MTH 42, Fall 2024

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Second Exam
Solutions and Answers

1 The take-home part

1. Let
$$X = \begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \\ x_{41} & x_{42} & x_{43} & x_{44} \end{pmatrix}.$$

   (a) Find a matrix A such that
$$A\,X = \begin{pmatrix} 3\,x_{41} - 2\,x_{21} & 3\,x_{42} - 2\,x_{22} & 3\,x_{43} - 2\,x_{23} & 3\,x_{44} - 2\,x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \\ x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \end{pmatrix}.$$

   Solution. The matrix
$$\begin{pmatrix} 3\,x_{41} - 2\,x_{21} & 3\,x_{42} - 2\,x_{22} & 3\,x_{43} - 2\,x_{23} & 3\,x_{44} - 2\,x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \\ x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \end{pmatrix}$$
   is obtained from X by performing the following row operations:

   1. Subtract 2 times the second row from 3 times the fourth:
$$\begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \\ 3\,x_{41} - 2\,x_{21} & 3\,x_{42} - 2\,x_{22} & 3\,x_{43} - 2\,x_{23} & 3\,x_{44} - 2\,x_{24} \end{pmatrix}$$
   2. Interchange the first and fourth rows:
$$\begin{pmatrix} 3\,x_{41} - 2\,x_{21} & 3\,x_{42} - 2\,x_{22} & 3\,x_{43} - 2\,x_{23} & 3\,x_{44} - 2\,x_{24} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \\ x_{11} & x_{12} & x_{13} & x_{14} \end{pmatrix}$$
   3. Interchange the second and fourth rows:
$$\begin{pmatrix} 3\,x_{41} - 2\,x_{21} & 3\,x_{42} - 2\,x_{22} & 3\,x_{43} - 2\,x_{23} & 3\,x_{44} - 2\,x_{24} \\ x_{11} & x_{12} & x_{13} & x_{14} \\ x_{31} & x_{32} & x_{33} & x_{34} \\ x_{21} & x_{22} & x_{23} & x_{24} \end{pmatrix}$$

4. Finally, interchange the second and third rows:

$$\begin{pmatrix} 3\,x_{41}-2\,x_{21} & 3\,x_{42}-2\,x_{22} & 3\,x_{43}-2\,x_{23} & 3\,x_{44}-2\,x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \\ x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \end{pmatrix}.$$

Performing the same operations to the identity matrix I_4 we get

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -2 & 0 & 3 \end{pmatrix} \sim \begin{pmatrix} 0 & -2 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & -2 & 0 & 3 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & -2 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

So,

$$A = \begin{pmatrix} 0 & -2 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

□

(b) Find a matrix B such that

$$X\,B = \begin{pmatrix} x_{14}-2\,x_{13} & x_{12} & x_{11} & x_{14} \\ x_{24}-2\,x_{23} & x_{22} & x_{21} & x_{24} \\ x_{34}-2\,x_{33} & x_{32} & x_{31} & x_{34} \\ x_{44}-2\,x_{43} & x_{42} & x_{41} & x_{44} \end{pmatrix}.$$

Answer. The matrix

$$\begin{pmatrix} x_{14}-2\,x_{13} & x_{12} & x_{11} & x_{14} \\ x_{24}-2\,x_{23} & x_{22} & x_{21} & x_{24} \\ x_{34}-2\,x_{33} & x_{32} & x_{31} & x_{34} \\ x_{44}-2\,x_{43} & x_{42} & x_{41} & x_{44} \end{pmatrix}.$$

is obtained from X by adding the fourth column to -2 times the third and then interchanging the first and third columns. Performing the same column operations to the identity matrix I_4 will give the desired B. Thus,

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$

□

2. Let

$$A = \begin{pmatrix} 2 & -1 & 1 \\ 6 & -3 & 4 \\ 3 & -2 & 3 \end{pmatrix}.$$

(a) Verify that A is a root of the polynomial

$$p(x) = x^3 - 2\,x^2 + 2\,x - 1.$$

Answer. We calculate:

$$A^2 = \begin{pmatrix} 1 & -1 & 1 \\ 6 & -5 & 6 \\ 3 & -3 & 4 \end{pmatrix}, \quad A^3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -3 & 4 \\ 0 & -2 & 3 \end{pmatrix}.$$

And so

$$p(A) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -3 & 4 \\ 0 & -2 & 3 \end{pmatrix} - 2\begin{pmatrix} 1 & -1 & 1 \\ 6 & -5 & 6 \\ 3 & -3 & 4 \end{pmatrix} + 2\begin{pmatrix} 2 & -1 & 1 \\ 6 & -3 & 4 \\ 3 & -2 & 3 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

□

(b) Find A^{-1}.

Solution. We have

$$p(A) = A^3 - 2A^2 + 2A - I$$

and since from Part (a) we have that A is a root of $p(x)$ we get

$$\begin{aligned} A^3 - 2A^2 + 2A - I = O &\implies A^3 - 2A^2 + 2A = I \\ &\implies \left(A^2 - 2A + 2I\right) A = I \\ &\implies A^2 - 2A + 2I = A^{-1}. \end{aligned}$$

Thus,

$$A^{-1} = \begin{pmatrix} 1 & -1 & 1 \\ 6 & -5 & 6 \\ 3 & -3 & 4 \end{pmatrix} - 2\begin{pmatrix} 2 & -1 & 1 \\ 6 & -3 & 4 \\ 3 & -2 & 3 \end{pmatrix} + 2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 1 & -1 \\ -6 & 3 & -2 \\ -3 & 1 & 0 \end{pmatrix}.$$

□

3. Let

$$\mathbb{Q}(\sqrt{3}) = \left\{a + b\sqrt{3} : a, b \in \mathbb{Q}\right\}.$$

(a) Prove that $\mathbb{Q}(\sqrt{3})$ is a subfield of the field of real numbers $\mathbb{R}$.

