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KAUFFMAN STATE SUMS AND BRACKET DEFORMATION

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ABSTRACT

We derive a formula expanding the bracket with respect to a natural deformation parameter. The expansion is in terms of a two-variable polynomial algebra of diagram resolutions generated by basic operations involving the Goldman bracket. A functorial characterization of this algebra is given. Differentiability properties of the star product underlying the Kauffman bracket are discussed.

Keywords: Kauffman bracket; state sum; deformation quantization; mapping class group; Goldman bracket; string topology.

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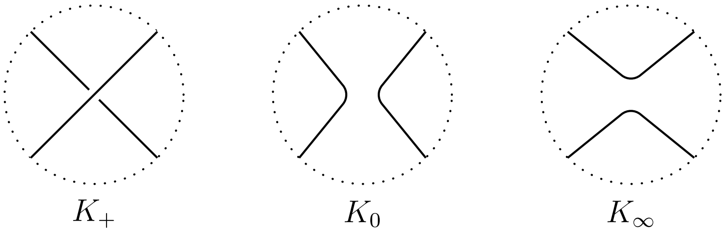
1. Introduction

Throughout let F be a compact connected oriented surface, $\amalg$ means disjoint union.

Let $\mathfrak{C}(F)$ be the set of isotopy classes of closed 1-dimensional submanifolds of the interior of F without inessential components. These are called *curve systems* on F. There is the empty curve system $\emptyset \in \mathfrak{C}(F)$.

Let $\mathfrak{L}(F)$ denote the set of isotopy classes of framed unoriented links in $F \times I$, including the empty link $\emptyset$. The set $\mathfrak{L}(F)$ is identified with the set of isotopy classes of diagrams $\mathfrak{D}(F)$ on F up to Reidemeister moves of type II and III (diagram will always mean regular diagram). The identification is given by regular projection and blackboard framing.

Let k be a commutative ring with 1. It is a result of Przytycki [13] that $\mathfrak{C}(F)$ is a module basis of the Kauffman bracket skein module $\mathfrak{K}(F)$ of F. By definition, $\mathfrak{K}(F)$ is the quotient of the free module $\mathsf{k}[t, t^{-1}]\mathfrak{L}(F)$ by the submodule generated by the elements $K_+ + tK_0 + t^{-1}K_\infty$ (resolution) and $K \amalg U + (t^2 + t^{-2})K$ (trivial component).



The over-under crossing information of K_+ and the orientation of F determine the resolutions K_0 and K_∞ in the usual way. U denotes a component whose projection is an embedded circle on F, which bounds a disk in the complement of the projection of K_+.

Let $[K]$ (respectively $[D]$) denote the image of a framed link (respectively diagram) in the Kauffman bracket module. The isomorphism

$$\mathfrak{K}(F) \to \mathsf{k}[t, t^{-1}]\mathfrak{C}(F)$$

is established using the Kauffman bracket state sum of diagrams

$$\langle D \rangle = \sum_{\sigma} (-t)^{\zeta(\sigma)-\iota(\sigma)}(-t^2 - t^{-2})^{\mu(\sigma)} D(\sigma) \in \mathsf{k}[t, t^{-1}]\mathfrak{C}(F)$$

for each diagram D on F. The sum is over all Kauffman states σ of D, i.e. assignments of state markers $0, \infty$ to each crossing of the diagram. The functions ζ (respectively ι) assign to each state its number of 0- (respectively ∞-) markers. Recall that the assignment of a state marker to a crossing defines a resolution of that crossing. Then μ assigns to the state σ the number of inessential circles in the resolution determined by σ. $D(\sigma)$ is the collection of essential components which appear in the resolution determined by σ. Note that $(\zeta + \iota)(\sigma)$ is equal to the number c of crossings of D for all σ.

It is easy to see that $\langle D \rangle$ only depends on $[D]$. Using the inclusion $\mathfrak{C}(F) \subset \mathfrak{D}(F)$, it follows that $[D] \mapsto \langle D \rangle$ defines a module isomorphism.

