Multipliers of Drury-Arveson Space: a survey

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Dedicated to Prof. Joseph Ball on the occasion of his 70th Birthday

Abstract. The Drury-Arveson Space, as a Hilbert function space, plays an important role in multivariable operator theory. We give a brief survey of some aspects of the multipliers on the space.

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1. Introduction

Let $\mathbb{B}^d$ denote the open unit ball $\{z : |z| < 1\}$ in $\mathbb{C}^d$. The Drury-Arveson space H^2_d ([9, 39]) is the reproducing kernel Hilbert space associated with the kernel

$$K_w(z) = \frac{1}{1 - \langle z, w\rangle}, \quad z, w \in \mathbb{B}^d, \quad \langle z, w\rangle = z_1 w_1 + \cdots + z_d w_d$$

which is a natural multivariable analogue of the Szegö kernel of the classical Hardy space H^2 of the unit disk. Note that H^2_d coincides with H^2 when $d = 1$.

An orthonormal basis of H^2_d is given by $\{e_\alpha\}$ where

$$e_\alpha = \sqrt{\frac{|\alpha|!}{\alpha!}} z^\alpha.$$

Here and in what follows, we use standard multivariable notations: for multi-integers $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{Z}^d_+$ and points $z = (z_1, \ldots, z_d) \in \mathbb{C}^d$ we set

$$|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_d, \qquad \alpha! = \alpha_1!\alpha_2! \ldots \alpha_d!, \qquad z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2} \ldots z_d^{\alpha_d}.$$

For functions $f, g \in H^2_d$ with Taylor expansions

$$f(z) = \sum_{\alpha\in\mathbb{Z}^d_+} c_\alpha z^\alpha \quad \text{and} \quad g(z) = \sum_{\alpha\in\mathbb{Z}^d_+} d_\alpha z^\alpha,$$

their inner product is given by

$$\langle f, g\rangle = \sum_{\alpha\in\mathbf{Z}_+^d} \frac{\alpha!}{|\alpha|!} c_\alpha \overline{d_\alpha}.$$

The Drury-Arveson space H_d^2 can be viewed in many different ways: It can be identified as the symmetric Fock space over $\mathbb{C}^d$; it is a member of the family of Besolv-Hardy-Sobolev spaces; it is a prototype for a complete Pick space; it is a free Hilbert module over the polynomial ring $\mathbb{C}[z_1, \ldots, z_d]$ with the identification of each variable z_i with each multiplication operator M_{z_i}. A panoramic view of most operator theoretic and function theoretic aspects of this space can be found in [67].

A holomorphic function f on $\mathbb{B}^d$ is said to be a *multiplier* of the Drury-Arveson space H_d^2 if $fg \in H_d^2$ for every $g \in H_d^2$. Every multiplier is in H_d^2 since $1 \in H_d^2$. Throughout the paper, we denote the collection of multipliers of H_d^2 by $\mathcal{M}_d$. For each $f \in \mathcal{M}_d$, the multiplication operator M_f defined by $M_f g = fg$ is necessarily bounded on H_d^2 [9], and the operator norm $\|M_f\|$ is also called the multiplier norm of f. We write $\|f\|_{\mathcal{M}_d}$ for its multiplier norm:

$$\|f\|_{\mathcal{M}_d} = \sup\{\|fg\| : g \in H_d^2, \|g\| \leq 1\}.$$

This norm gives $\mathcal{M}_d$ the structure of an operator algebra.

Multipliers are an important part of operator theory on H_d^2. For example, if $\mathcal{E}$ is a closed linear subspace of H_d^2 which is invariant under $M_{z_1}, \ldots, M_{z_d}$, then there exist $\{f_1, \ldots, f_k, \ldots\} \subset \mathcal{M}_d$ such that the operator

$$M_{f_1}M_{f_1}^* + \cdots + M_{f_k}M_{f_k}^* + \cdots$$

is the orthogonal projection from H_d^2 onto $\mathcal{E}$ (see p. 191 in [10]). It is known that $\mathcal{M}_d$ is the home for the multivariate von Neumann inequality hence plays a similar role of H^∞, the algebra of bounded holomorphic functions on the unit disk to higher dimensions. The multiplier algebra $\mathcal{M}_d$ is exactly the image of the free-semigroup algebra $\mathcal{F}_d$ (generated by d letters) after applying a point-evaluation map associated with points in the unit ball. Interpolation problems for Schur multipliers (contractive multipliers) H_d^2 related to multidimentional system theory have been intensively studied over the past few decades (see [12, 24, 41, 50, 54]). A corona theorem for the Drury-Averson space multipliers was proved by Costea, Sawyer and Wick [33]. Clouâtre and Davidson studied Henkin measures for $\mathcal{M}_d$ in [30] and the ideals of the closure of the polynomial multipliers on the Drury-Arveson space in [32].

In this paper we survey some results and methods related to multipliers of H_d^2. Results are presented below without proof but with references. This survey is not intended to be comprehensive in any way and we will have to limit the article to the aspects that are most familiar to us. The present paper is organized as follows. Following this introduction, in Section 2 we review an important class of multipliers - Schur class multipliers or the unit ball of $\mathcal{M}_d$. We revisit transfer function realization and related Nevanlinna-Pick type interpolation problems for these multipliers. In Section 3 we discuss polynomial multipliers and nonpolynomial multipliers. A corona theorem and some spectral properties for multipliers of H_d^2 are included in Section 4. In Section 5 we consider commutators involving multipliers and localizations. In the last section we discuss the problem of characterization of multipliers in $\mathcal{M}_d$.