Solution. We first note that

$$0 = 0 + 0\sqrt{3},$$

and therefore, since $0 \in \mathbb{Q}$ we have $0 \in \mathbb{Q}(\sqrt{3})$.
Similarly,

$$1 = 1 + 0\sqrt{3} \in \mathbb{Q}(\sqrt{3})$$

since $0, 1 \in \mathbb{Q}$.
Let $x = a_1 + b_1\sqrt{3}$ and $y = a_2 + b_2\sqrt{3}$ be two elements of $\mathbb{Q}(\sqrt{3})$. This means that $a_1, a_2, b_1, b_2 \in \mathbf{Q}$.
Then,

$$x + y = a_1 + b_1\sqrt{3} + a_2 + b_2\sqrt{3} = (a_1 + a_2) + (b_1 + b_2)\sqrt{3},$$

and since $\mathbb{Q}$ is closed under addition, $a_1 + a_2 \in \mathbb{Q}$ and $b_1 + b_2 \in \mathbb{Q}$ it follows that

$$x + y \in \mathbb{Q}(\sqrt{3})$$

So $\mathbb{Q}(\sqrt{3})$ is closed under addition.

Similarly,

$$\begin{aligned} x\,y &= (a_1 + b_1\sqrt{3})\,(a_2 + b_2\sqrt{3}) \\ &= a_1\,a_2 + a_1\,b_2\sqrt{3} + a_2\,b_1\sqrt{3} + 3\,b_1\,b_2 \\ &= (a_1\,a_2 + 3\,b_1\,b_2) + (a_1\,b_2 + a_2\,b_1)\,\sqrt{3}, \end{aligned}$$

And since $\mathbb{Q}$, being a field, is closed under addition and multiplication we have

$$a_1\,a_2 + 3\,b_1\,b_2 \in \mathbb{Q}, \quad a_1\,b_2 + a_2\,b_1 \in \mathbb{Q}$$

and so $x\,y \in \mathbb{Q}(\sqrt{3})$. Thus, $\mathbb{Q}(\sqrt{3})$ is closed under multiplication.
Also,

$$-x = -a_1 - b_1\sqrt{3} = (-a_1) + (-b_1)\sqrt{3}$$

and since $\mathbb{Q}$ is closed under opposites we have $-a_1 \in \mathbf{Q}$ and $-b_1 \in \mathbb{Q}$, and it follows that $-x \in \mathbb{Q}(\sqrt{3})$. So $\mathbb{Q}(3)$ is closed under opposites.
Finally[1] let $\bar{x} = a_1 - b_1\sqrt{3}$. Then,

$$x\,\bar{x} = (a_1 + b_1\sqrt{3})\,(a_1 - b_1\sqrt{3}) = a_1^2 - 3\,b_1^2. \tag{1}$$

I claim that $a_1^2 - 3\,b_1^2 \neq 0$. Indeed

$$a_1^2 - 3\,b_1^2 = 0 \implies 3 = \frac{a_1^2}{b_1^2} \implies \sqrt{3} = \frac{a_1}{b_1}.$$

But $a_1/b_1 \in \mathbb{Q}$ and so if $a_1^2 - 3\,b_1^2 = 0$ then $\sqrt{3}$ is a rational number, a contradiction.
Since $a_1^2 - 3\,b_1^2 \neq 0$, we can divide (1) by it to get

$$x\,\frac{\bar{x}}{a_1^2 - 3\,b_1^2} = 1.$$

It follows that

$$x^{-1} = \frac{\bar{x}}{a_1^2 - 3\,b_1^2},$$

or equivalently

$$x^{-1} = \frac{a_1}{a_1^2 - 3\,b_1^2} + \frac{b_1}{a_1^2 - 3\,b_1^2}\,\sqrt{3}. \tag{2}$$

Again since $\mathbb{Q}$ is a field and $a_1, b_1 \in \mathbb{Q}$ we have that

$$\frac{a_1}{a_1^2 - 3\,b_1^2}, \frac{b_1}{a_1^2 - 3\,b_1^2} \in \mathbb{Q}$$

and therefore, $x^{-1} \in \mathbb{Q}(\sqrt{3})$. Thus we established that $\mathbb{Q}(\sqrt{3})$ is closed under inverses.
Therefore $\mathbf{Q}$ is a subfield of $\mathbb{R}$. □

(b) Give an explicit formula for $(a + b\sqrt{3})^{-1}$.

Asnswer. Done already, see Equation (2). □

[1] Remember "rationalizing the denominator"?

4. Consider the set $\mathbb{F} = \{0, 1, a, b\}$ where $a \neq b$. Define addition and multiplication via the following tables

$+$	0	1	a	b
0	0	1	a	b
1	1	0	b	a
a	a	b	0	1
b	b	a	1	0

$\cdot$	0	1	a	b
0	0	0	0	0
1	0	1	a	b
a	0	a	b	1
b	0	b	1	a

.

Prove that $\mathbb{F}$ is a field.

Solution. We have to verify that all the *Field axioms* listed in Definition 40 hold.

- **Both addition and multiplication are commutative:** Indeed, the tables that define addition and multiplication are both symmetric.
- **Existence of zero:** From the addition table we see

$$0 + 0 = 0, \quad 1 + 0 = 1, \quad a + 0 = a, \quad b + 0 = b$$

  and therefore 0 is the zero element of $\mathbb{F}$.
- **Existence of opposites:** From the table we see that for all $x \in \mathbf{F}$

$$x + x = 0$$

  therefore every element is its own opposite.
- **Existence of one:** From the multiplication table we see

$$1 \cdot 0 = 0, \quad 1 \cdot 1 = 1, \quad 1 \cdot a = a, \quad 1 \cdot b = 1$$

  and therefore 1 is the neutral element of multiplication in $\mathbb{F}$.
- **Existence of inverses:** From the table we see that

$$1 \cdot 1 = 1, \quad a\, b = 1$$

  therefore every non-zero element has an inverse.
- **Multiplication is associative:** We have to verify that for all x, y, z

$$(x\, y)\, z = x\, (y\, z). \tag{3}$$

  If one of the x, y, z is 0 both sides of Equation (3) are 0 and so it holds.
  If one of the x, y, z is 1 then (3) is trivially true since both sides are equal to the product of the other two variables.
  If $x = y = z$ it reduces to

$$(x\, x)\, x = x\, (x\, x)$$

  and therefore it holds due to the commutativity of multiplication.

