MULTIPLIERS AND ESSENTIAL NORM ON THE DRURY-ARVESON SPACE

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Abstract. It is well known that for multipliers f of the Drury-Arveson space H^2_n, $\|f\|_\infty$ does not dominate the operator norm of M_f. We show that in general $\|f\|_\infty$ does not even dominate the essential norm of M_f. A consequence of this is that there exist multipliers f of H^2_n for which M_f fails to be essentially hyponormal, i.e., if K is any compact, self-adjoint operator, then the inequality $M_f^*M_f - M_fM_f^* + K \geq 0$ does not hold.

1. Introduction

Let $\mathbf{B}$ denote the open unit ball $\{z : |z| < 1\}$ in $\mathbf{C}^n$. In this paper, the complex dimension n is assumed to be greater than or equal to 2. An analogue of the classic Hardy space is the space H^2_n of analytic functions on $\mathbf{B}$ introduced by Drury [8] and Arveson [2]. Because of its connection to various topics in operator theory, e.g. the von Neumann inequality for commuting row contractions, H^2_n has been the subject of intense recent studies [1-7,9,10,12].

Recall that the Drury-Arveson space H^2_n is a reproducing kernel Hilbert space with

$$K(z,w) = \frac{1}{1-\langle z,w\rangle}, \quad z,w \in \mathbf{B},$$

as its kernel [2, 8]. Note that $K(z,w)$ is a multivariable-generalization of the one-variable Szegö kernel. An orthonormal basis of H^2_n is given by $\{(|\alpha|!/\alpha!)^{1/2}\zeta^\alpha : \alpha \in \mathbf{Z}^n_+\}$, where we use the standard multi-index notation. Thus for functions $f, g \in H^2_n$ with Taylor expansions

$$f(\zeta) = \sum_{\alpha\in\mathbf{Z}^n_+} c_\alpha \zeta^\alpha \quad \text{and} \quad g(\zeta) = \sum_{\alpha\in\mathbf{Z}^n_+} d_\alpha \zeta^\alpha,$$

the inner product is given by

$$\langle f, g\rangle = \sum_{\alpha\in\mathbf{Z}^n_+} \frac{\alpha!}{|\alpha|!} c_\alpha \overline{d_\alpha}.$$

With the identification of each variable ζ_i with each multiplication operator M_{ζ_i}, H^2_n is a free Hilbert module over the polynomial ring $\mathbf{C}[\zeta_1, \ldots, \zeta_n]$. See [2].

An analytic function f on $\mathbf{B}$ is said to be a *multiplier* of the Drury-Arveson space H^2_n if $fg \in H^2_n$ for every $g \in H^2_n$. Throughout the paper, we denote the collection of multipliers of H^2_n by $\mathcal{M}$. For each $f \in \mathcal{M}$, the multiplication operator M_f defined by $M_f g = fg$ is necessarily bounded on H^2_n [2], and the operator norm $\|M_f\|$ is also called the multiplier norm of f.

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Multipliers are an important part of operator theory on H^2_n. For example, if $\mathcal{E}$ is a closed linear subspace of H^2_n which is invariant under $M_{\zeta_1}, \dots, M_{\zeta_n}$, then there exist $\{f_1, \dots, f_k, \dots\} \subset \mathcal{M}$ such that the operator

$$M_{f_1} M^*_{f_1} + \cdots + M_{f_k} M^*_{f_k} + \cdots$$

is the orthogonal projection from H^2_n onto $\mathcal{E}$ (see page 191 in [3]).

Among the recent results related to multipliers, we would like to mention the following developments. Interpolation problems for multipliers and model theory related to the Drury-Arveson space have been intensely studied over the past decade or so [4, 5, 10, 12]. Recently, Arcozzi, Rochberg and Sawyer gave a characterization of the multipliers in terms of Carleson measures for H^2_n [1]. In [7], Costea, Sawyer and Wick proved a corona theorem for $\mathcal{M}$. More recently, we showed in [9] that for each $f \in \mathcal{M}$ and each $1 \le i \le n$, the commutator $[M^*_f, M_{\zeta_i}]$ belongs to the Schatten class $\mathcal{C}_p$, $p > 2n$.