2. Schur class multipliers

We will start with a special class of multipliers, the so-called Schur class multipliers. In single variable complex analysis the Schur class is the set of holomorphic functions $S(z)$ that are bounded by one on the unit disk. Schur functions are important, not only because they arise in diverse areas of classical analysis and operator theory, but also because they have connections with linear system theory and engineering.

Recall that the Schur class plays a prominent role in classical moment and interpolation problems. One of the best known examples is the **Nevanlinna-Pick interpolation problem** :

Given points $z_1, \cdots, z_n$ in the unit disk $\mathbb{D}$ and complex numbers $w_1, \cdots, w_d$. Find a Schur function $S(z)$ such that $S(z_i) = w_i$ for $i = 1, \cdots, n$.

A solution to this problem exists if and only if the associated Pick matrix

$$\left\{ \frac{1 - w_i \bar{w}_j}{1 - z_i \bar{z}_j} \right\}_{i,j=1}^{n}$$

is positive semidefinite. There are several approaches dealing with the problem. One operator-theoretic approach can be described as follows:

Let M_S be the operator of multiplication by a Schur function $S(z)$ on the Hardy space H^2 of the unit disk. Then M_S commutes with M_z, the operator of multiplication by the coordinate z (the shift operator) and any contraction on H^2 which commutes with M_z has this form for some Schur function. The commutation relation is preserved under compressions of the operators to any invariant subspace E of the backward shift M_z^*. The generalized interpolation theorem of Sarason [66] showed that every contraction on E which commutes with the compression of M_z to E is associated with some Schur function. Particular choice of the invariant subspace leads to solutions to the Nevanlinna-Pick problem. The celebrated commutant lifting theorem by Sz.Nagy-Foias [47] extends the conclusion to arbitrary Hilbert space contraction operators. In the book [20], Ball, Gohberg, and Rodman considered the interpolation of rational matrix functions. They emphasized the state space approach and transfer function realization.

When leaving the univariate setting there are different interesting multivariable counterparts of the classical Schur class [1]. Here we review some aspects of the class of contractive operator-valued multipliers for the Drury-Arveson space.

For a Hilbert space $\mathcal{Y}$, we use notation $H_{\mathcal{Y}}(k_d)$ for the Drury-Arveson space of $\mathcal{Y}$-valued functions. Given two Hilbert spaces $\mathcal{U}$ and $\mathcal{Y}$, we denote by $\mathcal{S}_d(\mathcal{U},\mathcal{Y})$ the class of $\mathcal{L}(\mathcal{U},\mathcal{Y})$-valued functions S on $\mathbb{B}^d$ such that the multiplication operator $M_S: \ f \mapsto S \cdot f$ defines a contraction from $H_{\mathcal{U}}(k_d)$ into $H_{\mathcal{Y}}(k_d)$, or equivalently, such that the de Branges-Rovnyak kernel

$$K_S(\lambda, z) = \frac{I_{\mathcal{Y}} - S(\lambda)S(z)^*}{1 - \langle \lambda, z\rangle} \tag{2.1}$$

is positive on $\mathbb{B}^d \times \mathbb{B}^d$.

It is readily seen that the class $\mathcal{S}_1(\mathcal{U},\mathcal{Y})$ is the classical Schur class. In general, it follows from $K_S \geq 0$ that S is holomorphic and takes contractive values on $\mathbb{B}^d$. However, for $d > 1$ there are holomorphic contractive-valued functions on $\mathbb{B}^d$ not in $\mathcal{S}_d$. The class $\mathcal{S}_d(\mathcal{U},\mathcal{Y})$ can be characterized in various ways similarly to the one-variable situation. Here we review the characterizations of these multipliers in terms of realizations due to Ball,Trent and Vinnikov in [24]. We review this result in the form we used in [17].

Theorem 2.1. *Let S be an $\mathcal{L}(\mathcal{U},\mathcal{Y})$-valued function defined on $\mathbb{B}^d$. The following are equivalent:*

1. *S belongs to $\mathcal{S}_d(\mathcal{U},\mathcal{Y})$.*
2. *The kernel*
$$K_S(\lambda, z) = \frac{I_{\mathcal{Y}} - S(\lambda)S(z)^*}{1 - \langle \lambda, z\rangle} \tag{2.2}$$
is positive on $\mathbb{B}^d \times \mathbb{B}^d$, i.e., there exists an operator-valued function $H\colon \mathbb{B}^d \to \mathcal{L}(\mathcal{H},\mathcal{Y})$ for some auxiliary Hilbert space $\mathcal{H}$ so that
$$K_S(\lambda, z) = H(\lambda)H(z)^*. \tag{2.3}$$
3. *There exists a Hilbert space $\mathcal{X}$ and a unitary connecting operator (or colligation) $\mathbf{U}$ of the form*
$$\mathbf{U} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A_1 & B_1 \\ \vdots & \vdots \\ A_d & B_d \\ C & D \end{bmatrix} : \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \mathcal{X}^d \\ \mathcal{Y} \end{bmatrix} \tag{2.4}$$
so that $S(\lambda)$ can be realized in the form

$$\begin{aligned} S(\lambda) &= D + C\left(I_{\mathcal{X}} - \lambda_1 A_1 - \cdots - \lambda_d A_d\right)^{-1}(\lambda_1 B_1 + \ldots + \lambda_d B_d) \\ &= D + C(I - Z(\lambda)A)^{-1}Z(\lambda)B, \end{aligned} \tag{2.5}$$

where we set

$$Z(\lambda) = \begin{bmatrix} \lambda_1 I_{\mathcal{X}} & \dots & \lambda_d I_{\mathcal{X}} \end{bmatrix}, \quad A = \begin{bmatrix} A_1 \\ \vdots \\ A_d \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ \vdots \\ B_d \end{bmatrix}. \tag{2.6}$$

4. *There exists a Hilbert space $\mathcal{X}$ and a contractive connecting operator* $\mathbf{U}$ *of the form* (2.4) *so that $S(\lambda)$ can be realized in the form* (2.5).