So we are left with verifying (3) when $\{x, y, z\} = \{a, b\}$ and either case $x = y \neq z$, or $x = z \neq y$, or $y = z \neq x$.

If $x = y$ then (3) becomes

$$(x\,x)\,z = x\,(x\,z).$$

If $x = a$ and $z = b$ the LHS equals $b\,b = a$ and the RHS equals $a\,(a\,b) = a\,1 = a$ and thus (3). If $x = b$ and $z = a$ then the LHS equals $a\,a = b$ and the RHS equals $b\,(b\,a) = b\,1 = b$ and (3) holds.

If $x = z$ then (3) reduces to

$$(x\,y)\,x = x\,(y\,x)$$

which is true due to commutativity.

Finally, if $y = z$ it becomes

$$(x\,y)\,y = x\,(y\,y) \iff x\,(y\,y) = (x\,y)\,y \iff (y\,y)\,x = y\,(y\,x)$$

which is equivalent, after renaming x to z and y to x, to the equation we got in the case $x = z$.

- **Addition is associative:** We need to prove that for all $x, y, z \in \mathbb{F}$ we have

$$(x + y) + z = x + (y + z). \tag{4}$$

If any of the x, y, z is zero then Equation (4) is trivially true. If $x = 0$, $y = 0$, or $z = 0$ it reduces to

$$y + z = y + z, \quad x + z = x + z, \text{ or } x + y = y + x$$

respectively.

So we need to verify Equation (4) for $x, y, z \in \{1, a, b\}$. If $x = y = z$ then it reduces to

$$(x + x) + x = x + (x + x)$$

which is true because $x + x = 0$.

If $x = y$ then, because $x + x = 0$, it reduces to

$$(x + x) + z = x + (x + z) \iff z = x + (x + z).$$

If $x = 1$ and $z = a$ this is equivalent to

$$a = 1 + (1 + a)$$

which holds because $1 + a = b$ and $1 + b = a$. If $x = 1$ and $z = b$ then it reduces to

$$b = 1 + (1 + b)$$

also true.

If $x = a$ and $z = 1$ it reduces to

$$1 = a + (1 + a)$$

which is true because $1 + a = b$ and $a + b = 1$.

If $x = a$ and $z = b$ it reduces to

$$a = b + (a + b)$$

which is true because $a + b = 1$ and $b + 1 = a$.

If $x = b$ and $z = 1$ we get

$$1 = b + (b + 1)$$

which is true because $b + 1 = a$ and $b + a = 1$.

Finally if $x = b$ and $z = a$ then it reduces to

$$a = b + (b + a),$$

which is true since $b + a = 1$ and $b + 1 = a$.

The only remaining case is when $\{x, y, z\} = \{1, a, b\}$. Because addition is commutative we can write Equation (4) as

$$(x + y) + z = (z + y) + x$$

and in this form it is clear that if it is true for (x, y, z) it will also be true for (z, y, x). So we will verify Equation (4) in two cases, when $x = 1$ and when $y = 1$.

If $x = 1$ (4) becomes

$$(1 + y) + z = 1 + (y + z)$$

but $y + z = 1$ and $1 + 1 = 0$ so the RHS is 0. The LHS is also 0 since it's either $(1 + a) + b = b + b$ or $(1 + b) + a = a + a$.

If $y = 1$ then it becomes, using

$$(x + 1) + z = x + (1 + z) \iff (x + 1) + z = (z + 1) + x$$

and thus by symmetry we only need to verify that

$$(a + 1) + b = a + (1 + b) \iff b + b = a + a$$

which is true.

- **Multiplication distributes over addition :**

  We need to prove that for all $x, y, z \in \mathbb{F}$ we have

$$x\,(y + z) = x\,y + x\,z. \tag{5}$$

  Since $0\,t = 0$ for all $t \in \mathbb{F}$, Equation (5) is true when $x = 0$. For the same reason it is also true if $y = 0$ or $z = 0$.

  Similarly, since $1\,t = 0$ for all $t \in \mathbb{F}$ (5) holds if $x = 1$.

  If $y = z$ then (5) becomes

$$x\,(y + y) = x\,y + x\,y.$$

  But for all $t \in \mathbb{F}$, we have $t + t = 0$ and so both sides are equal to 0.

  If $x = a$ the only cases not covered already are

$$a\,(1 + a) = a\,1 + a\,a \tag{6}$$
$$a\,(1 + b) = a\,1 + a\,b \tag{7}$$
$$a\,(a + b) = a\,a + a\,b. \tag{8}$$

  Equation (6) is equivalent to

$$a\,b = a + b$$

which is true.

Equation (8) is equivalent to

$$a\,a = a + 1$$

which is true.

Finally, Equation (8) is equivalent to

$$a\,1 = b + 1$$

which is also true.

Thus $\mathbb{F}$ is a field. □

5. Consider $\mathbb{R}^2$ with the usual addition

$$(a, b) + (c, d) = (a + c, b + d),$$

and multiplication given by

$$(a, b)\,(c, d) = (a\,c, b\,d).$$

Is $\mathbb{R}^2$ with these operations a field? Fully justify your answer.