Recall that the module $\mathfrak{K}(F)$ actually is a $\mathsf{k}[t, t^{-1}]$-algebra with multiplication $\star$ defined by stacking links. For two framed links K, K' in $F \times I$, we let $[K] \star [K']$ be the element of $\mathfrak{K}(F)$ represented by placing $K \subset F \times [1, 2]$ and $K' \subset F \times [0, 1]$, thus $K \amalg K' \subset F \times [0, 2]$ (and $[0, 2] \cong [0, 1]$ in a natural way). For two diagrams D, D' on F, we let $D \triangleright D'$ denote *some* diagram on F defined by having only crossings of D over D'. Note that $D \triangleright D'$ is *not* a well-defined diagram but $\langle D \triangleright D' \rangle \in \mathsf{k}[t, t^{-1}]\mathfrak{C}(F)$ is well-defined because of the isotopy invariance of the Kauffman bracket. For $\alpha, \beta \in \mathfrak{C}(F)$ let $\alpha \star \beta$ denote the result of multiplication in $\mathfrak{K}(F)$, and expanding using the Kauffman bracket. Thus $\alpha \star \beta \in \mathsf{k}[t, t^{-1}]\mathfrak{C}(F)$. Note that the $\star$-product is noncommutative except for F a disk, annulus or 2-sphere.

It is a difficult problem to relate the expansions $\langle D \rangle$ respectively $\alpha \star \beta$ with the geometry of the diagram respectively curves. For general diagrams this is a nontrivial question even in the commutative case. The $\star$-product of two curve systems on

F is trivially known in the commutative case but its computation is difficult in the non-commutative case. A complete answer indicating the relation of this problem with non-commutative geometry has been given for $F = S^1 \times S^1$ by Frohman and Gelca [6].

It is our goal to study the combinatorics of the *deformation theory* of the Kauffman bracket and the $\star$-product. Assume for the moment that k is a field of characteristic 0 (not necessarily algebraically closed). Then there is an embedding $\mathsf{k}[t, t^{-1}] \to \mathsf{k}[[h]]$ defined by mapping t to e^h. Using the inclusions

$$\mathsf{k}[t, t^{-1}]\mathfrak{C}(F) \to \mathsf{k}[[h]]\mathfrak{C}(F) \subset \mathsf{k}\mathfrak{C}(F)[[h]],$$

we can map $\langle D \rangle$ into $\mathsf{k}\mathfrak{C}(F)[[h]]$. (The second inclusion is proper because $\mathfrak{C}(F)$ is an infinite set.) The image of the bracket in $\mathsf{k}\mathfrak{C}(F)[[h]]$ is still denoted $\langle\ \rangle$ and we can write

$$\langle D \rangle = \sum_{j=0}^{\infty} \langle D \rangle_j h^j$$

with $\langle D \rangle_j \in \mathsf{k}\mathfrak{C}(F)$. In the case of $\alpha, \beta \in \mathfrak{C}(F)$ this defines

$$\alpha \star \beta = \sum_{j=0}^{\infty} \lambda_j(\alpha, \beta) h^j,$$

where for $j \geq 0$ the λ_j extend to k-bilinear mappings

$$\lambda_j\colon\ \mathsf{k}\mathfrak{C}(F) \otimes \mathsf{k}\mathfrak{C}(F) \to \mathsf{k}\mathfrak{C}(F).$$

Note that this sequence determines the $\star$-product.

For a given diagram D the contribution $\langle D \rangle_0$ of the state sum can be calculated by applying the skein relations for $t = 1$, thus $K_+ + K_0 + K_\infty = 0$ and $U + 2 = 0$ to the diagram. It has been shown by Bullock, Frohman and Kania-Bartoszynska [5] that

$$\langle D \rangle_1 = \sum_{\text{crossings } p} \langle D_{p,\infty} \rangle_0 - \langle D_{p,0} \rangle_0,$$

where $D_{p,0}$ (respectively $D_{p,\infty}$) are the diagrams resulting from the 0- (respectively ∞-) resolution of the crossing p. In fact in [5], the formula is only given for the first order contribution $\lambda_1(\alpha, \beta)$ of the $\star$-product of two simple closed curves. But it is easy to see that their combinatorial argument immediately applies to all diagrams.