In analogy with the univariate case, a realization of the form (2.5) is called *coisometric*, *isometric*, *unitary* or *contractive* if the operator $\mathbf{U}$ has the said property. It turns out that a more useful analogue of "coisometric realization" in the classical univariate case is not the condition that $\mathbf{U}^*$ be isometric, but rather that $\mathbf{U}^*$ be isometric on a certain subspace of $\mathcal{X}^d \oplus \mathcal{Y}$.

Definition 2.2. *A realization* (2.5) *of $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ is called* weakly coisometric *if the adjoint $\mathbf{U}^* : \mathcal{X}^d \oplus \mathcal{Y} \to \mathcal{X} \oplus \mathcal{U}$ of the connecting operator is contractive and isometric on the subspace $\begin{bmatrix} \mathcal{D} \\ \mathcal{Y} \end{bmatrix} \subset \begin{bmatrix} \mathcal{X}^d \\ \mathcal{Y} \end{bmatrix}$ where*

$$\mathcal{D} := \overline{\operatorname{span}}\{Z(z)^*(I_{\mathcal{X}} - A^*Z(z)^*)^{-1}C^*y\colon \ \ z \in \mathbb{B}^d,\, y \in \mathcal{Y}\} \subset \mathcal{X}^d. \tag{2.7}$$

For any $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$, the associated kernel K_S (2.1) is positive on $\mathbb{B}^d \times \mathbb{B}^d$ so we can associate with S the de Branges-Rovnyak reproducing kernel Hilbert space $\mathcal{H}(K_S)$. In parallel to the univariate case, $\mathcal{H}(K_S)$ is the state space of certain *canonical functional-model realization* for S (see [13]).

Definition 2.3. *We say that the contractive operator-block matrix*

$$\mathbf{U} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} \mathcal{H}(K_S) \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \mathcal{H}(K_S)^d \\ \mathcal{Y} \end{bmatrix}$$

is a canonical functional-model colligation *for the given function $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ if*

1. *The operator $A = \begin{bmatrix} A_1 \\ \vdots \\ A_d \end{bmatrix}$ solves the Gleason problem for $\mathcal{H}(K_S)$, i.e.,*

$$f(z) - f(0) = \sum_{j=1}^{d} z_j (A_j f)(z) \quad \textit{for all} \quad f \in \mathcal{H}(K_S).$$

2. *The operator $B = \begin{bmatrix} B_1 \\ \vdots \\ B_d \end{bmatrix}$ solves the Gleason problem for S:*

$$S(z)u - S(0)u = \sum_{j=1}^{d} z_j (B_j u)(z) \quad \textit{for all} \quad u \in \mathcal{U}.$$

3. *The operators $C : \mathcal{H}(K_S) \to \mathcal{Y}$ and $D : \mathcal{U} \to \mathcal{Y}$ are given by*

$$C\colon f \mapsto f(0), \quad D\colon u \mapsto S(0)u.$$

It was shown in [18] that any Schur-class function S with associated de Branges-Rovnyak space $H(K_S)$ finite-dimensional and not $(M_{z_1}^*, \dots, M_{z_d}^*)$-invariant does not admit a contractive commutative realization. Here a realization is said to be *commutative* if the state space operators $A_1, \dots, A_d$ commute with each other. The following result in [18] shows when a Schur-class function S admits a commutative weakly coisometric realization.

Theorem 2.4. *A Schur-class function $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ admits a commutative weakly coisometric realization if and only if the following conditions hold:*

1. *The associated de Branges-Rovnyak space $H(K_S)$ is $(M_{z_1}^*, \dots, M_{z_d}^*)$-invariant, and*
2. *the inequality*

$$\sum_{j=1}^{d} \|M_{z_j}^* f\|_{H(K_S)}^2 \le \|f\|_{H(K_S)}^2 - \|f(0)\|_{\mathcal{Y}}^2 \quad \textit{holds for all} \quad f \in H(K_S). \tag{2.8}$$

Furthermore, if conditions (1) and (2) are satisfied, then there exists a commutative canonical functional model colligation for S. Moreover, the state-space operators tuple is equal to the Drury-Arveson backward shift restricted to $H(K_S)$: $A_j = M_{z_j}^*|_{H(K_S)}$ for $j = 1, \dots, d$.