Answer. No. The zero element for addition is $(0, 0)$ and we have

$$(1, 0)\,(0, 1) = (0, 0)$$

with $(1, 0) \neq (0, 0)$ and $(0, 1) \neq (0, 0)$. This contradicts Item (g) of Theorem 4.1.1. □

6. Consider the following matrices[2] with complex entries

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Prove that these matrices satisfy the following relations:

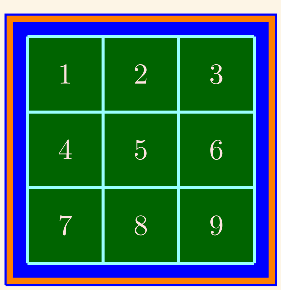
(a) $\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = I$.

(b) $\sigma_x\,\sigma_y = i\,\sigma_z, \quad \sigma_y\,\sigma_z = i\,\sigma_x, \quad \sigma_z\,\sigma_x = i\,\sigma_y.$

(c) $\sigma_x\,\sigma_y = -\sigma_y\,\sigma_x, \quad \sigma_x\,\sigma_z = -\sigma_z\,\sigma_x, \quad \sigma_y\,\sigma_z = -\sigma_z\,\sigma_y.$

Answer. All calculations are straightforward. □

7. Consider a 3×3 grid of squares, each either green or red. When we touch a square its color and the color of its neighbors change, where the neighbors of a square are all squares that share an edge with it.



[2] These matrices are called *Pauli spin matrices*. They are used in Quantum Mechanics to compute the spin of an electron.

Thus for example, if we touch the square numbered 1 the squares numbered 1, 2, and 4 change color, if we touch square 5 then all squares except 1, 3, 7, and 9 change color, and if we touch 8 then 5, 7, 8, and 9 change colors.

We start with all squares green. Find, if possible, a sequence of squares to touch so that all squares turn red.

Proof. As we did in Examples 80 and 81 in the notes represent each operation of changing the color of some of the squares by a 9×1 matrix with entries in the field $\mathbb{Z}/2$, with 0 standing for "don't change" and 1 for change. We want to find a solution $(x_i)_{i=1,\cdot,9}$ for the vector equation

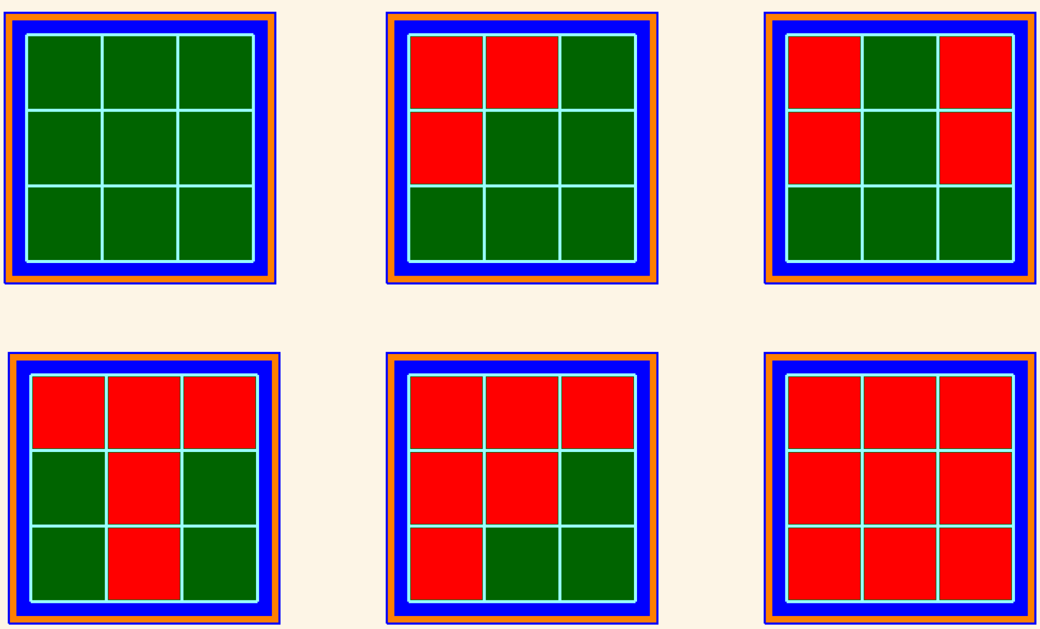
$$\sum_{i=1}^{9} x_i\,\mathbf{s}_i = \sum_{i=1}^{9} \mathbf{e}_i$$

where, for $i = 1, \ldots, 9$, $\mathbf{s}_i$ is the operation effected by touching square i, and $\mathbf{e}_i$ is the i-th element of the standard basis of $(\mathbb{Z}/2)^9$.

This vector equation is equivalent to a linear system with augmented matrix

$$\left(\begin{array}{ccccccccc|c}
1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1\\
1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1\\
0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1\\
1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1\\
0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1\\
0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1\\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1\\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1
\end{array}\right) \sim \left(\begin{array}{ccccccccc|c}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1\\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1\\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1
\end{array}\right).$$

Thus there is a unique solution: we have to touch the top-left, top-right, center, bottom-left and bottom-right squares.



□

8. Consider the following vectors in $\mathbb{C}^4$:

$$\mathbf{v}_1 = (1, i, 0, -i) \qquad \mathbf{v}_2 = (2+i, 3\,i, i, 1-4\,i)$$
$$\mathbf{v}_3 = (5+i, 2+6\,i, 1+2\,i, 7-9\,i) \qquad \mathbf{v}_4 = (0, 3-i, 1+i, 0).$$

Find a basis and state the dimension of the linear span $\mathbb{C}\left\langle \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4 \right\rangle$.