The interest in the two results above comes from its relation with the representation theory of the fundamental group $\pi_1(F)$ of the surface F. In fact let $\mathrm{Rep}(F)$ denote the universal $SL(2, \mathsf{k})$-character ring of $\pi_1(F)$. It has been shown by Bullock [4] and Przytycki-Sikora [14] that $\mathrm{Rep}(F)$ is naturally isomorphic with the k-algebra structure on $\mathsf{k}\mathfrak{C}(F)$ defined by λ_0. In the case of an algebraically closed field $\mathsf{k} = \mathsf{K}$, the algebra $\mathsf{K}\mathfrak{C}(F)$ can be identified with the ring of character functions $\mathcal{A}(F)$, i.e. regular functions on the variety of $SL(2, \mathsf{K})$-representations that are defined by evaluations and taking traces. More precisely the isomorphism of K-algebras is given by

$$\mathfrak{C}(F) \ni \alpha \mapsto n_\alpha \in \mathcal{A}(F),$$

where $n_\alpha(\rho) := -\mathrm{tr}(\rho(\alpha))$ for each representation $\rho\colon\ \pi \to SL(2,\mathsf{K})$.

It is the main result of [5] that for $\mathsf{K} = \mathbb{C}$,

$$\lambda_1(\alpha,\beta) = \{n_\alpha, n_\beta\},$$

where $\{\ ,\ \}$ is the Poisson bracket on $\mathcal{A}(F)$ defined from the complex symmetric bilinear form

$$B(x,y) = -\frac{1}{2}\mathrm{tr}(xy)$$

on the Lie algebra $sl(2,\mathbb{C})$ following Goldman [7]. Recall that for *closed* surfaces this Poisson bracket is defined from a complex symplectic structure derived from Poincare duality on F and B (see [8]).

In fact Bullock, Frohman and Kania-Bartoszynska prove that the $\star$-product on $\mathbb{C}\mathfrak{C}(F)[[h]]$ as above defines a deformation of the algebra $\mathcal{A}(F)$ in the sense of deformation quantization. (In [5] it is also shown that $\mathbb{C}\mathfrak{C}(F)[[h]]$ is isomorphic to a completed Kauffman bracket algebra $\hat{\mathfrak{K}}(F)$ defined from $\mathbb{C}\mathfrak{L}[[h]]$ dividing out by the closure of the submodule defined from the skein relations as before using the substitutions $t = e^h$.) We only like to point out that all results extend to the case of an algebraically closed field K.

It is our goal in this paper to prove the following result generalizing the combinatorial first order formula of [5].

Theorem (nontechnical version). *For all $j \geq 0$ and rings k of characteristic 0, the jth order term*

$$\langle D\rangle_j \in \mathsf{k}\mathfrak{C}(F)$$

is a sum of diagram resolutions of order $\leq j$, which are combinatorial generalizations of $\langle\ \rangle_0$ and $\langle\ \rangle_1$. Corresponding statements hold for the pairings λ_j.

Note that for all $j \geq 0$, $\langle\ \rangle_j$ is invariant under Reidemeister moves of type II and III. In Sec. 5, we will actually prove a relative version of Theorem 1 for diagrams which possibly contain proper arcs.

The result may seem surprising at first. But if one thinks about the identification of curve systems with regular functions, and observes that resolutions formally behave like derivatives, it could be expected, because regular functions are restrictions of polynomials.

The non-technical statement above will have to be refined in the following. In particular we will define the notion of *order*. Roughly a diagram resolution of order j is defined by state summations over j-element subsets of the the set of crossings with state-contributions depending only on the number of ∞-states of the state, followed by state-summation over the remaining crossings with contributions determined by the number of trivial components weighted with coefficients in k (obtained by expanding the unknot contribution in powers of h).

In Sec. 6, we will apply the theorem above to study differentiability properties of the deformation. It turns out that the loop correction terms in the bracket imply that the bilinear maps λ_j are *not* differential operators of order $\leq j$ in the usual sense. We will discuss a combinatorial version of differentiability in Sec. 6.

Problem. Interpret the higher order terms λ_j for $j \geq 2$ in terms of the geometry of the character variety.