Of particular relevance to this paper is the fact that under the assumption $n \ge 2$, $\mathcal{M}$ is strictly smaller than H^∞ [2]. Moreover, Arveson showed in [2] that, even for polynomials q, $\|q\|_\infty$ in general does not dominate the operator norm of M_q on H^2_n. This naturally brings up the question, what about the essential norm of M_f on H^2_n for general $f \in \mathcal{M}$?

Recall that the *essential norm* of a bounded operator A on a Hilbert space $\mathcal{H}$ is

$$\|A\|_{\mathcal{Q}} = \inf\{\|A + K\| : K \in \mathcal{K}(\mathcal{H})\},$$

where $\mathcal{K}(\mathcal{H})$ is the collection of compact operators on $\mathcal{H}$. Alternately, $\|A\|_{\mathcal{Q}} = \|\pi(A)\|$, where π denotes the quotient homomorphism from $\mathcal{B}(\mathcal{H})$ to the Calkin algebra $\mathcal{Q} = \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$.

Let $\mathcal{T}_n$ be the C^*-algebra generated by $M_{\zeta_1}, \cdots, M_{\zeta_n}$ on H^2_n, which was introduced by Arveson in [2]. In more ways than one, $\mathcal{T}_n$ is the analogue of the C^*-algebra generated by Toeplitz operators with *continuous* symbols. Indeed Arveson showed that there is an exact sequence

$$\{0\} \to \mathcal{K}(H^2_n) \to \mathcal{T}_n \xrightarrow{\tau} C(S) \to \{0\}, \tag{1.1}$$

where the homomorphism τ is an extension of the map

$$\tau(M_{\zeta_j}) = \zeta_j,$$

$j = 1, \dots, n$, and $S = \{z \in \mathbf{C}^n : |z| = 1\}$. It follows that if q is a *polynomial*, then

$$\|M_q\|_{\mathcal{Q}} = \|q\|_\infty. \tag{1.2}$$

This equality can also be understood from a slightly different point of view. Indeed by Proposition 5.3 in [2], for each polynomial q, the operator M_q is *essentially normal*, i.e., $[M^*_q, M_q]$ is compact. On the other hand, by Proposition 2.12 in [2], if q is a polynomial, then the spectral radius of M_q equals $\|q\|_\infty$. Since the norm and the spectral radius of any normal element in any C^*-algebra coincide, it follows that $\|M_q\|_{\mathcal{Q}} \le \|q\|_\infty$ whenever q is a polynomial. The reverse inequality, $\|M_q\|_{\mathcal{Q}} \ge \|q\|_\infty$, is easy once M^*_q is applied to the normalized reproducing kernel of H^2_n.

Equality (1.2) is particularly interesting in view of the fact that, even for polynomials, $\|q\|_\infty$ in general does not dominate the operator norm of M_q. The obvious question is, what happens if we consider a general $f \in \mathcal{M}$?

We report that (1.2) in general fails if we consider multipliers which are not polynomials. More specifically, the main results of the paper are Theorems 1.1 and 1.2 below.

Theorem 1.1. *There exists a sequence $\{\psi_k\} \subset \mathcal{M}$ such that*

$$\inf_{k\geq 1} \|M_{\psi_k}\|_{\mathcal{Q}} > 0 \qquad \textit{and} \qquad \lim_{k\to\infty} \|\psi_k\|_\infty = 0.$$

This has implications for other essential properties of multipliers.