Note that condition (2) in Theorem 2.4 means that M_z^* is a contractive solution to the Gleason problem for $H(K_S)$ ([48]). Weakly coisometric realizations for an $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ can be constructed in certain canonical way as follows. Upon applying Aronszajn's construction to the kernel K_S, (which is positive on $\mathbb{B}^d$ by Theorem 2.1), one gets the de Branges-Rovnyak space $H(K_S)$. A weakly coisometric realization for S with the state space equal to $H(K_S)$ (and output operator C equal to evaluation at zero on $H(K_S)$) will be called a *generalized functional-model realization.*

As shown in [17], any function $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ admits a generalized functional-model realization. In the univariate case, this reverts to the well known de Branges-Rovnyak functional-model realization [26, 27]. Another parallel to the univariate case is that *any* observable (i.e., the observability operator $\mathcal{O}_{C,\mathbf{A}}$ is injective: $C(I_{\mathcal{X}} - Z(\lambda)A)^{-1}x = 0$ implies $x = 0$.) weakly coisometric realization of a Schur-class function $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ is unitarily equivalent to some generalized functional-model realization (observability is a minimality condition that is fulfilled automatically for every generalized functional-model realization). However, in contrast to the univariate case, this realization is not unique in general (even up to unitary equivalence); moreover, a function $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ may admit generalized functional-model realizations with the same state space operators $A_1, \dots, A_d$ and different input operators B_j's.

Recently Jury and Martin studied the case when the realization is unique in [60, 61, 62]. They introduced the notion of a *quasi-extreme* multiplier of the Drury-Arveson space H_d^2 for a multiplier associated with a unique generalized functional-model realization. They gave some characterizations of these multipliers. Here are some characterizations of quasi-extremity, which imply that every quasi-extreme multiplier of H^2_d is in fact an extreme point of the unit ball of the multiplier algebra $\mathcal{M}_d$. (The converse statement, namely whether or not every extreme point is quasi-extreme, remains an open question.)

Theorem 2.5. *Let S be a contractive multiplier of H^2_d (to ease the notation assume $\mathcal{U} = \mathcal{Y} = \mathbb{C}$). The following are equivalent:*

1. *S is quasi-extreme.*
2. *the only multiplier T satisfying*
$$M_T^* M_T + M_S^* M_S \leq I \tag{2.9}$$
is $T \equiv 0$.
3. *There is a unique contractive solution $(X_1, \dots X_d)$ to the Gleason problem in $H(K_S)$.*
4. *There exists a contractive solution $(X_1, \dots X_d)$ such that the equality $\sum_{j=1}^d \|X_j f\|_f^2 = \|f\|^2_{H(K_S)} - |f(0)|^2$ holds for every $f \in H(K_S)$.*
5. *$H(K_S)$ does not contain the constant functions.*

Let $A = (A_1, \dots, A_d)$ be a commutative d-tuple of bounded, linear operators on the Hilbert space $\mathcal{X}$. If $C \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, then the pair (C, A) is said to be *output-stable* if the associated observability operator

$$\widehat{\mathcal{O}}_{C,A} \colon x \mapsto C(I - \lambda_1 A_1 - \cdots - \lambda_d A_d)^{-1} x$$

maps $\mathcal{X}$ into $H_{\mathcal{Y}}(k_d)$, or equivalently (by the closed graph theorem), the observability operator is bounded. Just as in the single-variable case, there is a system-theoretic interpretation (in the context of multidimensional systems) for this operator (see [15] for details). The following is a theorem about the left-tangential operator-argument interpolation (**LTOA**) problem formulated for the Drury-Arveson Schur-multiplier class $\mathcal{S}_d(\mathcal{U}, \mathcal{Y})$.

Theorem 2.6. *Suppose that we are given an auxiliary Hilbert space $\mathcal{X}$ together with commutative d-tuples*

$$Z^{(1)} = (Z_1^{(1)}, \dots, Z_d^{(1)}), \dots, Z^{(N)} = (Z_1^{(N)}, \dots, Z_d^{(N)}) \in \mathcal{L}(\mathcal{X})^d,$$

i.e., $Z_k^{(i)} \in \mathcal{L}(\mathcal{X})$ for $i = 1, \dots, N$ and $k = 1, \dots, d$ and for each fixed i, the operators $Z_1^{(i)}, \dots, Z_d^{(i)}$ commute pairwise, with the property that each d-tuple $Z^{(i)}$ has joint spectrum contained in $\mathbb{B}^d$ (or each $(X_i^*, Z^{*(i)})$ is an output stable pair). Assume in addition that we are given operators $X_1, \dots, X_N$ in $\mathcal{L}(\mathcal{Y}, \mathcal{X})$ and operators $Y_1, \dots, Y_N$ in $\mathcal{L}(\mathcal{U}, \mathcal{X})$. Then there is an $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ so that

$$(X_i S)^{\wedge L}(Z^{(i)}) := \sum_{n \in \mathbb{Z}_+^d} (Z^{(i)})^n X_i \mathcal{S}_n = Y_i \quad \text{for} \quad i = 1, \dots, N$$

if and only if the associated Pick matrix

$$\mathbb{P}_{LTOA} := \left[\sum_{n\in\mathbb{Z}_+^d} (Z^{(i)})^n (X_i X_j^* - Y_i Y_j^*)(Z^{(j)})^{n*} \right]_{i,j=1}^N$$

is positive semidefinite. Here $Z^n = Z_1^{n_1}\cdots Z_d^{n_d}$ *if* $Z = (Z_1,\ldots,Z_d) \in \mathcal{L}(\mathcal{C})^d$ *and* $n = (n_1,\ldots,n_d) \in \mathbb{Z}_+^d$.

