# MTH 23.5, Fall 2024

## Take home exam

### The Answers

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1. For the following data:

$$11,\ 29,\ 30,\ 41,\ 46,\ 46,\ 46,\ 47,\ 49,\ 52,\ 54,\ 54,\ 59,\ 60,\ 60.$$

   (a) Find he mode.

   *Answer.* The most frequently occurring value is 46 with frequency 3. Thus the mode is 46. □

   (b) Find the median.

   *Answer.* There are $n = 15$ values. Therefore the median will is at the position

$$\frac{n+1}{2} = \frac{16}{2} = 8.$$

   The eighth value is 47. Thus, the median is

$$\tilde{x} = 47.$$

   □

   (c) Find the first and third quartiles.

   *Answer.* The first quartile is the median of the lower part of data, that is the the median of the first 7 values. This is the 4-rd value. So

$$Q_1 = 41.$$

   The third quartile is the median of the upper half of data, that is the last 7 values. This is the 11-th value. So,

$$Q_3 = 54.$$

   □

   (d) What is the percentile rank of 46?

   *Answer.* The percentile rank of 46 is the percentage of data that is less or equal to 46. The are 7 out of 15 values $x$ with $x \leq 46$. Now,

$$\frac{7}{15} \approx 0.47 = 47\%.$$

   Thus the percentile rank of 46 is 47. □

2. A survey found that 60% of all the regular customers in an ice cream shop like vanilla ice cream, and 70% like chocolate. Of those who like chocolate, 50% also likes vanilla. If you walk into that store and encounter one of the regular customers what is the probability that

   (a) They like both chocolate and vanilla.

   (b) They like at least one of the flavors, chocolate or vanilla.

   *Answer.* Let $C$ be the event "Likes chocolate" and $V$ the event "Likes vanilla". We are given that

$$P(C) = 0.7, \quad P(V) = 0.6, \quad P(V \textbf{ given } C) = 0.5.$$

   (a) We have

$$\begin{aligned} P(C \textbf{ and } V) &= P(C) \cdot P(V \textbf{ given } C) \\ &= 0.7 \cdot 0.5 \\ &= 0.35. \end{aligned}$$

(b) We have

$$\begin{aligned} P(C\textbf{or } V) &= P(C) + P(V) - P(C\textbf{and } V) \\ &= 0.7 + 0.6 - 0.35 \\ &= 0.95. \end{aligned}$$

□

3. In a stack of selves I have 400 books. They are either in English or in Greek, and their content is either Math, Non Fiction other than math, or Fiction. The number of the books according their language and their category is shown in the table below

| | **Math** | **Non-Fiction** | **Fiction** | **Total** |
|---|---|---|---|---|
| **English** | 150 | 80 | 20 | 250 |
| **Greek** | 50 | 40 | 60 | 150 |
| Total | 200 | 120 | 80 | 400 |

If I pick up a book at random from that stack of selves what is the probability that the book is

(a) A Greek book.

*Answer.* Out of a total of 400 books 150 are Greek. So the probability that the randomly selected book is Greek is

$$P(G) = \frac{150}{400} = 0.375.$$

□

(b) A Greek book **given** that it is a Math book?

*Answer.* Since we are given that a Math book was selected we concentrate attention to the column of Math books. Out of a total of 200 Math books, 50 are Greek. Therefore,

$$P(G\,\underline{\textbf{given}}\ M) = \frac{50}{200} = 0.25$$

□

(c) A Greek book **or** a Math book?

*Answer.* We have

$$P(G\,\underline{\textbf{or}}\ M) = P(G) + P(M) - P(G\,\underline{\textbf{and}}\ M).$$

From Part (a) we know $P(G) = 0.375$.
Out of 400 books 200 are Math books. So,

$$P(M) = \frac{200}{400} = 0.5.$$

Out of 400 books, 50 are both Greek and Math. Thus,

$$P(G\,\underline{\textbf{and}}\ M) = \frac{50}{400} = 0.125.$$

Putting it all together we get

$$\begin{aligned} P(G\,\underline{\textbf{or}}\ M) &= 0.375 + 0.5 - 0.125 \\ &= 0.75. \end{aligned}$$

□

(d) **Not** a Math book?

*Answer.* We have

$$\begin{aligned} P(\underline{\textbf{not}}\ M) &= 1 - P(M) \\ &= 1 - 0.5 \\ &= 0.5. \end{aligned}$$

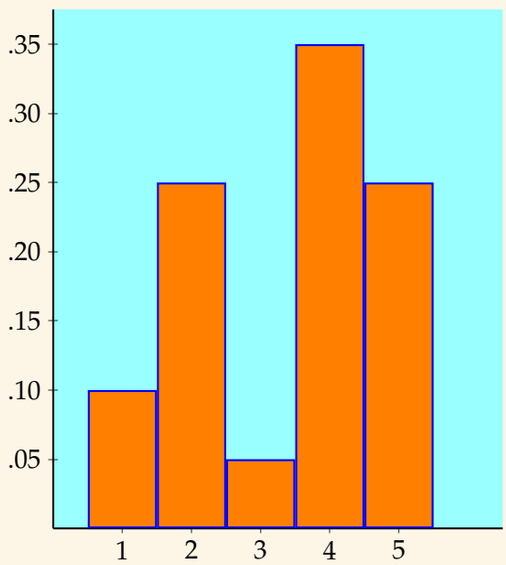
□

4. A box contains balls numbered 1 through 5 with relative frequencies as shown in the table below.

| $x$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $p$ | 0.10 | 0.25 | 0.05 | 0.35 | 0.25 |

(a) Construct the relative frequency histogram in the grid below.