Recall that an operator T is said to be *hyponormal* if $T^*T - TT^* \geq 0$. It is well known that the norm of a hyponormal operator coincides with its spectral radius. As we mentioned earlier, by Proposition 2.12 in [2], if q is a polynomial, then the spectral radius of M_q equals $\|q\|_\infty$. Therefore if q is a polynomial such that $\|M_q\| > \|q\|_\infty$, then M_q is not hyponormal. Thus there are plenty of multipliers $f \in \mathcal{M}$ for which M_f fails to be hyponormal on H^2_n. This is one phenomenon that sets the Drury-Arveson space apart from the Hardy space and the Bergman space. We will show that this phenomenon persists under compact perturbation.

Definition. An operator T is said to be *essentially hyponormal* if there is a compact self-adjoint operator K such that

$$T^*T - TT^* + K \geq 0.$$

Obviously, T is essentially hyponormal if and only if $\pi(T)$ is a hyponormal element in the Calkin algebra $\mathcal{Q}$, i.e., $\pi(T^*)\pi(T) - \pi(T)\pi(T^*) \geq 0$.

Theorem 1.2. *There exists a $\psi \in \mathcal{M}$ such that the multiplication operator M_ψ on H^2_n is not essentially hyponormal.*

Having introduced our results, the rest of this short paper consists of their proofs.

2. Estimates for Certain Multipliers

The proof of Theorem 1.1 involves Möbius transform. For each $z \in \mathbf{B}\backslash\{0\}$, let

$$\varphi_z(\zeta) = \frac{1}{1-\langle\zeta,z\rangle}\left\{z - \frac{\langle\zeta,z\rangle}{|z|^2}z - (1-|z|^2)^{1/2}\left(\zeta - \frac{\langle\zeta,z\rangle}{|z|^2}z\right)\right\}. \tag{2.1}$$

Then φ_z is an involution, i.e., $\varphi_z \circ \varphi_z = \mathrm{id}$ [13,Theorem 2.2.2]. Recall that the normalized reproducing kernel for H^2_n is given by

$$k_z(\zeta) = \frac{(1-|z|^2)^{1/2}}{1-\langle\zeta,z\rangle}, \qquad z,\zeta \in \mathbf{B}. \tag{2.2}$$

For each $z \in \mathbf{B}\backslash\{0\}$, define the operator U_z by the formula

$$(U_z g)(\zeta) = g(\varphi_z(\zeta))k_z(\zeta), \qquad g \in H^2_n. \tag{2.3}$$

It follows easily from Theorem 2.2.2 in [13] that if $z \in \mathbf{B}\backslash\{0\}$ and $x, y \in \mathbf{B}$, then

$$\langle U_z k_x, U_z k_y\rangle = \frac{(1-|x|^2)^{1/2}(1-|y|^2)^{1/2}}{1-\langle y,x\rangle} = \langle k_x, k_y\rangle.$$

Hence each U_z is a unitary operator on H_n^2. Moreover, we have

$$U_z M_f U_z^* = M_{f\circ\varphi_z} \tag{2.4}$$

for all $z \in \mathbf{B}\backslash\{0\}$ and $f \in \mathcal{M}$.

For each $j \in \mathbf{N}$, let E_j be the linear span of $\{\zeta^\alpha : |\alpha| \le j\}$ in H_n^2, and let $P_j : H_n^2 \to E_j$ be the orthogonal projection. Moreover, denote

$$Q_j = 1 - P_j.$$

Obviously, we have the strong convergence $Q_j \to 0$ as $j \to \infty$.

Lemma 2.1. *For each $j \in \mathbf{N}$, there is a constant $1 \le C_j < \infty$ such that*

$$\limsup_{|z|\uparrow 1} \|M_{f\circ\varphi_z} P_j\| \le C_j \|M_f\|_{\mathcal{Q}}$$

for every $f \in \mathcal{M}$.

