]$ which satisfy the conditions (1), (2), (3) above.

For the general LTOA problem, the analysis is similar. One can see that $S \in \mathcal{S}_d(\mathcal{U}_I, \mathcal{U}_O)$ solves the LTOA problem if and only if its graph $\mathcal{G} := \left[\begin{smallmatrix} S \\ I \end{smallmatrix}\right] \cdot H_{\mathcal{U}_I}(k_d)$ satisfies the following conditions:

1. $\mathcal{G} \subset \mathcal{M}$ where
$$\mathcal{M} = \left\{ f \in H_{\mathcal{U}_O \oplus \mathcal{U}_I}(k_d) : \left(\begin{bmatrix} X & -Y \end{bmatrix} f \right)^{\wedge L}(\mathbf{Z}) = \mathbf{0} \right\}, \tag{4.6}$$
where $H_{\mathcal{U}_O \oplus \mathcal{U}_I}(k_d) = \left[\begin{smallmatrix} H_{\mathcal{U}_O}(k_d) \\ H_{\mathcal{U}_I}(k_d) \end{smallmatrix}\right]$.
2. $\mathcal{G}$ is J-maximal negative subspace of $H_{\mathcal{U}_O \oplus \mathcal{U}_I}(k_d)$, where $J = I_{H_{\mathcal{U}_O}(k_d)} \oplus -I_{H_{\mathcal{U}_I}(k_d)}$.
3. $\mathcal{G}$ is invariant under $M_{\lambda_k}, k = 1, 2 \ldots d$.

Just as in the single-variable case, we see that a necessary condition for solutions to exist is that the analogue of (2.3) holds:

$$\mathcal{P} := H_{\mathcal{U}_O \oplus \mathcal{U}_I}(k_d) \ominus_J \mathcal{M} \text{ is a positive subspace of } H_{\mathcal{U}_O \oplus \mathcal{U}_I}(k_d). \tag{4.7}$$

Given that (4.7) holds, we see that solutions $\mathcal{G}$ of (1), (2), (3) above amount to subspaces $\mathcal{G}$ of $\mathcal{M}$ which are maximal negative as subspaces of $\mathcal{M}$ ($\mathcal{M}$-maximal negative) and which are shift invariant. These in turn can be parametrized if $\mathcal{M}$ has a suitable J-Beurling-Lax representer. For the Hilbert space setting ($J = I$), there is a Beurling-Lax representation theorem (see [6, 57, 46, 12, 15]): *given a closed shift-invariant subspace $\mathcal{M}$ of $H_{\mathcal{U}}(k_d)$, there is a suitable Hilbert space $\mathcal{U}'$ and a Schur-class multiplier $\mathcal{S}_d(\mathcal{U}', \mathcal{U})$ so that the orthogonal projection $P_{\mathcal{M}}$ of $H_{\mathcal{U}}(k_d)$ onto $\mathcal{M}$ is given by $P_{\mathcal{M}} = M_\Theta (M_\Theta)^*$.* Unlike the single-variable case ($d = 1$), in general one cannot take M_Θ to be an isometry, but rather, M_Θ is only a *partial isometry.*

An analogous result holds in the J-setting as follows, as can be seen by following the construction sketched in Subsection 3.1 for the single-variable case. In general we say that an operator T between two Kreĭn spaces $\mathcal{K}'$ and $\mathcal{K}$ is a (possibly unbounded) *Kreĭn-space partial isometry* if $T^{[*]}T$ and $TT^{[*]}$ (where $T^{[*]}$ is the adjoint of T with respect to the Kreĭn-spaces indefinite inner products) are bounded J-self-adjoint projection operators on $\mathcal{K}'$ and $\mathcal{K}$ respectively.

Theorem 4.1. *(See Theorem 3.3.2 in* [43]*.) Suppose that $\mathcal{M}$ is a regular subspace of $H_{\mathcal{U}_O \oplus \mathcal{U}_I}(k_d)$. Then there is a coefficient Kreĭn space $\mathcal{E}$ and a (possibly unbounded) Drury-Arveson multiplier Θ so that M_Θ is a (possibly unbounded) Kreĭn-space partial isometry with final projection operator (the bounded extension of $M_\Theta J_{\mathcal{E}} M_\Theta^* J$) equal to the J-orthogonal projection of $H_{\mathcal{U}_O \oplus \mathcal{U}_I}(k_d)$ onto $\mathcal{M}$. In case condition* (4.7) *holds, then one can take $\mathcal{E}$ to have the form $(\mathcal{U}_{O,aug} \oplus \mathcal{U}_O) \oplus \mathcal{U}_I$ with $J_{\mathcal{E}} = I_{\mathcal{U}_{0,aug} \oplus \mathcal{U}_O} \oplus -I_{\mathcal{U}_I}$ for a suitable augmentation Hilbert space $\mathcal{U}_{0,aug}$.*