Similarly one can pose right tangential operator-argument interpolation and bitangential operator-argument Nevanlinna-Pick problems. We note that this and related interpolation problems were studied in [12] by using techniques from reproducing kernel Hilbert spaces, Schur-complements and isometric extensions from the work of [40, 52, 51]. In [42, 19] we showed how the problem can be handled via the Grassmannian approach. We also refer to [22] for a comprehensive survey on the related topics and [14] for discussions of different approaches for bitangential matrix Nevanlinna-Pick interpolation problems.

3. Polynomial v. non-polynomial multipliers

It is easy to see that all polynomials are multipliers of the Drury-Arveson space: $\mathbb{C}[z_1,\cdots,z_d] \subseteq \mathcal{M}_d$. Naturally we would like to see how differently the non-polynomials multipliers behave compared to polynomial multipliers.

Recall that a commuting tuple of bounded operators $(A_1,\ldots,A_d)$ on a Hilbert space H is said to be a *row contraction* if it satisfies the inequality

$$A_1A_1^* + \cdots + A_dA_d^* \le 1.$$

The d-shift $(M_{z_1},\ldots,M_{z_d})$ on H_d^2 is a natural example of row contraction. In fact, the d-shift is the “master” row contraction in the sense that for each polynomial $p \in \mathbb{C}[z_1,\ldots,z_d]$, the von Neumann inequality

$$\|p(A_1,\ldots,A_d)\| \le \|p(M_{z_1},\ldots,M_{z_d})\|$$

holds whenever the commuting tuple $(A_1,\ldots,A_d)$ is a row contraction [9, 39].

In the single variable case, it is well known that the space of multipliers of the classic Hardy space H^2 is the space of bounded holomorphic functions on the unit disk, i.e. $\mathcal{M}_1 = H^\infty(\mathbb{D})$. The multiplier norm of a multiplier f, $\|M_f\|$ is equal to $\|f\|_\infty = \sup_{|z|<1}|f(z)|$. However, for $d \ge 2$, Arveson showed in [9] that $\mathcal{M}_d$ is strictly smaller than $H^\infty(\mathbb{B}_d)$, the space of bounded holomorphic functions on $\mathbb{B}_d$. Moreover, even for polynomials q, $\|q\|_\infty$ in general does not dominate the operator norm of M_q on H_d^2, see [9, 67].

Theorem 3.1. *For* $d > 1$ *the norms* $\|\cdot\|_\infty$ *and* $\|\cdot\|_{\mathcal{M}_d}$ *are not comparable on* $\mathcal{M}_d$. *There is a strict containment* $\mathcal{M}_d \subset H^\infty(\mathbb{B}_d)$, *and the d-shift* M_z *is not subnormal, that is,* M_z *does not have a joint normal extension.*

Note that it can be shown that if q is a *polynomial*, then

$$\|M_q\|_{\mathcal{Q}} = \|q\|_\infty, \tag{3.1}$$

where $\|M_q\|_{\mathcal{Q}}$ is the essential norm of q. Recall that the *essential norm* of a bounded operator A on a Hilbert space $\mathcal{H}$ is

$$\|A\|_{\mathcal{Q}} = \inf\{\|A+K\| : K \in \mathcal{K}(\mathcal{H})\},$$

where $\mathcal{K}(\mathcal{H})$ is the collection of compact operators on $\mathcal{H}$. Alternately, $\|A\|_{\mathcal{Q}} = \|\pi(A)\|$, where π denotes the quotient homomorphism from $\mathcal{B}(\mathcal{H})$ to the Calkin algebra $\mathcal{Q} = \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$. Indeed by Proposition 5.3 in [9], for each polynomial q, the operator M_q is *essentially normal*, i.e., $[M_q^*, M_q]$ is compact. On the other hand, by Proposition 2.12 in [9], if q is a polynomial, then the spectral radius of M_q equals $\|q\|_\infty$. Since the norm and the spectral radius of any normal element in any C^*-algebra coincide, it follows that $\|M_q\|_{\mathcal{Q}} \le \|q\|_\infty$ whenever q is a polynomial. The reverse inequality, $\|M_q\|_{\mathcal{Q}} \ge \|q\|_\infty$, can be achieved simply by applying M_q^* to the normalized reproducing kernel of H_d^2.

It turns out that (3.1) in general fails if we consider multipliers which are not polynomials ([44]):

Theorem 3.2. *There exists a sequence $\{\psi_k\} \subset \mathcal{M}_d$ such that*

$$\inf_{k\ge 1} \|M_{\psi_k}\|_{\mathcal{Q}} > 0 \qquad \textit{and} \qquad \lim_{k\to\infty} \|\psi_k\|_\infty = 0.$$

This has implications for other essential properties of multipliers. Recall that an operator T is said to be *hyponormal* if $T^*T - TT^* \ge 0$ and an operator T is said to be *essentially hyponormal* if there is a compact self-adjoint operator K such that

$$T^*T - TT^* + K \ge 0.$$

Obviously, T is essentially hyponormal if and only if $\pi(T)$ is a hyponormal element in the Calkin algebra $\mathcal{Q}$, i.e., $\pi(T^*)\pi(T) - \pi(T)\pi(T^*) \ge 0$. It is well known that the norm of a hyponormal operator coincides with its spectral radius. As we mentioned earlier, by Proposition 2.12 in [9], if q is a polynomial, then the spectral radius of M_q equals $\|q\|_\infty$. Therefore if q is a polynomial such that $\|M_q\| > \|q\|_\infty$, then M_q is not hyponormal. Thus there are plenty of multipliers $f \in \mathcal{M}_d$ for which M_f fails to be hyponormal on H_d^2. This is one phenomenon that sets the Drury-Arveson space H_d^2 apart from the Hardy space and the Bergman space. Note that the phenomenon persists under compact perturbation too.

