HANDOUT ABOUT THE "TABLE OF SIGNS" METHOD

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The method

To find the sign of a rational (including polynomial) function

$$f(x) = \frac{p(x)}{q(x)}$$

where $p(x)$ and $q(x)$ have no common factors proceed as follows:

(1) Factor the numerator and the denominator of the function into irreducible factors of degree ≤ 2, i.e. linear or quadratic factors. Each *linear* factor of the numerator gives a root of the function, and each *linear* factor of the denominator gives a vertical asymptote.

(2) List the zeros of the numerator and denominator in increasing order and plot their relative positions in the real number line. Use solid dots to indicate roots of the numerator and hollow roots to indicate roots of the denominator. The real line will break into subintervals and these subintervals will label the columns of the "table of signs". The rows of the table will be labeled by the irreducible factors of the numerator and the denominator. There is one last row labelled by f(x). To distinguish between roots and asymptotes use solid lines as row separators at the roots and dotted or dashed lines at the asymptotes.

(3) Fill the the table by determining the sign of each factor in each interval. This can be quickly accomplished by using the following facts:

   (a) A factor of the form $x - a$ is negative to the left of a and positive to the right of a. A factor of the form $a - x$ is positive to the left of a and negative to the right of a.

   (b) Each quadratic factor has always the same sign which coincides with the sign of its constant term. (Why is that?)

(4) Finally fill the last row by taking the product of all the signs in each column, in other words, if there is an even number of negative signs in that column we get a $+$ otherwise we get a $-$.

Examples

Example 1. Solve: $(x-2)(x+3) < 0$

Solution. We need to find when the polynomial function $f(x) := (x-2)(x-3)$ is *negative*. We have two roots $x = 2$ and $x = -3$ and two linear factors. So we have the following table:

	$-\infty$	-3		2	∞
$x+3$	$-$	0	$+$		$+$
$x-2$	$-$		$-$	0	$+$
$f(x)$	$+$	0	$-$	0	$+$

So we see that the solution is:

$$(-3, 2)$$

□

Example 2. Solve: $x^2 - 6x + 7 \geq 0$

Solution. We are asked to find when the polynomial function $f(x) := x^2 - 6x + 7$ is *positive or zero*. So we first need to factor $f(x)$. Using the quadratic formula we get the following roots:

$$\begin{aligned} x &= \frac{6 \pm \sqrt{36 - 28}}{2} \\ &= \frac{6 \pm \sqrt{8}}{2} \\ &= \frac{6 \pm 2\sqrt{2}}{2} \\ &= 3 \pm \sqrt{2} \end{aligned}$$

So we have the following table of signs:

	$-\infty$	$3 - \sqrt{2}$		$3 + \sqrt{2}$	∞
$x - 3 + \sqrt{2}$	$-$	0	$+$		$+$
$x - 3 - \sqrt{2}$	$-$		$-$	0	$+$
$f(x)$	$+$	0	$-$	0	$+$

So the solution is

$$(-\infty, 3 - \sqrt{2}] \cup [3 + \sqrt{2}, \infty)$$

□

Notice that in the solution to the last example we included the endpoints, since the points where the function is zero is part of the solution.

Example 3. Solve: $(1 - x)(x + 4)(x^2 - 4x + 20) > 0$

Solution. In this case besides the linear factors we also have an irreducible quadratic factor. [1] We have the following table:

	$-\infty$	-4		1	∞
$x + 4$	$-$	0	$+$		$+$
$1 - x$	$+$		$+$	0	$-$
$x^2 - 4x + 20$	$+$		$+$		$+$
$f(x)$	$-$	0	$+$	0	$-$

So the solution is

$$(-4, 1)$$

□

[1] To see that the quadratic factor is indeed irreducible notice that its [illegible] is $b^2 - 4ac = -64$, so the quadratic factor has no real solutions.

Notice that in the table of the last example we could have omitted the quadratic factor and all the signs on the last row would be the same. *In general factors that are always positive can be omitted from the table of signs.*

Example 4. Solve: $\dfrac{x(x+3)(x-2)}{(x+5)(x-3)} \leq 0$

Solution. In this case we have three roots $x = 0$, $x = -3$ and $x = 2$ and two asymptotes $x = -5$ and $x = 3$. So we have the following table:

	$-\infty$	-5		-3		0		2		3	∞
$x+5$	$-$	0	$+$		$+$		$+$		$+$		$+$
$x+3$	$-$		$-$	0	$+$		$+$		$+$		$+$
x	$-$		$-$		$-$	0	$+$		$+$		$+$
$x-2$	$-$		$-$		$-$		$-$	0	$+$		$+$
$x-3$	$-$		$-$		$-$		$-$		$-$	0	$+$
$f(x)$	$\ominus$		$+$	0	$\ominus$	0	$+$	0	$\ominus$		$+$

Therefore the solution set is:

$$(-\infty, -5) \cup [-3, 0] \cup [2, 3)$$

□

Notice that in the last example, we were careful not to include the points where the denominator vanishes into the solution set!

When we have repeated factors we don't need to use a different row for each apperance of a factor, instead we can use one row for each power of a factor. The signs can then computed by observing that an odd power of a factor has the same sign as the factor while an even power is always positive or zero.

Example 5. Solve: $(x+2)^2(x+1)^3(x-1)^8(x-2)^5 < 0$

Solution. We have the following table:

	$-\infty$	-2		-1		1		2	∞
$(x+2)^2$	$+$	0	$+$		$+$		$+$		$+$
$(x+1)^3$	$-$		$-$	0	$+$		$+$		$+$
$(x-1)^8$	$+$		$+$		$+$	0	$+$		$+$
$(x-2)^5$	$-$		$-$		$-$		$-$	0	$+$
$f(x)$	$+$	0	$+$	0	$-$	0	$-$	0	$+$

So the solution set is

$$(-1,1) \cup (1,2)$$

□

Example 6. Solve: $(x+2)^2(x+1)^3(x-1)^8(x-2)^5 \leq 0$

Solution. The polynomial is the same as in Example 5, but the inequality is not strict. Therefore we have to include the zeros of the function in the solution set:

$$\{-2\} \cup [-1,2]$$

□

Exercise Solve the following inequalities:

(1) $(x-4)(x+3)(x+5) < 0$
(2) $(x+2)^2(x+1)^3(x-1)^8(x-2)^5 > 0$
(3) $(x+2)^2(x+1)^3(x-1)^8(x-2)^5 \geq 0$
(4) $x^2-2x-15 \leq 0$
(5) $x^2-4x+5 < 0$
(6) $x^4-13x^2+36 > 0$
(7) $x^3+2x^2-11x-12 \leq 0$