Answer. We have

$$\begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 \end{pmatrix} = \begin{pmatrix} 1 & 2+i & 5+i & 0 \\ i & 3\,i & 2+6\,i & 3-i \\ 0 & i & 1+2\,i & 1+i \\ -i & 1-4\,i & 7-9\,i & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

It follows that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is linearly independent and since $\dim \mathbb{C}^4 = 4$ we have $\mathbb{C}\left\langle \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4 \right\rangle = \mathbb{C}^4$. So,

$$\dim \mathbb{C}\left\langle \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4 \right\rangle = 4$$

and a basis is, for example, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$. □

9. Consider $\mathbb{R}$ as a vector space over $\mathbb{Q}$. Prove that

$$\left\{\sqrt{2}, \sqrt{3}, \sqrt{5}\right\}$$

is linearly independent. You may consider Item (a) of Example 93 in the notes known.

Solution. Assume, that for some $a, b, c \in \mathbb{Q}$ we have

$$a\,\sqrt{2} + b\,\sqrt{3} + c\,\sqrt{5} = 0. \tag{9}$$

Then

$$\begin{aligned} a\,\sqrt{2} + b\,\sqrt{3} + c\,\sqrt{5} = 0 &\implies c\,\sqrt{5} = -a\,\sqrt{2} - b\,\sqrt{3} \\ &\implies 5\,c^2 = 2\,a^2 + 3\,b^2 + 2\,a\,b\,\sqrt{6} \\ &\implies 2\,a\,b\,\sqrt{6} = 5\,c^2 - 2\,a^2 - 3\,b^2. \end{aligned}$$

If $c \neq 0$ then we must have $a\,b = 0$ otherwise Equation 9 gives

$$\sqrt{6} = \frac{5\,c^2 - 2\,a^2 - 3\,b^2}{2\,a\,b} \in \mathbb{Q},$$

a contradiction.

Thus $a = 0$ or $b = 0$. Assume $a = 0$ then we get

$$5\,c^2 - 3\,b^2 = 0 \implies \frac{5}{3} = \frac{b^2}{c^2} \implies \sqrt{\frac{5}{3}} = \frac{b}{c} \in \mathbb{Q}$$

a contradiction.

Similarly, $b = 0$ leads to the contradiction

$$\sqrt{\frac{5}{2}} \in \mathbb{Q}.$$

Thus $c = 0$ and Equation 9 gives

$$a\sqrt{2} + b\sqrt{3} = 0$$

and since (Item (a) of Example 93 in the notes) $\left\{\sqrt{2}, \sqrt{3}\right\}$ is linearly independent we have that $a = b = 0$ as well.

We conclude that Equation 9 holds if and only if $a = b = c = 0$ and therefore $\left\{\sqrt{2}, \sqrt{3}, \sqrt{5}\right\}$ is linearly independent. □

10. Let $\mathbf{S}_n$ denote the set of $n \times n$ symmetric matrices over $\mathbb{R}$ (see Definition 29 in the notes).

 (a) Prove that $\mathbf{S}_n$ is a vector subspace of $\mathbf{M}_n$.

 Answer. By definition

$$A \in \mathbf{S}_n \iff A^* = A.$$

 The zero matrix O is in $\mathbf{S}_n$ and therefore $\mathbf{S}_n \neq \emptyset$.
 If $A, B \in \mathbf{S}_n$ and $\lambda, \mu \in \mathbb{R}$ then

$$(\lambda A + \mu B)^* = \lambda A^* + \mu B^* = \lambda A + \mu B$$

 and thus $\lambda A + \mu B \in \mathbf{S}_n$. □

 (b) Find a basis and the dimension of $\mathbf{S}_n$.

 Solution. Let $S_{k\ell}$ for $1 \leq k \leq l \leq n$ be the matrix with all entries 0 except the (k, ℓ) and (ℓ, k) entries that are equal to 1. In other words, if $\{E_{k\ell} : k, \ell \in \{1, \ldots, n\}\}$ then

$$S_{k\ell} = \begin{cases} E_{k\ell} + E_{\ell k} & k \neq \ell \\ E_{kk} & k = \ell \end{cases}..$$

 Then $B = \{S_{k\ell} : k, \ell \in \mathbb{N},\ 1 \leq k < l \leq n\}$ is a basis of $\mathbf{S}_n$. Indeed let

$$A = \sum_{\substack{1 \leq k \leq n \\ 1 \leq \ell \leq n}} a_{k\ell} E_{k\ell} \tag{10}$$

 be a symmetric matrix. Then for all k, ℓ we have $a_{k\ell} = a_{\ell k}$ and so if we split the sum in the RHS of Equation (10) into two to parts: the diagonal terms plus the off-diagonal terms we have

$$A = \sum_{\substack{1 \leq k \leq n \\ 1 \leq \ell \leq n \\ k \neq \ell}} a_{k\ell} E_{k\ell} + \sum_{k=1}^{n} a_{kk} E_{kk} = \sum_{\substack{1 \leq k \leq n \\ 1 \leq \ell \leq n \\ k < \ell}} a_{k\ell} \left(E_{k\ell} + E_{\ell k}\right) + \sum_{k=1}^{n} a_{kk} E_{kk} = \sum_{S_{k\ell} \in B} a_{k\ell} S_{k\ell}.$$

 Thus B is spanning.

Now notice that

$$\sum_{S_{k\ell}\in B} a_{k\ell}\, S_{k\ell} = \sum_{1\le k\le \ell\le n} a'_{k\ell}\, E_{k\ell}$$

where

$$a'_{k\ell} = \begin{cases} a_{k\ell} & k \le \ell \\ a_{\ell k} & k > \ell \end{cases},$$

and so

$$\begin{aligned} \sum_{S_{k\ell}\in B} a_{k\ell}\, S_{k\ell} = O &\implies \sum_{1\le k\le \ell\le n} a'_{k\ell}\, E_{k\ell} = O \\ &\implies a'_{k\ell} = 0 \quad \forall k, \ell \\ &\implies a_{k\ell} = 0 \quad \forall k, \ell \end{aligned}$$

.