Theorem 3.3. *There exists a $\psi \in \mathcal{M}_d$ such that the multiplication operator M_ψ on H_d^2 is not essentially hyponormal.*

Let $\mathcal{A}_d$ be the norm closure of the polynomials in $\mathcal{M}_d$. We can see that $\mathcal{A}_d \subset A(\mathbb{B}^d)$, the ball algebra. Thus all multipliers in $\mathcal{A}_d$ are continuous on $\overline{\mathbb{B}^d}$. Note that there are continuous multipliers which are not in $\mathcal{A}_d$. Since the multiplier norm and the supremum norm are not comparable, the image of $\mathcal{A}_d$ inside of $A(\mathbb{B}^d)$ is not closed. It can be shown that the maximal ideal space of $\mathcal{A}_d$ is homeomorphic to $\overline{\mathbb{B}_d}$.

In [31], Clouâtre and Davidson identified $\mathcal{A}_d$ as a direct sum of the preduals of $\mathcal{M}_d$ and of a commutative von Neumann algebra $\mathfrak{W}$,

$$\mathcal{A}_d^* \simeq \mathcal{M}_{d*} \oplus_1 \mathfrak{W}_*.$$

They established analogues of several classical results concerning the dual space of the ball algebra. These developments are deeply intertwined with the problem of peak interpolation for multipliers. It is also worth mentioning that they shed light on the nature of the extreme points of the unit ball of $\mathcal{A}_d^*$. The following results (Theorem 7.5/Theorem 7.7 in [31]) ensure the existence of many extreme points in the closed unit ball of $\mathcal{M}_{d*}$, thus showing a sharp contrast with the more classical situation of the closed unit ball of H^∞ [6].

Theorem 3.4. *Let $f \in \mathcal{A}_d$ with $\|f\|_\infty < \|f\|_{\mathcal{A}_d} = 1$. The set*

$$\mathcal{F} = \{\Psi \in \mathcal{M}_{d*} : \|\Psi\|_{\mathcal{A}_d^*} = 1 = \Psi(f)\}$$

has extreme points, which are also extreme points of the closed unit ball of $\mathcal{M}_{d*}$.

Theorem 3.5. *The following statements hold.*

1. *The set of weak-$*$ exposed points of $\overline{b_1(\mathcal{A}_d^*)}$ that lie in $\mathfrak{W}_*$ is $\{\lambda\tau_\zeta : \lambda \in \mathbb{T}, \zeta \in \mathbb{S}^d\}$, where $\mathbb{T}$ is the unit circle and $\mathbb{S}^d$ is the unit sphere. This set is weak-$*$ compact and it coincides with the extreme points of $\overline{b_1(\mathfrak{W}_*)}$.*
2. *Let $\Phi \in \overline{b_1(\mathcal{M}_{d*})}$ be a weak-$*$ exposed point of $\overline{b_1(\mathcal{A}_d^*)}$, and let $f \in \overline{b_1(\mathcal{A}_d)}$ such that*
$$\mathrm{Re}\Psi(f) < 1 = \mathrm{Re}\Phi(f) \quad \textit{for all} \quad \Psi \in \overline{b_1(\mathcal{A}_d^*)}, \Psi \neq \Phi.$$
Then, $1 = \|f\|_{\mathcal{A}_d} > \|f\|_\infty$.
3. *If $1 = \|f\|_{\mathcal{A}_d} > \|f\|_\infty$ and $N = \{\xi \in H_d^2 : \|f\xi\|_{H_d^2} = \|\xi\|_{H_d^2}\}$ is one dimensional, then the functional $[\xi(f\xi)^*]$ is a weak-$*$ exposed point of $\overline{b_1(\mathcal{A}_d^*)}$.*
4. *The extreme points of $\overline{b_1(\mathcal{M}_{d*})}$ are contained in the weak-$*$ closure of the set*
$$\big\{[\xi(f\xi)^*] : 1 = \|\xi\|_{H_d^2} = \|f\xi\|_{H_d^2} = \|f\|_{\mathcal{A}_d} > \|f\|_\infty\big\}.$$

4. Corona theorem and Spectral theory

Carleson's corona theorem for H^∞ in [29] states that the open unit disk is dense in the maximal ideal space of H^∞. Costea, Sawyer and Wick extended this to multiplier algebras of certain Besov-Sobolev spaces on the unit ball including the multiplier algebra of the Drury-Arveson space. Here is the version of the Corona theorem for $\mathcal{M}_d$ ([33]):

Theorem 4.1. *The corona theorem holds for the multiplier algebra $\mathcal{M}_d$ of the Drury-Arveson space. That is, for $g_1, \ldots, g_k \in \mathcal{M}_d$, if there is a $c > 0$ such that*

$$|g_1(z)| + \cdots + |g_k(z)| \geq c$$

for every $z \in \mathbb{B}^d$, then there exist $f_1, \ldots, f_k \in \mathcal{M}_d$ such that

$$f_1 g_1 + \cdots + f_k g_k = 1.$$

An immediate consequence of this theorem is the so-called *one function corona theorem.*

Theorem 4.2. *Let $f \in \mathcal{M}_d$. If there is a $c > 0$ such that $|f(z)| \geq c$ for every $z \in \mathbb{B}^d$, then $1/f \in \mathcal{M}_d$.*

There have been several different proofs of this one function corona theorem without invoking the general corona theorem [45, 64, 28]. Also we note that it is true that $1/f \in H^2_d$ for any $f \in H^2_d$ with a lower bound when $d \leq 3$. This is because for any $f \in H^2_d$, $\|f\|_{H^2_d}$ is equivalent to the norm of Rf in the Bergman space if $d = 2$ and in the Hardy space if $d = 3$, where R is the radial derivative. The problem is completely open for $d \geq 4$ since for $d \geq 4$, the norm in the Drury-Arveson space involves higher radial derivatives. There have been related discussions ([4], [64]) but the problem for the general case still requires new ideas.

