MTH 42, Fall 2024

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Take Home Exam

1. Find a polynomial of degree at most 4 that satisfies the following conditions

$$p(0) = -5, \quad p(-1) = -10, \quad p(1) = 0, \quad p(2) = 29, \quad p(-2) = -15.$$

Hint. This is similar to Examples 1.9 and 1.10 in the notes, and Exercise 6 in the First Homework. The polynomial will be of the form $p(x) = a_4\,x^4 + a_3\,x^3 + a_2\,x^2 + a_1\,x + a_0$.

2. Let V be the subspace of $\mathbb{R}^5$ spanned by the vectors

$$\begin{aligned} \mathbf{v}_1 &= (1, 2, -1, 3, 4), & \mathbf{v}_2 &= (2, 4, -2, 6, 8), \\ \mathbf{v}_3 &= (1, 3, 2, 2, 6), & \mathbf{v}_4 &= (1, 4, 5, 1, 8), \\ \mathbf{v}_5 &= (2, 7, 3, 3, 9), & \mathbf{v}_6 &= (4, 9, -1, 11, 18). \end{aligned}$$

Find a basis and the dimension of V.

Hint. See Examples 5.16 and 5.17 in the notes.

3. Let S_1 and S_2 be two subsets of $\mathbb{R}^n$ with $S_1 \subseteq S_2$. Prove
   (a) If S_1 is spanning then S_2 is also spanning.
   (b) If S_1 is linearly dependent then S_2 is also linearly dependent.
   (c) If S_2 is linearly independent then S_1 is also linearly independent.

4. Let

$$B = \{(1, 1, 1, 1, 1), (0, 1, 1, 1, 1), (0, 0, 1, 1, 1), (0, 0, 0, 1, 1), (0, 0, 0, 0, 1)\}.$$

   (a) Prove that B is a basis of $\mathbb{R}^5$.
   (b) Express the elements of the standard basis of $\mathbb{R}^5$ as linear combinations of elements of B.

Hint. This is a combination of Exercises 3 and 4 of the Third homework. You can answer all questions with one calculation. See Example 5.18 in the notes.

5. Let $T\colon\ \mathbb{R}^3 \to \mathbb{R}^4$ be defined by

$$T(x, y, z) = (x + 2\,y + z, x + y, y - 3\,z, 4\,x - 3\,y + 2\,z).$$

   (a) Prove that T is linear.
   (b) Find the matrix of T.

Hint. For the first part see Example 6.11 in the notes. For the second part see Example 6.18 in the notes.

6. Let

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix}.$$

(a) Prove that the columns of A form a basis of $\mathbb{R}^3$.

(b) Express each of the vectors in the standard basis of $\mathbb{R}^3$ as linear combinations of the columns of A.

(c) Let T be the linear function that sends the i-th column to the i-th row of A. That is if $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$ are the columns of A then T is defined by

$$T\,\mathbf{a}_1 = (1, 2, 3), \quad T\,\mathbf{a}_2 = (0, 2, 1), \quad T\,\mathbf{a}_3 = (1, 0, 3).$$

Find the matrix of T.

Hint. See Example 6.19 in the notes.