Therefore B is also linearly independent. The dimension of $\mathbf{S}_n$ is therefore the cardinality of B. Now, if we set

$$B_0 = \{S_{k\ell} : 1 \le k < \ell \le n\}, \quad B_1 = \{S_{kk} : 1 \le k \le n\}$$

then

$$B = B_0 \cup B_1, \quad B_0 \cap B_1 = 0, \quad |\, B_0 \,| = \binom{n}{2}, \quad |\, B_1 \,| = n.$$

Therefore

$$\begin{aligned} |\, B \,| &= |\, B_0 \,| + |\, B_1 \,| \\ &= \binom{n}{2} + n \\ &= \frac{n\,(n-1)}{2} + n \\ &= \frac{n\,(n+1)}{2}. \end{aligned}$$

In conclusion

$$\dim \mathbf{S}_n = \frac{n(n+1)}{2}.$$

□

11. Consider the vector space $\mathbb{R}[x]$ of polynomials with real coefficients. Which of the following subsets is a vector subspace of $\mathbb{R}[x]$?

(a) $V = \{p(x) \in \mathbb{R}[x] : p(42) = 0\}$.

Answer. The zero polynomial $0 \in V$, and thus $V \neq \emptyset$.
Now if $p(x), q(x) \in V$ and $\lambda, \mu \in \mathbb{R}$ then

$$\lambda\, p(42) + \mu\, q(42) = \lambda\, 0 + \mu\, 0 = 0$$

and therefore $\lambda\, p(x) + \mu\, q(x) \in V$.
It follows that V is a vector subspace of $\mathbb{R}[x]$. □

(b) $U = \{p(x) \in \mathbb{R}[x] : p(42) \geq 0\}$.

Answer. U is not a vector subspace. For, every vector subspace is closed under taking opposites, and U isn't. For example the constant polynomial $p(x) = 42$ is in U while its opposite $-p(x) = -42$ is not in U. □

(c) $W = \{p(x) \in \mathbb{R}[x] : p(42) = p(0)\}$.

Answer. W is a vector subspace. Indeed for the zero polynomial we have $0(42) = 0(0)$ and therefore $0 \in W$. Thus $W \neq \emptyset$.
Now let $p(x), q(x) \in W$ and $\lambda, \mu \in \mathbb{R}$. Then

$$\lambda\, p(42) + \mu\, q(42) = \lambda\, p(0) + \mu\, q(0)$$

and therefore $\lambda\, p(x) + \mu\, q(x) \in W$. □

(d) $X = \{p(x) \in \mathbb{R}[x] : \deg p(x) = 8\}$.

Answer. X is not a vector subspace because it does not contain the zero polynomial. □

Fully justify your answers.

12. Let $\mathbf{P}_3$ be the set of real polynomials of degree at most 3:

$$\mathbf{P}_3 = \{p(x) \in \mathbb{R}[x] : \deg p(x) \leq 3\}\,.$$

Prove that

$$B = \left\{1, x-1, (x-1)^2, (x-1)^3\right\},$$

is a basis of $\mathbf{P}_3$.

Solution. Let $p_i(x)$, $i = 0, 1, 2, 3$ be the given polynomials in the given order. To prove that $B = \{p_0(x), p_1(x), p_2(x), p_3(x)\}$ is a basis of $\mathbf{P}_3$, we need to prove that every polynomial $p(x) = c_0 + c_1\, x + c_2\, x^2 + c_3\, x^3$ can be expressed uniquely as a linear combination of elements of B. In other words we need to prove that given $p(x)$ there are unique $\lambda_i \in \mathbb{R}$, $i = 0, 1, 2, 3$ such that

$$p(x) = \sum_{i=0}^{3} \lambda_i\, p_i(x). \tag{11}$$

We have

$$(x-1)^2 = x^2 - 2\,x + 1, \quad (x-1)^3 = x^3 - 3\,x^2 + 3\,x - 1$$

.

And therefore

$$\begin{aligned}\lambda_0\, p_0(x) + \lambda_1\, p_1(x) + \lambda_2\, p_2(x) + \lambda_3\, p_3(x) &= \lambda_0\, 1 + \lambda_1\, (x-1) + \lambda_2\, (x-1)^2 + \lambda_3\, (x-1)^3 \\ &= (\lambda_0 - \lambda_1 + \lambda_2 - \lambda_3) \\ &\quad + (\lambda_1 - 2\,\lambda_2 + 3\,\lambda_3)\, x \\ &\quad + (\lambda_2 - 3\,\lambda_3)\, x^2 \\ &\quad + \lambda_3\, x^3.\end{aligned}$$

.

So Equation (11) is equivalent to the system

$$\begin{cases} \lambda_0 - \lambda_1 + \lambda_2 - \lambda_3 &= c_0 \\ \lambda_1 - 2\,\lambda_2 + 3\,\lambda_3 &= c_1 \\ \lambda_2 - 3\,\lambda_3 &= c_2 \\ \lambda_3 &= 1 \end{cases}.$$

The matrix of the system

$$\begin{pmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

is invertible. It follows that the system has a unique solution and therefore B is a basis. □

13. Let $S = \{A, B, C, D\} \subseteq \mathbf{M}_2$ where

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

(a) Prove that S is a basis of $\mathbf{M}_2$.

(b) Express $X = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ as a linear combination of elements of S.

Answer. We have

$$x\,A + y\,B + z\,C + w\,D = \begin{pmatrix} x & x+y \\ y+z & z+w \end{pmatrix}.$$

So,

$$\begin{pmatrix} a & b \\ c & c \end{pmatrix} = x\,A + y\,B + z\,C + w\,D \iff \begin{cases} x &= a \\ x + y &= b \\ y + z &= c \\ z + w &= d \end{cases} \iff \begin{cases} x &= a \\ y &= -a + b \\ z &= a - b + c \\ w &= -a + b - c + d \end{cases}.$$

Thus any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{M}_2$ can be uniquely expressed as a linear combination of S and therefore S is a basis of $\mathbf{M}_2$.