Remark. In [2] it is shown that the Poisson bracket on $\mathcal{A}(F)$ is inherited from a Poisson algebra of chord diagrams. Moreover for surfaces with nonempty boundary the bracket deformation is inherited from the Kontsevitch integral as constructed in [2]. Interestingly in this case the representation variety has no global symplectic structure. Thus it seems particularly interesting to get a better understanding of the situation for closed surfaces.

In Sec. 2, we will discuss invariance properties of the bracket deformation, in particular invariance under the mapping class group. This is to motivate our approach to the diagram resolution algebra in Sec. 3, where we first give an *intrinsic* description from functorial properties. In Sec. 4, we will prove existence and identify this algebra with a polynomial algebra in two variables. In Sec. 5, we prove the technical version of the theorem. In Sec. 6, we discuss the question of differentiability of the star product defined by the Kauffman bracket.

2. Invariance Properties of the Bracket Deformation

The mapping class group $\mathcal{M}(F) = \pi_0(\mathrm{Diff}^+(F))$ of the surface F acts on the set of curve systems $\mathfrak{C}(F)$ in a natural way. This action obviously extends to the Kauffman bracket algebra and is compatible with the multiplication, that is

$$g(\alpha \star \beta) = (g\alpha) \star (g\beta)$$

for $\alpha, \beta \in \mathfrak{C}(F)$ and $g \in \mathcal{M}(F)$. Note that in the calculation of bracket of a diagram D on F, we have

$$\langle gD \rangle = g\langle D \rangle = \sum_{\sigma} (-t)^{(\zeta-\iota)(\sigma)} (-t^2 - t^{-2})^{\mu(\sigma)} (g\delta(\sigma))$$

reducing the action on diagrams to the action on curve systems. Note that $\mathcal{M}$ acts trivially on inessential components. The observation above means that the deformation of the commutative product on $\mathcal{A}(F)$ defined by the bracket is invariant under the action of the mapping class group. It has been observed by Goldman that the symplectic structure and thus the Poisson bracket are invariant under the action of $\mathcal{M}(F)$.

The ring $\mathsf{k}[t, t^{-1}]$ has the natural involution defined by $t \mapsto t^{-1}$. This defines an anti-involution of the module $\mathsf{k}[t, t^{-1}]\mathfrak{C}(F)$. Let $\tau\colon \mathfrak{K}(F) \to \mathfrak{K}(F)$ be the anti-involution defined by changing all crossings of diagrams. This can be interpreted as the action of the element of $\pi_0(\mathrm{Diff}(F \times I))$, which is defined by the reflection $I \times I, t \mapsto (1-t)$. It follows immediately from the definition of the bracket that τ corresponds to the ring involution under the bracket isomorphism, i.e.

$$\langle \tau(D) \rangle = \tau \langle D \rangle$$

for all diagrams D. In particular for any two $x, y \in \mathfrak{K}(F)$, we have the relation

$$\tau(x \star y) = \tau(y) \star \tau(x).$$

For elements in $\mathfrak{C}(F)$ this simplifies to

$$\tau(\alpha \star \beta) = \beta \star \alpha$$

because $\mathfrak{C}(F)$ is invariant under the action of τ. This is a *hermitian* property of the deformation defined by the bracket (see [16]).

Note that $g\tau = \tau g$ thus we have a naturally action of $\mathcal{M}(F) \times \mathbb{Z}_2 \subset \pi_0(\mathrm{Diff}(F \times I))$ on $\mathfrak{K}(F)$.

The ring homomorphism $\mathsf{k}[t, t^{-1}] \to \mathsf{k}[[h]]$ defined for rings k of characteristic 0 (thus $k \supset \mathbb{Q}$ and $e^h = \sum_{j=0}^{\infty}(h^j/j!)$ is defined) is equivariant with respect to the involution on $\mathsf{k}[t, t^{-1}]$ and the involution $h \mapsto -h$ on $\mathsf{k}[[h]]$. Note also that for each j, $\langle\ \rangle_j$ is invariant with respect to $\mathcal{M}(F)$ meaning that $\langle gD \rangle_j = g\langle D \rangle_j$, where the action of $\mathcal{M}(F)$ on the right-hand side is the classical action of $\mathcal{M}(F)$ on curve systems.