Recently, in the context of more general Hardy-Sobolev spaces, Cao, He and Zhu developed some spectral theory for multipliers of these spaces ([28]). For $\mathcal{M}_d$ the following results hold. The proofs use one function corona theorem and estimates of higher order radial derivatives.

Theorem 4.3. *Suppose $f \in \mathcal{M}_d$.*

1. *The spectrum of M_f is the closure of $f(\mathbb{B}^d)$ in the complex plane.*
2. *The essential spectrum of M_f is given by*
$$\sigma_e(M_f) = \bigcap_{r \in (0,1)} \overline{f(\mathbb{B}^d - r\mathbb{B}^d)},$$
where $r\mathbb{B}^d = \{z \in \mathbb{C}^d : |z| < r\}$
3. *M_f is Fredholm if and only if there exist $r \in (0,1)$ and $\delta > 0$ such that $|f(z)| \geq \delta$ for all $z \in \mathbb{B}^d - r\mathbb{B}^d$. Moreover, when M_f is Fredholm, its Fredholm index is always 0 for $d > 1$ and is equal to minus the winding number of the mapping $e^{it} \mapsto f(re^{it})$, where $r \in (0,1)$ is sufficiently close to 1.*

5. Commutators and localization

If we take a list of Hardy-space results and try to determine which ones have analogues on H^2_d and which ones do not, commutators are certainly very high on any such list. One prominent part of the theory of the Hardy space is the Toeplitz operators on it. Since there is no L^2 associated with H^2_d, the only analogue of Toeplitz operators on H_d^2 are the multipliers. We can consider the commutators of the form $[M_f^*, M_{z_i}]$, where f is a multiplier for the Drury-Arveson space.

Recall that for each $1 \le p < \infty$, the Schatten class $\mathcal{C}_p$ consists of operators A satisfying the condition $\|A\|_p < \infty$, where the p-norm is given by the formula

$$\|A\|_p = \{\mathrm{tr}((A^*A)^{p/2})\}^{1/p}.$$

Arveson showed in his seminal paper [9] that commutators of the form $[M_{z_j}^*, M_{z_i}]$ on H_d^2 all belong to $\mathcal{C}_p$, $p > d$. As the logical next step, one certainly expects a Schatten class result for commutators on H_d^2 involving multipliers other than the simplest coordinate functions. The following result was proved in [43].

Theorem 5.1. *Let f be a multiplier for the Drury-Arveson space H_d^2. For each $1 \le i \le d$, the commutator $[M_f^*, M_{\zeta_i}]$ belongs to the Schatten class $\mathcal{C}_p$, $p > 2d$. Moreover, for each $2d < p < \infty$, there is a constant C which depends only on p and n such that*

$$\|[M_f^*, M_{z_i}]\|_p \le C\|M_f\|$$

for every multiplier f of H_d^2 and every $1 \le i \le d$.

This Schatten-class result has C^*-algebraic implications.

Let $\mathcal{T}_d$ be the C^*-algebra generated by $M_{z_1}, \cdots, M_{z_d}$ on H_d^2. Recall that $\mathcal{T}_d$ was introduced by Arveson in [9]. In more ways than one, $\mathcal{T}_d$ is the analogue of the C^*-algebra generated by Toeplitz operators with *continuous* symbols. Indeed Arveson showed that there is an exact sequence

$$\{0\} \to \mathcal{K} \to \mathcal{T}_d \xrightarrow{\tau} C(\mathbb{S}^d) \to \{0\}, \tag{5.1}$$

where $\mathcal{K}$ is the collection of compact operators on H_d^2. But there is another natural C^*-algebra on H_d^2 which is also related to "Toeplitz operators", where the symbols are not necessarily continuous. We define

$$\mathcal{TM}_d = \text{the } C^*\text{-algebra generated by } \{M_f : fH_d^2 \subset H_d^2\}.$$

Theorem 1.1 tells us that $\mathcal{T}_d$ is contained in the essential center of $\mathcal{TM}_d$, in analogy with the classic situation on the Hardy space of the unit sphere S. This opens the door for us to use the classic localization technique [38] to analyze multipliers.

Let S_w be a class of Schur multipliers defined as follows: For each $w \in \mathbb{B}^d$, let

$$S_w(z) = \frac{1 - |w|}{1 - \langle z, w\rangle}. \tag{5.2}$$

Note that the norm of the operator M_{S_w} on H_d^2 is 1. Here is a localization result shown in [43].