For the given X we have $x = 1$, $y = 1$, $z = 2$, $z = 2$ and so

$$X = A + B + 2\,C + 2\,D.$$

□

14. Find conditions on the complex number z so that the vectors

$$\mathbf{v}_1 = (z, 0, 1), \quad \mathbf{v}_1 = (0, 1, z^3), \quad \mathbf{v}_3 = (z, 1, 1 + z)$$

form a basis of $\mathbb{C}^3$.

Answer. $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis of $\mathbb{C}^3$ if and only if the matrix with columns those vectors is invertible. We have

$$\begin{aligned}\begin{pmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3\end{pmatrix} &= \begin{pmatrix} z & 0 & 1 \\ 0 & 1 & z^3 \\ z & 1 & 1+z \end{pmatrix} \\ &\sim \begin{pmatrix} z & 0 & 1 \\ 0 & 1 & z^3 \\ 0 & 1 & z \end{pmatrix} \\ &\sim \begin{pmatrix} z & 0 & 1 \\ 0 & 1 & z^3 \\ 0 & 0 & z-z^3 \end{pmatrix}.\end{aligned}$$

The last matrix is invertible if and only if $z \neq 0$ and $z - z^3 \neq 0$. Thus these vectors form a basis if and only if $z \notin \{0, 1, -1\}$. □

15. Consider the vector space $\mathbf{M}_n$ of real $n \times n$ matrices, and let B be a basis of $\mathbf{M}_n$. Prove that

$$B^* = \{X^* : X \in B\}$$

is also a basis of $\mathbf{M}_n$, where X^* stands for the transpose of a matrix X.

Solution. Let $B = \{X_1, X_2, \ldots, X_{n^2}\}$ so that $B^* = \left\{X_1^*, X_2^*, \ldots, X_{n^2}^*\right\}$. We will prove that B^* is linearly independent and spanning.

Consider then a linear dependency

$$\lambda_1 X_1^* + \lambda_2 X_2^* + \cdots + \lambda_{n^2} X_{n^2}^* = O$$

with $\lambda_i \in \mathbb{R}$ for $i = 1, \ldots, n^2$. Then,

$$(\lambda_1 X_1^* + \lambda_2 X_2^* + \cdots + \lambda_{n^2} X_{n^2}^*)^* = O^*$$

or equivalently

$$\lambda_1 \, (X_1^*)^* + \lambda_2 \, (X_2^*)^* + \cdots + \lambda_{n^2} \, (X_{n^2}^*)^* = O,$$

which is equivalent to

$$\lambda_1 X_1 + \lambda_2 X_2 + \cdots + \lambda_{n^2} X_{n^2} = O.$$

But B, being a basis, is linearly independent and therefore we conclude that

$$\lambda_1 = \lambda_2 = \cdots = \lambda_{n^2} = 0.$$

Thus the only linear combination of B^* equal to the zero matrix is the trivial linear combination, and so B^* is linearly independent.

Now, let $A \in \mathbf{M}_n$. Since B is spanning we can express A^* as a linear combination of elements of B, that is for some $\lambda_1, \ldots, \lambda_{n^2} \in \mathbb{R}$ we have

$$A^* = \lambda_1 X_1 + \cdots + \lambda_{n^2} X_{n^2}.$$

But then

$$(A^*)^* = (\lambda_1 X_1 + \cdots + \lambda_{n^2} X_{n^2})^*,$$

or equivalently

$$A = \lambda_1 X_1^* + \cdots + \lambda_{n^2} X_{n^2}^*,$$

establishing that A can be expressed as a linear combination of elements of B^*. Therefore B^* is spanning. $\square$

2 The in-class part

1. Let A be a 3×4 and B and 4×3 matrix. Prove that $A\,B$ is not invertible.

   Answer. The linear map $A\colon \mathbb{R}^4 \to R^3$ is not injective by the rank-nullity theorem. Indeed $\operatorname{rank} A \leq 3$ and therefore

   $$4 = A + \operatorname{rank} A \implies A \geq 1,$$

   and therefore $\ker A \neq \{\mathbf{0}\}$.

   Now we have that

   $$\begin{aligned} \mathbf{x} \in \ker A &\implies A\,\mathbf{x} = \mathbf{0} \\ &\implies B\,(A\,\mathbf{x}) = \mathbf{0} \\ &\implies (B\,A)\,\mathbf{x} = \mathbf{0} \\ &\implies \mathbf{x} \in \ker B\,A, \end{aligned}$$

   and therefore

   $$\ker A \subseteq \ker(B\,A).$$

   Therefore $\ker(B\,A) \neq \{\mathbf{0}\}$ establishing that $B\,A$ is not injective.

   $\square$

   Note. $A\dot{B}$ *can* be invertible. For example, if

   $$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

   Then, $A\,B = I_3$, and thus invertible.

2. Consider $\mathbb{R}$ as a vector space over $\mathbb{Q}$. Show that
   (a) $\{\sqrt{12}, \sqrt{27}\}$ is linearly dependent.
   (b) $\{\sqrt{5}, \sqrt{7}\}$ is linearly independent.

3. Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 0 \\ 0 & 6 & 7 \end{pmatrix}$, and $B = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 3 & 12 \\ 0 & 0 & 17 \end{pmatrix}$. Find an invertible 3×3 matrix X such that $X\,A = B$.

Solution. B is obtained from A by adding 4 times the first row from to the opposite of the second, then adding two times the second row to the opposite of the third.

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 0 \\ 0 & 6 & 7 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & 3 & 12 \\ 0 & 6 & 7 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & 3 & 12 \\ 0 & 0 & 17 \end{pmatrix} = B.$$

Applying the same row operations to the identity matrix we obtain

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 4 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 4 & -1 & 0 \\ 8 & -2 & -1 \end{pmatrix}.$$

Thus

$$X = \begin{pmatrix} 1 & 0 & 0 \\ 4 & -1 & 0 \\ 8 & -2 & -1 \end{pmatrix}.$$

□

4. Let $\mathbf{P}_3$ be the vector space of real polynomials of degree at most 3. Prove that

$$B = \{1, x, x\,(x+1), x\,(x+1)\,(x+2)\}$$

is a basis of $\mathbf{P}_3$.