If $\mathcal{M}$ comes from a **LTOA** *interpolation problem as in* (4.6)*, then condition* (4.7) *holds if and only if*

$$\Lambda := \left(\widehat{\mathcal{O}}_{\left[\begin{smallmatrix} X^* \\ Y^* \end{smallmatrix}\right], \mathbf{Z}^*}\right)^* J \widehat{\mathcal{O}}_{\left[\begin{smallmatrix} X^* \\ Y^* \end{smallmatrix}\right], \mathbf{Z}^*} \geq 0. \tag{4.8}$$

Then $\mathcal{M}$ is regular if and only if Λ is strictly positive and then the set of all solutions S of the **LTOA** *interpolation problem is given by formula* (2.8) *where now the free parameter G sweeps the Drury-Arveson Schur class $\mathcal{S}_d(\mathcal{U}_I, \mathcal{U}_{O,aug} \oplus \mathcal{U}_O)$. Moreover, a realization formula for the representer Θ is given by*

$$\Theta(\boldsymbol{\lambda}) = \mathbf{D} + C(I - \lambda_1 A_1 - \cdots - \lambda_d A_d)^{-1}(\lambda_1 \mathbf{B}_1 + \cdots + \lambda_d \mathbf{B}_d)$$

where the nonbold components of the matrix

$$\mathbf{U} = \begin{bmatrix} A & \mathbf{B} \\ C & \mathbf{D} \end{bmatrix} = \begin{bmatrix} A_1 & \mathbf{B}_1 \\ \vdots & \vdots \\ A_d & \mathbf{B}_d \\ C & \mathbf{D} \end{bmatrix}$$

are given by

$$A = \begin{bmatrix} Z_1^* \\ \vdots \\ Z_d^* \end{bmatrix}, \quad C = \begin{bmatrix} X^* \\ Y^* \end{bmatrix}$$

while the bold components are given via solving the J-Cholesky factorization problem:

$$\begin{bmatrix} \mathbf{B} \\ \mathbf{D} \end{bmatrix} J \begin{bmatrix} \mathbf{B}^* & \mathbf{D}^* \end{bmatrix} = \begin{bmatrix} \oplus_{k=1}^d \Lambda^{-1} & 0 \\ 0 & J \end{bmatrix} - \begin{bmatrix} A \\ C \end{bmatrix} \Lambda^{-1} \begin{bmatrix} A^* & C^* \end{bmatrix}.$$

Remark 4.2. The major new feature in the multivariable setting compared to the single-variable case is that Θ is only a (possibly unbounded) partial J-isometry rather a J-unitary map. Nevertheless, there is still a correspondence (2.8) between maximal negative subspaces in the model (or parameter) Kreĭn space $(\mathcal{U}_{O,\mathrm{aug}} \oplus \mathcal{U}_O) \oplus \mathcal{U}_I$ and $\mathcal{M}$-maximal negative subspaces of $\mathcal{M} \subset H_{\mathcal{U}_O \oplus \mathcal{U}_I}(k_d)$, but with the price that the solution S no longer uniquely determines the associated free parameter G. Roughly, what makes this work is that the construction guarantees that $\operatorname{Ker} M_\Theta$ is necessarily a positive subspace of $H_{(\mathcal{U}_{O,\mathrm{aug}} \oplus \mathcal{U}_O) \oplus \mathcal{U}_I}(k_d)$. Verification of this correspondence for the unbounded case can be done analogously to the single-variable case by use of the Drury-Arveson-space Leech theorem which in turn follows from the Commutant Lifting Theorem for the Drury-Arveson-spaces multipliers (see [62, 31, 36]).

4.2. Interpolation problems for multianalytic functions on the Fock space

Recently there has been much interest in noncommutative function theory and associated multivariable operator theory and multidimensional system theory, spurred on by a diverse collection of applications too numerous to mention in any depth here. Let us just point out that there are at least three points of view: (1) formal power series in freely noncommuting indeterminates [21, 63, 62, 32, 33, 13], (2) functions in d noncommuting operators acting on some fixed infinite-dimensional separable Hilbert space [22, 23, 10], and (3) functions of d $N \times N$-matrix arguments where the size $N = 1, 2, 3, \dots$ is arbitrary [3, 52, 41, 50, 51].