Proof. For each $j \in \mathbf{N}$, since $\dim(E_j) < \infty$, any two norms on E_j are equivalent. Since E_j consists of polynomials, we have $\|M_g\| < \infty$ for each $g \in E_j$. Hence there is a C_j such that

$$\|M_g\| \le C_j \|g\|_{H_n^2} \tag{2.5}$$

for every $g \in E_j$. Now let $f \in \mathcal{M}$. Using the unitary U_z, we have $\|f \circ \varphi_z\|_{H_n^2} = \|U_z(f\circ\varphi_z)\|_{H_n^2} = \|fk_z\|_{H_n^2}$. Since $k_z \to 0$ weakly as $|z| \uparrow 1$, we have

$$\limsup_{|z|\uparrow 1} \|f\circ\varphi_z\|_{H_n^2} = \limsup_{|z|\uparrow 1} \|fk_z\|_{H_n^2} \le \limsup_{|z|\uparrow 1} \|(M_f + K)k_z\|_{H_n^2}$$

for every compact operator K. Consequently,

$$\limsup_{|z|\uparrow 1} \|f\circ\varphi_z\|_{H_n^2} \le \|M_f\|_{\mathcal{Q}}. \tag{2.6}$$

Now if $g \in E_j$, then, using (2.5), we have

$$\|M_{f\circ\varphi_z} g\|_{H_n^2} = \|M_g(f\circ\varphi_z)\|_{H_n^2} \le \|M_g\|\|f\circ\varphi_z\|_{H_n^2} \le C_j\|f\circ\varphi_z\|_{H_n^2}\|g\|_{H_n^2}.$$

Hence $\|M_{f\circ\varphi_z}P_j\| \le C_j\|f\circ\varphi_z\|_{H_n^2}$. Combining this with (2.6), the lemma follows. $\square$

The next lemma is so elementary that its proof will be omitted.

Lemma 2.2. *For each bounded operator A on H_n^2, we have*

$$\limsup_{j\to\infty} \|Q_j A\| \le \|A\|_{\mathcal{Q}} \qquad \textit{and} \qquad \limsup_{j\to\infty} \|AQ_j\| \le \|A\|_{\mathcal{Q}}.$$

Proof of Theorem 1.1. By Theorem 3.3 in [2], there is a sequence of polynomials $\{p_i\}$ such that

$$\|M_{p_i}\| = 1 \tag{2.7}$$

for every i and

$$\lim_{i\to\infty} \|p_i\|_\infty = 0. \tag{2.8}$$

We will find a sequence of natural numbers $\{i(j)\}_{j=1}^\infty$ and a sequence $\{z_j\} \subset \mathbf{B}\backslash\{0\}$ such that the desired multipliers $\{\psi_k\}$ will have the form

$$\psi_k = \sum_{j=k}^\infty p_{i(j)} \circ \varphi_{z_j}, \tag{2.9}$$

$k \in \mathbf{N}$. To do this, we note that (2.8) enables us to inductively select an ascending sequence of natural numbers

$$\ell(1) < \ell(2) < \cdots < \ell(m) < \cdots$$

such that

$$C_m \|p_{\ell(m)}\|_\infty \leq \frac{1}{2^m} \tag{2.10}$$

for each $m \in \mathbf{N}$, where C_m is the constant provided by Lemma 2.1. Since each p_i is a polynomial, by (1.2) this implies

$$C_m \|M_{p_{\ell(m)}}\|_{\mathcal{Q}} \leq \frac{1}{2^m}. \tag{2.11}$$

By (2.11) and Lemma 2.1, for each $m \in \mathbf{N}$ there is a $w_m \in \mathbf{B}\backslash\{0\}$ such that

$$\|M_{p_{\ell(m)} \circ \varphi_{w_m}} P_m\| \leq \frac{2}{2^m}. \tag{2.12}$$

It follows from Lemma 2.2 that for each $m \in \mathbf{N}$ there is a natural number $r(m) > m$ such that

$$\|M_{p_{\ell(m)} \circ \varphi_{w_m}} Q_{r(m)}\| \leq 2\|M_{p_{\ell(m)} \circ \varphi_{w_m}}\|_{\mathcal{Q}} = 2\|M_{p_{\ell(m)}}\|_{\mathcal{Q}},$$

where the $=$ is a consequence of (2.4). By (2.11) and the fact that $C_m \geq 1$, we have

$$\|M_{p_{\ell(m)} \circ \varphi_{w_m}} Q_{r(m)}\| \leq \frac{2}{2^m}. \tag{2.13}$$