Theorem 5.2. *Let $A \in \mathcal{TM}_d$. Then for each $\xi \in \mathbb{S}^d$, the limit*

$$\lim_{r\uparrow 1} \|AM_{\mathcal{S}_{r\xi}}\| \tag{5.3}$$

exists. Moreover, we have

$$\|A\|_{\mathcal{Q}} = \sup_{\xi\in\mathbb{S}^d} \lim_{r\uparrow 1} \|AM_{\mathcal{S}_{r\xi}}\|.$$

Alternatively, we can state this result in a version which may be better suited for applications:

Theorem 5.3. *For each $A \in \mathcal{TM}_d$, we have*

$$\|A\|_{\mathcal{Q}} = \lim_{r\uparrow 1} \sup_{r\le|w|<1} \|AM_{S_w}\|.$$

In addition to the analogue of Toeplitz operators and the Toeplitz algebra, it is also interesting to consider possible analogues of Hankel operators in the setting of H^2_d. This is more difficult than the Toeplitz case, since there isn't an L^2 associated with H^2_d. But there are analogous problems. For example, we may ask the following questions: Suppose $f \in \mathcal{M}_d$. Under what condition on f is the commutator $[M_f^*, M_f]$ compact? Under what condition on f does the commutator $[M_f^*, M_f]$ belong to the Schatten class $\mathcal{C}_p$, $p > d$?

The C^*-algebra $\mathcal{TM}_d$ itself is quite interesting. To see why, let us consider the analogous situation on the Hardy space. Let $H^2(\mathbb{S}^d)$ be the Hardy space on the unit sphere. On $H^2(\mathbb{S}^d)$, we naturally have

$$\mathcal{T}(H^\infty(\mathbb{S}^d)) = C^*\text{-algebra generated by } \{M_f : f \in H^\infty(\mathbb{S}^d)\}.$$

Note that $H^\infty(\mathbb{S}^d)$ is precisely the collection of the multipliers for $H^2(\mathbb{S}^d)$. In this sense, $\mathcal{TM}_d$ is as close to an analogue of $\mathcal{T}(H^\infty(\mathbb{S}^d))$ as we can get on H^2_d. The significance of this becomes clear when we consider the essential commutants. It is well known [34, 37, 49] that the essential commutant of $\mathcal{T}(H^\infty(\mathbb{S}^d))$ is $\mathcal{T}(\mathrm{QC})$, the C^*-algebra generated by the Toeplitz operators

$$\{T_f : f \in \mathrm{QC} = L^\infty \cap \mathrm{VMO}\}$$

on $H^2(\mathbb{S}^d)$. In this light, it will be interesting to see what the essential commutant of $\mathcal{TM}_d$ is.

6. Characterizations of multipliers

Due to the importance of multipliers, it is natural to ask whether we have a nice characterization for these multipliers. As we have seen that there have been some work on the characterization of special multipliers such as Schur multipliers and quasi-extreme multipliers. But the determination of which $f \in H^2_d$ is a multiplier in general is still very challenging.

Let m be an integer such that $2m \geq d$. Then given any $f \in H_d^2$, one can define the measure $d\mu_f$ on $\mathbf{B}$ by the formula

$$d\mu_f(z) = |(R^m f)(z)|^2(1-|z|^2)^{2m-d}dv(z), \tag{6.1}$$

where R is the radial derivative and dv is the normalized volume measure on $\mathbb{B}^d$.

In [63] Ortega and Fàbrega proved the following characterization:

Theorem 6.1. *$f \in \mathcal{M}_d$ if and only if $d\mu_f$ is a Carleson measure for H_d^2. That is, $f \in \mathcal{M}_d$ if and only if there is a C such that*

$$\int |h(z)|^2 d\mu_f(z) \leq C\|h\|^2$$

for every $h \in H_d^2$.

In [7] Arcozzi, Rochberg and Sawyer gave a characterization for all the H_d^2-Carleson measures on $\mathbb{B}^d$. See Theorem 34 in that paper. For a given Borel measure on $\mathbb{B}^d$, the conditions in [7] are not easy to verify. More to the point, Theorem 34 in [7] deals with all Borel measures on $\mathbb{B}^d$, not just the class of measures $d\mu_f$ of the form (6.1). A natural question is to ask the following: Let k_z be the normalized reproducing kernel for H_d^2, i.e., $k_z(w) = \frac{(1-|z|^2)^{1/2}}{1-\langle z,w\rangle}, |z| < 1, |w| < 1$. For $f \in H_d^2$, does the condition

$$\sup_{|z|<1} \|fk_z\| < \infty$$

imply that f is a multiplier for H_d^2?

What makes this question particularly tempting is that an affirmative answer would give a very simple characterization of the membership $f \in \mathcal{M}$. But that would be too simple a characterization, as it turns out. Actually the answer is negative as shown in [46].

Theorem 6.2. *There exists an $f \in H_d^2$ satisfying the conditions $f \notin \mathcal{M}_d$ and $\sup_{|z|<1} \|fk_z\| < \infty$.*

In [4], a more general result of Aleman, Hartz, McCarthy and Richter for complete Pick space implies the following sufficient condition for $f \in \mathcal{M}_d$:

Theorem 6.3. *If $f \in H_d^2$ and satisfies*

$$\sup_{|z|<1} \text{Re } \langle f, K_z f\rangle_{H_d^2} < \infty,$$

then $f \in \mathcal{M}_d$.

In [4] it is proved that the condition is not a necessary condition for some complete Pick space. The characterization problem still remains to be challenging.

Remark: Note that there have been numerous studies on relevant noncommutative generalizations. To list few: [55, 56, 57, 58, 59, 36, 11, 16, 23]. Unfortunately we have to omit the discussions here. Interested readers may find information from these papers and the references therein.

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