We restrict our discussion here to the noncommutative version of the Drury-Arveson Schur class, elements of which first appeared in the work of Popescu [61] as the characteristic functions of row contractions. This Schur class consists of formal power series in a set of noncommuting indeterminates which define contractive multipliers between (unsymmetrized) vector-valued Fock spaces. To introduce this setting, let $\{1, \dots, d\}$ be an alphabet consisting of d letters and let $\mathcal{F}_d$ be the associated free semigroup generated by the letters $1, \dots, d$ consisting of all words γ of the form $\gamma = i_N \cdots i_1$, where each $i_k \in \{1, \dots, d\}$ and where $N = 1, 2, \dots$. For $\gamma = i_N \cdots i_1 \in \mathcal{F}_d$ we set $|\gamma| := N$ to be the *length* of the word γ. Multiplication of two words $\gamma = i_N \cdots i_1$ and $\gamma' = j_{N'} \cdots j_1$ is defined via concatenation:

$$\gamma\gamma' = i_N \cdots i_1 j_{N'} \cdots j_1.$$

The empty word $\emptyset$ is included in $\mathcal{F}_d$ and acts as the unit element for this multiplication; by definition $|\emptyset| = 0$. We let $z = (z_1, \dots, z_d)$ be a d-tuple of freely noncommuting indeterminates with associated noncommutative formal monomials $z^\gamma = z_{i_N} \cdots z_{i_1}$ if $\gamma = i_N \cdots i_1 \in \mathcal{F}_d$.

For a Hilbert space $\mathcal{U}$, we define the associated *Fock space* $H^2_{\mathcal{U}}(\mathcal{F}_d)$ to consist of formal power series in the set of noncommutative indeterminates $z = (z_1, \dots, z_d)$

$$\widehat{u}(z) = \sum_{\gamma \in \mathcal{F}_d} u(\gamma) z^\gamma$$

satisfying the square-summability condition on the coefficients:

$$\sum_{\gamma \in \mathcal{F}_d} \|u(\gamma)\|^2_{\mathcal{U}} < \infty.$$

Given two coefficient Hilbert spaces $\mathcal{U}$ and $\mathcal{Y}$, we define the noncommutative Schur class $\mathcal{S}_{nc,d}(\mathcal{U}, \mathcal{Y})$ to consist of formal power series with operator coefficients

$$S(z) = \sum_{\gamma \in \mathcal{F}_d} S_\gamma z^\gamma$$

such that the noncommutative multiplication operator

$$M_S \colon \widehat{u}(z) = \sum_{\gamma \in \mathcal{F}_d} u(\gamma) z^\gamma \mapsto S(z) \cdot \widehat{u}(z) := \sum_{\gamma \in \mathcal{F}_d} \left(\sum_{\alpha, \beta \in \mathcal{F}_d \colon \alpha\beta = \gamma} S_\alpha u(\beta) \right) z^\gamma$$

defines a contraction operator from $H^2_{\mathcal{U}}(\mathcal{F}_d)$ into $H^2_{\mathcal{Y}}(\mathcal{F}_d)$.

One can view elements S of the noncommutative Schur class $\mathcal{S}_{nc,d}(\mathcal{U}, \mathcal{Y})$ as defining functions of d noncommuting arguments and then set up noncommutative analogues of Nevanlinna-Pick interpolation problems as follows. Given a (not necessarily commutative) d-tuple of bounded operators $\mathbf{A} = (A_1, \dots, A_d)$ on a

Hilbert space $\mathcal{X}$ together with an output operator $C\colon \mathcal{X} \to \mathcal{Y}$, let us say that the output-pair $(C, \mathbf{A})$ is *output stable* if the noncommutative observation operator

$$\widehat{\mathcal{O}}^{nc}_{C,\mathbf{A}} \colon x \mapsto C(I - z_1 A_1 - \cdots - z_d A_d)^{-1} x = \sum_{\gamma \in \mathcal{F}_d} (C\mathbf{A}^\gamma x) z^\gamma$$

maps $\mathcal{X}$ into the Fock space $H^2_{\mathcal{Y}}(\mathcal{F}_d)$; here we use the noncommutative multivariable notation:

$$\mathbf{A}^\gamma = A_{i_N} \cdots A_{i_1} \text{ if } \gamma = i_N \cdots i_1 \in \mathcal{F}_d \text{ with } A^\emptyset = I_{\mathcal{X}}.$$

If $(\mathbf{Z} = (Z_1, \dots, Z_d), X)$ is a multivariable input-pair (so Z_j acts on a state space $\mathcal{X}$ and X is an input operator mapping an input space $\mathcal{U}_I$ into $\mathcal{X}$) such that the output-pair $(X^*, \mathbf{Z}^* = (Z_1^*, \dots, Z_d^*))$ is output-stable, then $\widehat{\mathcal{O}}^{nc}_{X^*,\mathbf{Z}^*}$ maps $\mathcal{X}$ boundedly into $H^2_{\mathcal{U}_I}(\mathcal{F}_d)$ and hence its adjoint $\left(\widehat{\mathcal{O}}^{nc}_{X^*\mathbf{Z}^*}\right)^*$ maps $H^2_{\mathcal{U}_I}(\mathcal{F}_d)$ boundedly into $\mathcal{X}$: in this case we say that the input pair $(\mathbf{Z}, X)$ is *input-stable*. We can use such operators to define interpolation conditions on a noncommutative Schur-class function.