By a similar argument, for each $m \in \mathbf{N}$, there is an $s(m) > m$ such that

$$\|Q_{s(m)} M_{p_{\ell(m)} \circ \varphi_{w_m}}\| \leq \frac{2}{2^m}. \tag{2.14}$$

Note that for each m, the subspace $Q_m H^2_n$ is invariant under $\{M_f : f \in \mathcal{M}\}$. That is,

$$M_f g = Q_m M_f g \quad \text{if} \quad g \in Q_m H^2_n \quad \text{and} \quad f \in \mathcal{M}.$$

Using this fact and the relation $P_i = 1 - Q_i$, it follows from simple algebra that

$$\begin{aligned} M_{p_{\ell(m)} \circ \varphi_{w_m}} = {} & M_{p_{\ell(m)} \circ \varphi_{w_m}} P_m + Q_{s(m)} M_{p_{\ell(m)} \circ \varphi_{w_m}} (Q_m - Q_{r(m)}) + M_{p_{\ell(m)} \circ \varphi_{w_m}} Q_{r(m)} \\ & + (P_{s(m)} - P_m) M_{p_{\ell(m)} \circ \varphi_{w_m}} (P_{r(m)} - P_m). \end{aligned}$$

Thus if we set

$$Y_m = M_{p_{\ell(m)} \circ \varphi_{w_m}} P_m + Q_{s(m)} M_{p_{\ell(m)} \circ \varphi_{w_m}} (Q_m - Q_{r(m)}) + M_{p_{\ell(m)} \circ \varphi_{w_m}} Q_{r(m)},$$

then

$$M_{p_{\ell(m)} \circ \varphi_{w_m}} = Y_m + (P_{s(m)} - P_m) M_{p_{\ell(m)} \circ \varphi_{w_m}} (P_{r(m)} - P_m), \tag{2.15}$$

$m \in \mathbf{N}$. Note that by (2.12), (2.13) and (2.14), we have

$$\|Y_m\| \leq \frac{6}{2^m}. \tag{2.16}$$

Set $m_1 = 5$. We then inductively select a sequence of integers $m_1 < m_2 < \cdots < m_j < \cdots$ such that the inequality

$$m_{j+1} > \max\{r(m_j), s(m_j)\} \tag{2.17}$$

holds for every $j \geq 1$. Now set

$$i(j) = \ell(m_j) \quad \text{and} \quad z_j = w_{m_j} \tag{2.18}$$

for each $j \in \mathbf{N}$. With this notation, from (2.15) we obtain

$$M_{p_{i(j)} \circ \varphi_{z_j}} = Y_{m_j} + (P_{s(m_j)} - P_{m_j}) M_{p_{i(j)} \circ \varphi_{z_j}} (P_{r(m_j)} - P_{m_j}). \tag{2.19}$$

With $i(j)$ and z_j determined as above, we now define ψ_k by (2.9) for each $k \geq 1$.

Next we show that the sequence $\{\psi_k\}$ has the desired properties. First we need to show that $\psi_k \in \mathcal{M}$ for each k. By the relations $s(m) > m, r(m) > m$ and (2.17), we have

$$P_{s(m_j)} - P_{m_j} \perp P_{s(m_{j'})} - P_{m_{j'}} \quad \text{and} \quad P_{r(m_j)} - P_{m_j} \perp P_{r(m_{j'})} - P_{m_{j'}} \tag{2.20}$$

whenever $j < j'$. Recall that $\|M_{p_i \circ \varphi_z}\| = \|M_{p_i}\|$ by (2.4) and that $\|M_{p_i}\| = 1$ by choice. Combining these facts with (2.20), we see that the norm of the operator