*Answer.* The histogram is shown below.




□

(b) A ball is drawn randomly from that box. What is the probability that its number is even?

*Answer.* Let $E$ be the event "The number of the drawn ball is even". In order for $E$ to happen the number of the ball has to be 2 or 4. Thus,

$$P(E) = P(2) + P(4) = 0.25 + 0.35 = 0.60.$$

□

(c) A ball is randomly drawn from the box and its number is odd. What's the probability that its number is 1?

*Answer.* Let $O$ be the event "the number of the ball is odd". Then from Part (b) we have

$$P(O) = 1 - P(E) = 1 - 0.60 = 0.40.$$

We are asked to find $P(1 \ \underline{\textbf{given}} \ O)$. We have

$$P(1 \ \underline{\textbf{given}} \ O) = \frac{P(1)}{P(O)} = \frac{0.1}{0.4} = 0.25.$$

□

5. Find the mean and the standard deviation of the following sample data:

$$0 \quad 2 \quad 3 \quad 5.$$

*Answer.* We perform the calculations in the following table:

| | $x$ | $x - \bar{x}$ | $(x - \bar{x})^2$ |
|---|---|---|---|
| | 0 | $-2.5$ | 6.25 |
| | 2 | $-0.5$ | 0.25 |
| | 3 | 0.5 | 0.25 |
| | 5 | 2.5 | 6.25 |
| $\Sigma$ | 10 | | 13 |

We first calculate the mean:

$$\bar{x} = \frac{\sum x}{n} = \frac{10}{4} = 2.5.$$

We then computed $x - \bar{x}$ and $(x - \bar{x})^2$ for all the values. The variance is

$$s^2 = \frac{\sum (x - \bar{x})^2}{n - 1} = \frac{13}{3} \approx 4.33.$$

So, the standard deviation is

$$s = \sqrt{4.33} \approx 2.08.$$

□

6. The GPAs of all currently registered students at a large university have a bell-shaped distribution with mean 2.7 and standard deviation 0.6. Students with a GPA below 1.5 are placed on academic probation. Approximately what percentage of currently registered students at the university are on academic probation?

   *Answer.* We are given that the data have a bell-shaped distribution, so we will use the Empirical Rule. Since $\mu = 2.7$ and $\sigma = 0.6$ we have

   $$x = 1.5 \implies z = \frac{1.5 - 2.7}{0.6} = -2.$$

   Thus GPAs below 1.5 correspond to $z$-scores less than $-2$. The Empirical Rule says that approximately 95% of the $z$-scores lie between $-2$ and 2. It follows that approximately 5% of the data have $z$-scores outside that range, and by symmetry we have that approximately 2.5% of the data have $z$-score less than $-2$.

   Therefore, approximately 2.5% of the students have GPA below 1.5. □

7. Alice and Bob took the same Statistics course in different semesters with different final exams. Alice's score on the final exam was 85, and on that exam the mean was 80 and the standard deviation 5. Bob's score on the final exam that he took was 84, and on that exam the mean was 74 and the standard deviation 8.

   (a) Who did relatively better, Alice or Bob?

   *Answer.* We compare the $z$-scores of their grades. The $z$-score of Alice's grade is

   $$x = 85 \implies z = \frac{85 - 80}{5} = \frac{5}{5} = 1.$$

   The $z$-score of Bob's grade is

   $$x = 84 \implies z = \frac{84 - 74}{8} = \frac{10}{8} = 1.25.$$

   Thus Bob did relatively better. □

   (b) What score in Alice's version of the exam, does Bob's score correspond to?

   *Answer.* The $z$-score of Bob's grade is 1.25. In the exam that Alice took that $z$-score corresponds to the grade

   $$z = 1.25 \implies x = 80 + 1.25 \cdot 5 = 80 + 6.25 = 86.25.$$

   □

8. An animal shelter has a 65% adoption rate for kittens. Of all kittens in the shelter, 80% live to be 8 years or older. Of the kittens who are adopted, 90% live to be 8 years or older.

   (a) What is the probability that a randomly selected kitten in the shelter will get adopted **and** live 8 or more years?

   (b) What is the probability that a randomly selected kitten in the shelter will get adopted **or** live 8 or more years?

   *Answer.* Let $A$ be the event that a randomly selected kitten will get adopted, and $L$ the event that a randomly selected kitten will live to be 8 years or older. We are given that

   $$P(A) = 0.65, \quad P(L) = 0.80, \quad P(L \text{ } \underline{\textbf{given}} \text{ } A) = 0.90.$$

(a) We have

$$P(A \ \underline{\textbf{and}} \ L) = P(A) \cdot P(L \ \underline{\textbf{given}} \ A) = 0.65 \cdot 0.90 = 0.585.$$

(b) We have

$$P(A \ \underline{\textbf{or}} \ L) = P(A) + P(L) - P(A \ \underline{\textbf{and}} \ L) = 0.65 + 0.80 - 0.585 = 0.865.$$

□