Problem ncLTOA (noncommutative Left Tangential Operator Argument Interpolation Problem): Let $\mathcal{U}_I, \mathcal{U}_O$, $\mathcal{X}$ be Hilbert spaces. Suppose that we are given the data set $(\mathbf{Z}, x, Y)$ with $\mathbf{Z} = (Z_1, \dots, Z_d)$ with each $Z_j \in \mathcal{L}(\mathcal{X})$, $X \in \mathcal{L}(\mathcal{U}_O, \mathcal{X})$, $Y \in \mathcal{L}(\mathcal{U}_I, \mathcal{X})$ such that $(\mathbf{Z}, X)$ is a stable input pair. Find $S \in \mathcal{S}_{nc,d}(\mathcal{U}_I, \mathcal{U}_O)$ such that

$$\left(\widehat{\mathcal{O}}^{nc}_{X^*,\mathbf{Z}^*}\right)^* M_S = \left(\widehat{\mathcal{O}}^{nc}_{Y^*,\mathbf{Z}^*}\right)^*, \tag{4.9}$$

or equivalently,

$$(\widehat{XS})^{\wedge L,nc}(\mathbf{Z}) = Y \tag{4.10}$$

where the noncommutative left tangential operator-argument point-evaluation is given by

$$(\widehat{XS})^{\wedge L,nc}(\mathbf{Z}) = \sum_{\gamma \in \mathcal{F}_d} \mathbf{Z}^{\gamma^\top} X S_\gamma \text{ if } S(z) = \sum_{\gamma \in \mathcal{F}_d} S_\gamma z^\gamma.$$

Here we use the notation $\gamma^\top$ for the *transpose* of the word γ: $\gamma^\top = i_1 \cdots i_N$ if $\gamma = i_N \cdots i_1$.

Problems of this sort have been studied in the literature, e.g. in [64, 35, 10]. The solution of the **ncLTOA** problem via the Grassmannian approach proceeds in a completely analogous fashion as in the commutative case. In this setting, the shift-invariant subspaces are subspaces of $H^2_{\mathcal{U}_O \oplus \mathcal{U}_I}(\mathcal{F}_d)$ which are invariant under the right creation operators

$$R_{z_k} \colon f(z) = \sum_{\gamma \in \mathcal{F}_d f_\gamma z^\gamma} \mapsto f(z) z_k = \sum_{\gamma \in \mathcal{F}_d} f_\gamma z^{\gamma \cdot k}$$

for $k = 1, \dots, d$. We view $H^2_{\mathcal{U}_O \oplus \mathcal{U}_I}(\mathcal{F}_d)$ is a Kreĭn space in the indefinite inner product induced by $J = \left[\begin{smallmatrix} I_{\mathcal{U}_O} & 0 \\ 0 & -I_{\mathcal{U}_I} \end{smallmatrix}\right]$. Graph spaces $\mathcal{G} = \left[\begin{smallmatrix} M_S \\ I \end{smallmatrix}\right] H^2_{\mathcal{U}_I}(\mathcal{F}_d)$ of solutions

S of the **ncLTOA** interpolation problem are characterized by the condition: $\mathcal{G}$ *is a* $H^2_{\mathcal{U}_O \oplus \mathcal{U}_I}(\mathcal{F}_d)$*-maximal negative subspace of*

$$\mathcal{M} := \left\{ f \in H^2_{\mathcal{U}_O \oplus \mathcal{U}_I}(\mathcal{F}_d) \colon \left(\begin{bmatrix} X & -Y \end{bmatrix} f \right)^{\wedge L, nc}(\mathbf{Z}) = 0 \right\} \tag{4.11}$$

which is also shift-invariant. The Pick matrix condition

$$H^2_{\mathcal{U}_O \oplus \mathcal{U}_I} \ominus_J \mathcal{M} \text{ is a positive subspace} \tag{4.12}$$

is necessary for solutions to exist; conversely, if (4.12) holds, then it suffices to look for any shift-invariant subspace $\mathcal{G}$ contained in $\mathcal{M}$ ($\mathcal{M}$ as in (4.11)) which is maximal negative as a subspace of $\mathcal{M}$. Such subspaces $\mathcal{G} = \left[\begin{smallmatrix} M_S \\ I \end{smallmatrix}\right] \cdot H^2_{\mathcal{U}_I}(\mathcal{F}_d)$ can be parametrized via the linear-fractional formula (2.8) (where now the free parameter G is in the noncommutative Schur class $\mathcal{S}_{nc,d}(\mathcal{U}_I, \mathcal{U}_{O,\text{aug}} \oplus \mathcal{U}_O(\mathcal{F}_d))$ if there is a suitable J-Beurling-Lax representation for $\mathcal{M}$. For the case $J = I$, such Beurling-Lax representations (with M_Θ isometric rather than merely partially isometric) have been known for some time (see [61, 65]); we note that the paper [13] derives the $J = I$ Beurling-Lax theorem for the Fock-space setting from the point of view which we have here, where the shift-invariant subspace $\mathcal{M}$ is presented as the kernel of an operator of the form $\left(\widehat{\mathcal{O}}^{nc}_{C,A}\right)^*$. Adaptation of this construction to the J-case (with the complication that M_Θ, while J-isometric, may be unbounded) is carried out in [43]. The following theorem summarizes the results for solving the **ncLTOA** interpolation problem via the Grassmannian approach.