$$B_k = \sum_{j=k}^{\infty} (P_{s(m_j)} - P_{m_j}) M_{p_{i(j)} \circ \varphi_{z_j}} (P_{r(m_j)} - P_{m_j})$$

does not exceed 1. By (2.16) and the choice that $m_1 = 5$, the norm of the operator

$$A_k = \sum_{j=k}^{\infty} Y_{m_j}$$

does not exceed 1/2. By (2.9) and (2.19), $M_{\psi_k} = A_k + B_k$. Thus the norm of the operator M_{ψ_k} is at most 3/2. That is, $\psi_k \in \mathcal{M}$ for each $k \in \mathbf{N}$. Applying (2.18) and (2.10), we have

$$\|\psi_k\|_\infty \leq \sum_{j=k}^{\infty} \|p_{i(j)} \circ \varphi_{z_j}\|_\infty = \sum_{j=k}^{\infty} \|p_{i(j)}\|_\infty \leq \sum_{j=k}^{\infty} \frac{1}{2^{m_j}}.$$

Hence

$$\lim_{k \to \infty} \|\psi_k\|_\infty = 0.$$

What remains for the proof is the inequality

$$\inf_{k \geq 1} \|M_{\psi_k}\|_{\mathcal{Q}} > 0.$$

Since $M_{\psi_k} = A_k + B_k$ and $\|A_k\| \leq 1/2$, it suffices to show that

$$\inf_{k \geq 1} \|B_k\|_{\mathcal{Q}} \geq 1.$$

Since $\|M_{p_{i(j)} \circ \varphi_{z_j}}\| = \|M_{p_{i(j)}}\| = 1$ and $\lim_{j \to \infty} \|Y_{m_j}\| = 0$, by (2.19) we have

$$\lim_{j \to \infty} \|(P_{s(m_j)} - P_{m_j}) M_{p_{i(j)} \circ \varphi_{z_j}} (P_{r(m_j)} - P_{m_j})\| = 1.$$

Therefore there exists a sequence of unit vectors $\{g_j\}$ such that

$$g_j \in (P_{r(m_j)} - P_{m_j}) H_n^2 \tag{2.21}$$

for every j and

$$\lim_{j \to \infty} \|(P_{s(m_j)} - P_{m_j}) M_{p_{i(j)} \circ \varphi_{z_j}} (P_{r(m_j)} - P_{m_j}) g_j\|_{H_n^2} = 1. \tag{2.22}$$

By (2.21) and (2.20), $g_j \to 0$ weakly as $j \to \infty$. By this weak convergence and (2.22), we have

$$\lim_{j \to \infty} \|(B_k + K) g_j\|_{H_n^2} = 1$$

for every compact operator K. This implies $\|B_k\|_{\mathcal{Q}} \geq 1$, completing the proof of Theorem 1.1. $\square$

3. Spectral radius

To prove Theorem 1.2, we begin with a simple fact about multipliers.

Lemma 3.1. *Let $f \in \mathcal{M}$. If there is a $c > 0$ such that $|f(z)| \geq c$ for every $z \in \mathbf{B}$, then $1/f$ is also a multiplier of H^2_n.*

Proof. This certainly follows from the recently proved corona theorem for $\mathcal{M}$ [7]. But it also follows from an earlier, much simpler result due to Chen [6]. By Theorem 2 in [6], there are constants $0 < A \leq B < \infty$ such that $A\|g\|_{H^2_n} \leq \|g\|_{\#} \leq B\|g\|_{H^2_n}$ for every $g \in H^2_n$, where

$$\|g\|^2_{\#} = |g(0)|^2 + \iint \frac{|g(z) - g(w)|^2}{|1 - \langle z, w\rangle|^{2n+1}} dv(z)dv(w). \tag{3.1}$$

Let $f \in \mathcal{M}$ be such that $|f| \geq c > 0$ on $\mathbf{B}$. Then for each $g \in H^2_n$,

$$\frac{g(z)}{f(z)} - \frac{g(w)}{f(w)} = \frac{g(z) - g(w)}{f(z)} + \frac{g(z) - g(w)}{f(w)} + \frac{f(w)g(w) - f(z)g(z)}{f(z)f(w)}, \tag{3.2}$$

