MTH 42, Fall 2024

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Take Home Part of Exam 2

1. Let
$$X = \begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \\ x_{41} & x_{42} & x_{43} & x_{44} \end{pmatrix}.$$

   (a) Find a matrix A such that
$$A\,X = \begin{pmatrix} 3\,x_{41} - 2\,x_{21} & 3\,x_{42} - 2\,x_{22} & 3\,x_{43} - 2\,x_{23} & 3\,x_{44} - 2\,x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \\ x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \end{pmatrix}.$$

   (b) Find a matrix B such that
$$X\,B = \begin{pmatrix} x_{14} - 2\,x_{13} & x_{12} & x_{11} & x_{14} \\ x_{24} - 2\,x_{23} & x_{22} & x_{21} & x_{24} \\ x_{34} - 2\,x_{33} & x_{32} & x_{31} & x_{34} \\ x_{44} - 2\,x_{43} & x_{42} & x_{41} & x_{44} \end{pmatrix}.$$

2. Let
$$A = \begin{pmatrix} 2 & -1 & 1 \\ 6 & -3 & 4 \\ 3 & -2 & 3 \end{pmatrix}.$$

   (a) Verify that A is a root of the polynomial
$$p(x) = x^3 - 2\,x^2 + 2\,x - 1.$$

   (b) Find A^{-1}.

3. Let
$$\mathbb{Q}(\sqrt{3}) = \left\{ a + b\,\sqrt{3} : a, b \in \mathbb{Q} \right\}.$$

   (a) Prove that $\mathbb{Q}(\sqrt{3})$ is a subfield of the field of real numbers $\mathbb{R}$.

   (b) Give an explicit formula for $(a + b\,\sqrt{3})^{-1}$.

   Hint. See Example 75 in the Lecture notes.

4. Consider the set $\mathbb{F} = \{0, 1, a, b\}$ where $a \neq b$. Define addition and multiplication via the following tables

$+$	0	1	a	b
0	0	1	a	b
1	1	0	b	a
a	a	b	0	1
b	b	a	1	0

$\cdot$	0	1	a	b
0	0	0	0	0
1	0	1	a	b
a	0	a	b	1
b	0	b	1	a

.

Prove that $\mathbb{F}$ is a field.

5. Consider $\mathbb{R}^2$ with the usual addition

$$(a, b) + (c, d) = (a + c, b + d),$$

and multiplication given by

$$(a, b)\,(c, d) = (a\,c, b\,d).$$

Is $\mathbb{R}^2$ with these operations a field? Fully justify your answer.

6. Consider the following matrices[1] with complex entries

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

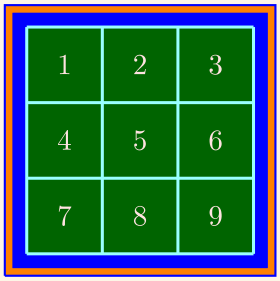
Prove that these matrices satisfy the following relations:

(a) $\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = I$.

(b) $\sigma_x\,\sigma_y = i\,\sigma_z, \quad \sigma_y\,\sigma_z = i\,\sigma_x, \quad \sigma_z\,\sigma_x = i\,\sigma_y$.

(c) $\sigma_x\,\sigma_y = -\sigma_y\,\sigma_x, \quad \sigma_x\,\sigma_z = -\sigma_z\,\sigma_x, \quad \sigma_y\,\sigma_z = -\sigma_z\,\sigma_y$.

7. Consider a 3×3 grid of squares, each either green or red. When we touch a square its color and the color of its neighbors change, where the neighbors of a square are all squares that share an edge with it.



Thus for example, if we touch the square numbered 1 the squares numbered 1, 2, and 4 change color, if we touch square 5 then all squares except 1, 3, 7, and 9 change color, and if we touch 8 then 5, 7, 8, and 9 change colors.

We start with all squares green. Find, if possible, a sequence of squares to touch so that all squares turn red.

Hint. See Examples 80 and 81 in the notes.

8. Consider the following vectors in $\mathbb{C}^4$:

$$\mathbf{v}_1 = (1, i, 0, -i) \qquad \mathbf{v}_2 = (2 + i, 3\,i, i, 1 - 4\,i)$$
$$\mathbf{v}_3 = (5 + i, 2 + 6\,i, 1 + 2\,i, 7 - 9\,i) \qquad \mathbf{v}_4 = (0, 3 - i, 1 + i, 0).$$

Find a basis and state the dimension of the linear span $\mathbb{C}\,\langle \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4 \rangle$.

[1]These matrices are called *Pauli spin matrices*. They are used in Quantum Mechanics to compute the spin of an electron.

9. Consider $\mathbb{R}$ as a vector space over $\mathbb{Q}$. Prove that

$$\left\{\sqrt{2}, \sqrt{3}, \sqrt{5}\right\}$$

is linearly independent. You may consider Item (a) of Example 93 in the notes known.

10. Let $\mathbf{S}_n$ denote the set of $n \times n$ symmetric matrices over $\mathbb{R}$ (see Definition 29 in the notes).
    (a) Prove that $\mathbf{S}_n$ is a vector subspace of $\mathbf{M}_n$.
    (b) Find a basis and the dimension of $\mathbf{S}_n$.

    Hint. See Example 95 in the notes.

11. Consider the vector space $\mathbb{R}[x]$ of polynomials with real coefficients. Which of the following subsets is a vector subspace of $\mathbb{R}[x]$?
    (a) $V = \{p(x) \in \mathbb{R}[x] : p(42) = 0\}$.
    (b) $U = \{p(x) \in \mathbb{R}[x] : p(42) \geq 0\}$.
    (c) $W = \{p(x) \in \mathbb{R}[x] : p(42) = p(0)\}$.
    (d) $X = \{p(x) \in \mathbb{R}[x] : \deg p(x) = 8\}$.

    Fully justify your answers.

12. Let $\mathbf{P}_3$ be the set of real polynomials of degree at most 3:

$$\mathbf{P}_3 = \{p(x) \in \mathbb{R}[x] : \deg p(x) \leq 3\}.$$

Prove that

$$B = \left\{1, x-1, (x-1)^2, (x-1)^3\right\},$$

is a basis of $\mathbf{P}_3$.

13. Let $S = \{A, B, C, D\} \subseteq \mathbf{M}_2$ where

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

    (a) Prove that S is a basis of $\mathbf{M}_2$.
    (b) Express $X = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ as a linear combination of elements of S.

14. Find conditions on the complex number z so that the vectors

$$\mathbf{v}_1 = (z, 0, 1), \quad \mathbf{v}_1 = (0, 1, z^3), \quad \mathbf{v}_3 = (z, 1, 1+z)$$

form a basis of $\mathbb{C}^3$.

15. Consider the vector space $\mathbf{M}_n$ of real $n \times n$ matrices, and let B be a basis of $\mathbf{M}_n$. Prove that

$$B^* = \{X^* : X \in B\}$$

is also a basis of $\mathbf{M}_n$, where X^* stands for the transpose of a matrix X.