Theorem 4.3. *Suppose that $\mathcal{M}$ is a regular subspace of $H^2_{\mathcal{U}_O \oplus \mathcal{U}_I}(\mathcal{F}_d)$. Then there is a coefficient Kreĭn space $\mathcal{E}$ and a (possibly unbounded) noncommutative Schur-class multiplier S so that M_S is a (possibly unbounded) Kreĭn-space isometry with the bounded extension of $M_\Theta M_\mathcal{E} M_\Theta^* J$ equal to the (bounded) J-orthogonal projection of $H^2_{\mathcal{U}_O \oplus \mathcal{U}_I}(\mathcal{F}_d)$ onto $\mathcal{M}$. In case condition* (4.12) *holds, then one can take $\mathcal{E}$ to have the form $(\mathcal{U}_{O,aug} \oplus \mathcal{U}_O) \oplus \mathcal{U}_I$ with $J_\mathcal{E} = I_{\mathcal{U}_{O,aug} \oplus \mathcal{U}_)} \oplus -I_{\mathcal{U}_I}$.*

If $\mathcal{M}$ comes from a **ncLTOA** *interpolation problem as in* (4.11)*, then condition* (4.12) *holds if and only if*

$$\Lambda := \left(\widehat{\mathcal{O}}^{nc}_{\left[\begin{smallmatrix} X^* \\ Y^* \end{smallmatrix}\right], \mathbf{Z}^*} \right)^* J \widehat{\mathcal{O}}^{nc}_{\left[\begin{smallmatrix} X^* \\ Y^* \end{smallmatrix}\right], \mathbf{Z}^*} \geq 0. \tag{4.13}$$

Then $\mathcal{M}$ is regular if and only if Λ is strictly positive and then the set of all solutions S of the **ncLTOA** *interpolation problem is given by formula* (2.8) *where now the free parameter G is in the noncommutative Schur class $\mathcal{S}_{nc,d}(\mathcal{U}_I, \mathcal{U}_{O,aug} \oplus \mathcal{U}_O)$. Moreover, a realization formula for the representer Θ is given by*

$$\Theta(z) = \mathbf{D} + C(I - z_1 A_1 - \cdots - z_d A_d)^{-1}(z_1 \mathbf{B}_1 + \cdots + z_d \mathbf{B}_d)$$

where the associated colligation matrix

$$\mathbf{U} = \begin{bmatrix} A & \mathbf{B} \\ C & \mathbf{D} \end{bmatrix} = \begin{bmatrix} A_1 & \mathbf{B}_1 \\ \vdots & \vdots \\ A_d & \mathbf{B}_d \\ C & \mathbf{D} \end{bmatrix}$$

is constructed via the same recipe as given in Theorem 4.1*, the one distinction now being that the d-tuple* $\mathbf{Z} = (Z_1, \dots, Z_d)$ *is no longer assumed to be commutative.*

We note that not all multivariable interpolation problems succumb to the Grassmannian/Beurling-Lax approach. Indeed, the lack of a Beurling theorem in the polydisk setting (see e.g. [69]) is the tipoff to the more complicated structures that one can encounter. To get state-space formulas for solutions as we are getting here, one must work with the Schur-Agler class rather than the Schur class; moreover, without imposing additional apparently contrived moment conditions, it is often impossible to get a single linear-fractional formula which parametrizes the set of all solutions; for a recent survey we refer to [28].

References

[1] V.M. Adamjan, D.Z. Arov, and M.G. Kreĭn, *Infinite block Hankel matrices and their connection with interpolation problem*, Amer. Math. Soc. Transl. (2) **111**, 133–156 [Russian original: 1971].

[2] J. Agler and J.E. McCarthy, *Complete Nevanlinna-Pick kernels*, J. Funct. Anal., **175** (2000),111–124.

[3] D. Alpay and D. Kaliuzhnyĭ-Verbovetzkiĭ, *Matrix-J-unitary non-commutative rational formal power series*, in: The State Space Method Generalizations and Applications (Ed. D. Alpay and I. Gohberg), pp. 49–113, **OT161** Birkhäuser-Verlag, Basel, 2006.

[4] A. Arias and G. Popescu, *Noncommutative interpolation and Poisson transforms*, Israel J. Math. **115** (2000), 205–234.

[5] D.Z. Arov, *γ-generating matrices, J-inner matrix-functions and related extrapolation problems*, Teor. Funktsii Funktsional. Anal. i Prilozhen., I, **51** (1989), 61–67; II, **52** (1989), 103-109; III, **53** (1990), 57–64; English transl., J. Soviet Math. I, **52** (1990), 3487–3491; II, **52** (1990), 3421-3425; III, **58** (1992), 532, 537.