$z, w \in \mathbf{B}$. From (3.1) and (3.2) we see that $1/f \in \mathcal{M}$. □

From this lemma we immediately obtain

Proposition 3.2. *For each $f \in \mathcal{M}$, the spectrum of the operator M_f on H^2_n is contained in the closure of $\{f(z) : z \in \mathbf{B}\}$. Consequently the spectral radius of M_f does not exceed $\|f\|_\infty$.*

Remark 1. In the case where f has the property that there is a sequence of polynomials $\{p_k\}$ such that $\lim_{k\to\infty} \|M_f - M_{p_k}\| = 0$, Proposition 3.2 was proved by Arveson. See Proposition 2.12 in [2].

Remark 2. It follows from (3.1) and (3.2) that if $f \in \mathcal{M}$ and if $\inf_{z\in\mathbf{B}} |f(z)| > 0$, then

$$\|M_{1/f}\| \leq C(\|1/f\|_\infty + \|1/f\|^2_\infty \|M_f\|).$$

Surprisingly, Chen himself did not seem to notice this fact in [6].

Remark 3. The referee observes that an alternate proof for Lemma 3.1 is to use the Carleson-measure characterization for functions in $\mathcal{M}$ [1]. In this approach, what replaces (3.2) is the formula for high-order derivatives of $1/f$.

Proposition 3.3. *Let $f \in \mathcal{M}$. If f has the property that $\|M_f\|_{\mathcal{Q}} > \|f\|_\infty$, then the operator M_f on H^2_n is not essentially hyponormal.*

Proof. Recall that we denote the quotient map from $\mathcal{B}(H^2_n)$ to the Calkin algebra $\mathcal{Q}$ by π. Let Φ be the GNS representation of $\mathcal{Q}$ on a Hilbert space $\mathcal{H}$. If M_f were essentially hyponormal, then $\{\pi(M_f)\}^*\pi(M_f) - \pi(M_f)\{\pi(M_f)\}^* \geq 0$ in $\mathcal{Q}$. Consequently $\Phi(\pi(M_f))$ would be a hyponormal operator on $\mathcal{H}$.

Write $\mathrm{rad}(T)$ for the spectral radius of any operator T. It is well known that if T is a hyponormal operator, then $\|T\| = \mathrm{rad}(T)$. See Problem 205 in [11]. Thus we would have

$$\|\Phi(\pi(M_f))\| = \mathrm{rad}(\Phi(\pi(M_f))).$$

Since $\Phi \circ \pi$ is a C^*-algebraic homomorphism, the spectrum of $\Phi(\pi(M_f))$ is contained in the spectrum of M_f. Therefore $\mathrm{rad}(\Phi(\pi(M_f))) \leq \mathrm{rad}(M_f)$. Applying Proposition 3.2, we have

$$\mathrm{rad}(\Phi(\pi(M_f))) \leq \mathrm{rad}(M_f) \leq \|f\|_\infty.$$

On the other hand, since Φ is a faithful representation, we have

$$\|\Phi(\pi(M_f))\| = \|\pi(M_f)\| = \|M_f\|_{\mathcal{Q}}.$$

These three displayed lines together contradict the assumption $\|M_f\|_{\mathcal{Q}} > \|f\|_\infty$. $\square$

Proof of Theorem 1.2. By Theorem 1.1, there is a $\psi \in \mathcal{M}$ with $\|M_\psi\|_{\mathcal{Q}} = c$ for some $c > 0$ while $\|\psi\|_\infty < c/2$. That is, we have a $\psi \in \mathcal{M}$ with $\|M_\psi\|_{\mathcal{Q}} > \|\psi\|_\infty$. For such a ψ, Proposition 3.3 tells us that M_ψ is not essentially hyponormal on H^2_n. $\square$

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