Solution. Let $p_i(x)$, for $i = 0, 1, 2, 3$ be the elements of B in the order listed above. We have to show that any polynomial $p(x) \in \mathbf{P}_3$ can be written, uniquely, as a linear combination of B. Let then

$$p(x) = c_0 + c_1\,x + c_2\,x^2 + c_3\,x^4$$

with $c_i \in \mathbb{R}$, for $i = 0, 1, 2, 3$. We have to show that there are unique $\lambda_i \in \mathbb{R}$ such that

$$p(x) = \sum_{i=0}^{3} \lambda_i\,p_i(x). \tag{12}$$

Expressing the elements of B in the standard basis of $\mathbf{P}_3$ we have

$$\begin{aligned} p_0(x) &= 1 \\ p_1(x) &= x \\ p_2(x) &= x + x^2 \\ p_3(x) &= (x+2)\,(x+x^2) \\ &= 2\,x + 3\,x^2 + x^3. \end{aligned}$$

Thus,

$$\begin{aligned} \sum_{i=0}^{3} \lambda_i\,p_i(x) &= \lambda_0 + \lambda_1\,x + \lambda_2\,(x + x^2) + \lambda_3\,(2\,x + 3\,x^2 + x^3) \\ &= \lambda_0 + (\lambda_1 + \lambda_2 + 2\,\lambda_3)\,x + (\lambda_2 + 3\,\lambda_3)\,x^2 + \lambda_3\,x^3, \end{aligned}$$

and so Equation (11) is equivalent to the linear system:

$$\begin{cases} \lambda_0 & = c_0 \\ \lambda_1 + \lambda_2 + 2\,\lambda_3 & = c_1 \\ \lambda_2 + 3\,\lambda_3 & = c_2 \\ \lambda_3 & = c_3 \end{cases}.$$

The matrix of the system is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

which is in echelon form, with no free columns. Thus the system has a unique solution establishing that B is indeed a basis. □

5. Recall that a square matrix A is *antisymmetric* if $A^* = -A$. Let $\mathbf{A}_n$ be the set of antisymmetric $n \times n$ matrices. Prove:

   (a) $\mathbf{A}_3$ is a vector subspace of $\mathbf{M}_n$.

   Solution. The zero matrix $O \in \mathbf{A}_n$ and assuming $A, B \in \mathbf{A}_n$, $\lambda, \mu \in \mathbb{R}$ we have

   $$(\lambda\, A + \mu\, B)^* = \lambda\, A^* + \mu\, B^* = -\lambda\, A - \mu\, B = -(\lambda\, A + \mu\, B),$$

   establishing that $\lambda\, A + \mu\, B \in \mathbf{A}_n$.
   Therefore $\mathbf{A}_n$ is indeed a vector subspace of $\mathbf{M}_n$. □

   (b) Find a basis of $\mathbf{A}_n$.

   Solution. Let E_{ij}, $1 \leq i, j \leq n$ be the standard basis of $\mathbf{M}_n$, then the set

   $$B = \{E_{ij} - E_{ji} : 1 \leq i < j \leq n\}$$

   forms a basis of $\mathbf{A}_n$.
   Indeed, let $A = \sum a_{ij}\, E_{ij} \in \mathbf{M}_n$. Then

   $$A \in \mathbf{A}_n \implies a_{ij} = -a_{ji}, \text{ for all } i, j$$

   and in particular for the diagonal elements we have $a_{ii} = 0$ for all $i = 1 \ldots, n$. Therefore if $A \in \mathbf{A}_n$ we can write

   $$\begin{aligned} A &= \sum_{1 \leq i < j \leq n} a_{ij}\, E_{ij} + \sum_{1 \leq j < i \leq n} a_{ij}\, E_{ij} \\ &= \sum_{1 \leq i < j \leq n} a_{ij}\, E_{ij} + \sum_{1 \leq i < j \leq n} (-a_{ij})\, E_{ji} \\ &= \sum_{1 \leq i < j \leq n} a_{ij}\, (E_{ij} - E_{ji}). \end{aligned}$$

   We have hence established that B is spanning.

B is also linearly independent since

$$\sum_{1\le i<j\le n} a_{ij}\left(E_{ij}-E_{ji}\right)=O \implies \sum_{1\le i\le j\le n} a'_{ij}\,E_{ij}$$

where

$$a'_{ij}=\begin{cases} a_{ij} & i<j \\ 0 & i=j \\ -a_{ji} & i>j \end{cases}.$$

And since the standard basis of M_n is linearly independent we conclude that all $a'_{ij}=0$, hence all $a_{ij}=0$.

Thus, B is spanning and linearly independent and therefore a basis. □

6. Let $S=\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3,\mathbf{v}_4\}\subseteq(\mathbb{Z}/3)^4$, where

$$\mathbf{v}_1=(1,1,0,1),\quad \mathbf{v}_2=(2,2,2,0),\quad \mathbf{v}_3=(0,0,2,1),\quad \mathbf{v}_4=(1,2,1,2).$$

Find $\dim\langle S\rangle$, the dimension of the linear span of S.

Solution. We have

$$\begin{pmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4\end{pmatrix}=\begin{pmatrix}1&2&0&1\\1&2&0&2\\0&2&2&1\\1&0&1&2\end{pmatrix}\sim\begin{pmatrix}1&2&0&1\\0&1&0&0\\0&2&2&1\\0&1&1&1\end{pmatrix}\sim\begin{pmatrix}1&2&0&1\\0&1&0&0\\0&2&2&1\\0&0&0&2\end{pmatrix}.$$

There are three basic columns and therefore $\dim\langle S\rangle=3$. □