[6] W. Arveson, *Subalgebras of C^*-algebras. III. Multivariable operator theory*, Acta Math. **181** (1998), no. 2, 159–228.

[7] W. Arveson, *The curvature invariant of a Hilbert module over* $\mathbb{C}[z_1, \dots, z_d]$, J. Reine Angew. Math. **522** (2000), 173–236.

[8] T.Y. Azizov, I.S. Iokhidov, *Linear Operators in Spaces with an Indefinite Metric*, Wiley & sons Ltd. Chichester, 1989.

[9] J.A. Ball, *Nevanlinna-Pick interpolation: generalizations and applications*, in: Recent Results in Operator Theory Vol. I (Ed. J.B. Conway and B.B. Morrel), Longman Scientific and Tech., Essex, 1988, pp. 51–94.

[10] J.A. Ball and V. Bolotnikov, *Interpolation in the noncommutative Schur-Agler class*, J. Operator Theory **58** (2007), no. 1, 83–126.

[11] J.A. Ball and V. Bolotnikov, *Interpolation problems for Schur multipliers on the Drury-Arveson space: from Nevanlinna-Pick to Abstract Interpolation Problem*, Integral Equations and Operator Theory **62** (2008), 301–349.

[12] J.A. Ball, V. Bolotnikov and Q. Fang, *Multivariable backward-shift invariant subspaces and observability operators*, Multidimens. Syst. Signal Process **18** (2007), 191–248.

[13] J.A. Ball, V. Bolotnikov and Q. Fang, *Schur-class multipliers on the Fock space: de Branges-Rovnyak reproducing kernel spaces and transfer-function realizations*, in: Operator Theory, Structured Matrices, and Dilations, Tiberiu Constantinescu Memorial Volume, Theta Foundation in Advance Mathematics, 2007, 85–114.

[14] J.A. Ball, V. Bolotnikov and Q. Fang, *Transfer-function realization for multipliers of the Arveson space*, J. Math. Anal. Appl. **333** (2007), no. 1, 68–92.

[15] J.A. Ball, V. Bolotnikov and Q. Fang, *Schur-class multipliers on the Arveson space: de Branges-Rovnyak reproducing kernel spaces and commutative transfer-function realizations*, J. Math. Anal. Appl. **341** (2008), no. 1, 519–539.

[16] J.A. Ball, N. Cohen and A.C.M. Ran, *Inverse spectral problems for regular improper rational matrix functions*, in: Topics in interpolation theory of rational matrix-valued functions (Ed. I. Gohberg) OT 33, Birkhäuser Verlag, Basel, 1988, pp. 123–173.

[17] J.A. Ball, I. Gohberg and L. Rodman, *Interpolation of Rational Matrix Functions*, **OT45** Birkhäuser-Verlag, Basel-Boston, 1990.

[18] J.A. Ball, I. Gohberg and M.A. Kaashoek, *Nevanlinna-Pick interpolation for time-varying input-output maps: the discrete case*, in: Time-Variant Systems and Interpolation (Ed. I. Gohberg), OT 56 Birkhäuser-Verlag, Basel, 1992, pp. 1–51.

[19] J.A. Ball, I. Gohberg and M.A. Kaashoek, *Bitangential interpolation for input-output operators of time-varying systems: the discrete time case*, in: New Aspects in Interpolation and Completion Theories (Ed. I. Gohberg), OT 64 Birkhäuser Verlag, Basel, 1993, pp. 33–72.

[20] J.A. Ball, I. Gohberg and M.A. Kaashoek, *Two-sided Nudelman interpolation for input-output operators of discrete time-varying systems*, Integral Equations and Operator Theory **21** (1995), 174–211.

[21] J.A. Ball, G. Groenewald and T. Malakorn, *Structured noncommutative multidimensional linear systems*, SIAM J. Control and Optimization **44** no. 4 (2005), 1474–1528.

[22] J.A. Ball, G. Groenewald and T. Malakorn, *Conservative structured noncommutative multidimensional linear systems*, in: The State Space Method Generalizations and Applications (Ed. D. Alpay and I. Gohberg), pp. 179–223, **OT161** Birkhäuser-Verlag, Basel, 2006.

[23] J.A. Ball, G. Groenewald and T. Malakorn, *Bounded real lemma for structured noncommutative multidimensional linear systems and robust control*, Multidimensional Systems and Signal Processing **17** (2006), 119–150.

[24] J.A. Ball and J.W. Helton, *A Beurling-Lax theorem for the Lie group $U(m,n)$ which contains most classical interpolation*, J. Operator Theory **9** (1983), 107–142.

[25] J.A. Ball and J.W. Helton, *Beurling-Lax representations using classical Lie groups with many applications II: $GL(n,\mathbb{C})$ and Wiener-Hopf factorization*, Integral Equations and Operator Theory **7** (1984), 291–309.

[26] J.A. Ball and J.W. Helton, *Interpolation problems of Pick-Nevanlinna and Loewner types for meromorphic matrix functions: parametrization of the set of all solutions*, Integral Equations and Operator Theory **9** (1986), 155–203.

[27] J.A. Ball and J.W. Helton, *Shift invariant subspaces, passivity, reproducing kernels and H^∞-optimization*, in: Contributions to Operator Theory and its Applications (Ed. J.W. Helton and L. Rodman), **OT35** Birkhäuser Verlag, Basel, 1988, pp. 265–310.

[28] J.A. Ball and S. ter Horst, *Multivariable operator-valued Nevanlinna-Pick interpolation: a survey*, in: Operator Algebras, Operator Theory and Applications (Ed. J.J. Grobler, L.E. Labuschagne, and M. Möller), pages 1–72, **OT195** Birkhäuser, Basel-Berlin, 2009.

[29] J.A. Ball, K.M. Mikkola and A.J. Sasane, *State-space formulas for the Nehari-Takagi problem for nonexponentially stable infinite-dimensional systems*, SIAM J. Control and Optimization **44** (2005) no. 2, 531–563.

[30] J.A. Ball and M.W. Raney, *Discrete-time dichotomous well-posed linear systems and generalized Schur-Nevanlinna-Pick interpolation*, Complex Analysis and Operator Theory **1** (2007), 1–54.

[31] J.A. Ball, T. T. Trent and V. Vinnikov, *Interpolation and commutant lifting for multipliers on reproducing kernel Hilbert spaces*, in: Operator Theory and Analysis (Ed. H. Bart, I. Gohberg and A.C.M. Ran), pp. 89–138, **OT122**, Birkhäuser, Basel, 2001.

[32] J.A. Ball and V. Vinnikov, *Formal reproducing kernel Hilbert spaces: the commutative and noncommutative settings*, in: Reproducing Kernel Spaces and Applications (Ed. D. Alpay), pp. 77–134, **OT143** Birkhäuser-Verlag, Basel, 2003.

[33] J.A. Ball and V. Vinnikov, *Lax-Phillips scattering and conservative linear systems: A Cuntz-algebra multidimensional setting*, Memoirs of the American Mathematical Society, Volume **178** Number 837, American Mathematical Society, Providence, 2005.

[34] J. Bognár, *Indefinite Inner Product Spaces*, Springer-Verlag, Berlin-New York, 1974.

[35] T. Constantinescu and J.L. Johnson, *A note on noncommutative interpolation*, Can. Math. Bull. **46** (2003) no. 1, 59–70.

[36] K.R. Davidson and T. Le, *Commutant lifting for commuting row contractions.* Bull. Lond. Math. Soc. **42** (2010) no. 3, 506-516.

[37] K.R. Davidson and D.R. Pitts, *Nevanlinna-Pick interpolation for non-commutative analytic Toeplitz algebras*, Integral Equations & Operator Theory **31** (1998) no. 2, 401–430.

[38] S.W. Drury, *A generalization of von Neumann's inequality to the complex ball*, Proc. Amer. Math. Soc. **68** (1978), 300–304.

[39] H. Dym, *J Contractive Matrix Functions, Reproducing Kernel Hilbert Spaces and Interpolation*, CBMS Regional Conference series **71**, American Mathematical Society, Providence, 1989.

[40] H. Dym, *Linear fractional transformations, Riccati equations and bitangential interpolation, revisited*, in: Reproducing Kernel Spaces and Applications (Ed. D. Alpay), pp. 171–212, **OT143** Birkhäuser-Verlag, Basel, 2003.

[41] H. Dym, J.W. Helton, and S. McCullough, *The Hessian of a noncommutative polynomial has numerous negative eigenvalues*, J. Anal. Math. **102** (2007), 29-76.

[42] J. Eschmeier and M. Putinar, *Spherical contractions and interpolation problems on the unit ball*, J. Reine Angew. Math. **542** (2002), 219–236.

[43] Q. Fang, *Multivariable Interpolation Problems*, PhD dissertation, Virginia Tech, 2008.

[44] C. Foias and A.E. Frazho, *The Commutant Lifting Approach to Interpolation Problems*, **OT44** Birkhäuser Verlag, Basel-Boston, 1990.

[45] C. Foias, A.E. Frazho, I. Gohberg, and M.A. Kaashoek, *Metric Constrained Interpolation, Commutant Lifting and Systems*, **OT100** Birkhäuser Verlag, Basel-Boston, 1998.

[46] D.C. Greene, S. Richter, and C. Sundberg, *The structure of inner multipliers on spaces with complete Nevanlinna-Pick kernels*, J. Funct. Anal. **194** (2002), 311–331.

[47] P.R. Halmos, *Shifts on Hilbert space*, J. für die Reine und Angewandte Math. **208** (1961), 102–112.

[48] J.W. Helton, *Orbit structure of the Möbius transformation semigroup acting on H^∞ (broadband matching)*, Topics in Functional Analysis (essays dedicated to M.G. Kreĭn on the occasion of his 70th birthday), pp. 129–157, Advances in Math. Suppl. Stud., **3**, Academic Press, New York-London, 1978.

[49] J.W. Helton, J.A. Ball, C.B. Johnson, and M.A. Kaashoek, *Operator Theory, Analytic Functions, Matrices, and Electrical Engineering*, CBMS Regional Conference Series **68**, American Mathematical Society, Providence, 1987.

[50] J.W. Helton, I. Klep, and S. McCullough, *Proper analytic free maps*, J. Funct. Anal. **260** (2011) no. 5, 1476–1490.

[51] J.W. Helton, I. Klep, and S. McCullough, *Analytic mappings between noncommutative pencil balls*, J. Math. Anal. Appl. **376** (2011) no. 2, 407-428.

[52] J.W. Helton, S.A. McCullough, and V. Vinnikov, *Noncommutative convexity arises from linear matrix inequalities*, J. Funct. Anal. **240** (2006) no. 1, 105-191

[53] O. Iftume, M.A. Kaashoek, and A. Sasane, *A Grassmannian band method approach to the Nehari-Takagi problem*, J. Math. Anal. Appl. **310** (2005), 97–115.

[54] V. Katsnelson, A. Kheifets, and P. Yuditskii, *An abstract interpolation problem and extension theory of isometric operators*, in: Operators in Spaces of Functions and Problems in Function Theory (Ed. V.A. Marchenko), pp. 83–96, **146** Naukova Dumka, Kiev, 1987; English translation in: Topics in Interpolation Theory (Ed. H. Dym, B. Fritzsche, V. Katsnelson, and B. Kirstein), pp. 283–298, **OT95**, Birkhäuser, Basel, 1997.

[55] I.V. Kovalishina and V.P. Potapov, *Seven Papers Translated from the Russian*, Amer. Math. Soc. Transl. (2) **138**, Providence, RI, 1988.

[56] M. Möller, *Isometric and contractive operators in Kreĭn spaces*, *St. Petersburg Mathematics J.* **3** (1992) no. 3, 595–611.

[57] S. McCullough and T. T. Trent, *Invariant subspaces and Nevanlinna-Pick kernels*, J. Funct. Anal. 178 (2000), no. 1, 226–249.

[58] G. Meinsma, *Polynomial solutions to H_∞ problems*, Int. J. Robust and Nonlinear Control **4** (1994), 323–351.

[59] V. Olshevsky and V. Pan, *A unified superfast algorithm for boundary rational tangential interpolation problems and for inversion and factorization of dense structure matrices*, in: Proc. 39th IEEE Symposium on Foundations of Computer Science", pp. 192–201, 1998.

[60] V. Olshevsky and A. Shokrollahi, *A superfast algorithm for confluent rational tangential interpolation problem via matrix-vector multiplication for confluent Cauchy-like matrices*, in: Structured Matrices in Mathematics, Computer Science, and Engineering, I (Boulder, CO, 1999), pp. 31-45, Contemp. Math. **280**, Amer. Math. Soc., Providence, RI, 2001.

[61] G. Popescu, *Characteristic functions for infinite sequences of noncommuting operators*, J. Operator Theory **22** (1989), 51–71.

[62] G. Popescu, *Isometric dilations for infinite sequences of noncommuting operators*, Trans. Amer. Math. Soc. **316** (1989) no. 2, 523–536.

[63] G. Popescu, *Multi-analytic operators on Fock spaces*, Math. Ann. **303** (1995), 31–46.

[64] G. Popescu, *Multivariable Nehari problem and interpolation*, J. Funct. Anal. **200** (2003) no. 2, 536–581.

[65] G. Popescu, *Operator theory on noncommutative varieties*, Indiana Univ. Math. J. **55** (2006), no. 2, 389-442.

[66] R. Nevanlinna, *Über beschränkte analytische Funktionen*, Ann. Acad. Sci Fenn. **A32** no. 7 (1929).

[67] A.A.. Nudelman, *On a new problem of moment type*, Soviet Math. Doklady **18** (1977), 507–510.

[68] M. Rosenblum and J. Rovnyak, *Hardy Classes and Operator Theory*, Oxford University PRess, New York, 1985.

[69] W. Rudin, *Function Theory on Polydisks*, Benjamin, New York, 1969.

[70] D. Sarason, *Generalized interpolation in H^∞*, Trans. Amer. Math. Soc. **127** (1967), 179–203.

[71] H.L. Trentelman and J.C. Willems, *H_∞ control in a behavioral context: the full information case*, IEEE Transactions on Automatic Control **44** (1999) no. 3, 521–536.

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