Remark. It follows immediately from the state sum definition that we have the symmetries

$$\langle \tau(D) \rangle_k = (-1)^k \langle D \rangle$$

for all $k \geq 0$.

3. The Kauffman Resolution Algebra

For each finite set S let $|S|$ be the number of elements of S.

We will consider connected compact oriented surfaces F with $r \geq 0$ boundary components equipped with a fixed oriented diffeomorphism (parametrization)

$$\bigcup_r S^1 \to \partial F.$$

This is briefly called a *surface*. The image of the ith S^1 is denoted $\partial_i F$. The mapping class group $\mathcal{M}(F)$ is the group of isotopy classes of diffeomorphisms of F fixing ∂F pointwise.

A diagram on F is a pair (D, C) with D a diagram of a regularly immersed proper 1-manifold in F with the usual under-over crossing information at each crossing, $C \subset F$ is a subset of the set of crossings of D. If the ith boundary component $\partial_i F$ contains j_i boundary points of D, then we assume that

$$D \cap \partial_i F = \bigcup_{\ell=1}^{j_i} \left\{ e^{\frac{2\pi\sqrt{-1}\ell}{j_i}} \right\}.$$

Let (D, C) be isotopic to (D', C') if there is an isotopy of F fixing ∂F pointwise and mapping D to D' and C to C'. The set of isotopy classes of diagrams is denoted $\mathcal{D}(F)$ and we will denote the isotopy class of a pair by $[D, C]$.

The set $\mathcal{D}(F)$ naturally decomposes according to $|C|$ and by the number of boundary points contained in each of the r components. Thus

$$\mathcal{D}(F) = \bigcup_{k\geq 0, j \geq 0} \mathcal{D}(F)[k]\{j\},$$

where k is the number elements in C and $j = (j_1, \ldots, j_r)$ is a multi-index with j_i the number of boundary points of D in $\partial_i F$. Thus $(1/2)(j_1 + \cdots + j_r)$ is the number of arc components of the diagram D. As usual $j \geq 0$ means $j_i \geq 0$ for all i. There is also a unique isotopy class of diagram $\emptyset \in \mathcal{D}(F)[0]\{0\}$ with $0 = (0, \ldots, 0)$.

For $k \geq 0$ let $\mathcal{D}(F)[k] := \cup_j \mathcal{D}(F)[k]\{j\}$. Similar notation applies to the other grading. Then $\mathfrak{D}(F)$ from Sec. 1 can naturally be identified with the subset of those $(D, C) \in \mathcal{D}(F)[k]\{0\}$ with $k = |C|$. This set is naturally contained in the subset $\mathfrak{D}^a(F)$ of all diagrams $[D, C]$ on F for which C is the set of *all crossings*. We call the elements of $\mathfrak{D}^a(F)$ *real diagrams*. (Representatives of elements of $\mathfrak{D}^a(F)$ possibly contain arc components.)

The above decompositions naturally define the structure of a bi-graded (with the second grading a multi-grading itself) on the free k-module with basis $\mathcal{D}(F)$. Also there is defined the graded submodule $\mathsf{k}\mathfrak{D}(F) \subset \mathsf{k}\mathfrak{D}^a(F)$ spanned by isotopy classes of k-crossing diagrams in degree $k \geq 0$.

Definition and Remarks. We define the *Kauffman bracket module* $\mathfrak{K}^a(F)$ by quotiening the free module $\mathsf{k}[t, t^{-1}]\mathfrak{D}^a(F)$ by the usual skein relations. A k-module basis is given by the set $\mathfrak{C}^a(F)$ of isotopy classes of curve systems on F consisting of arbitrary properly embedded 1-manifolds with specified boundary conditions as described above, but without inessential closed components. This set is by definition $\mathfrak{D}^a(F)[0] \subset \mathcal{D}(F)[0]$. The right-hand side here contains the elements with $C = \emptyset$ but the diagrams still can have crossings while the left-hand side only contains the real diagrams of this set. (Note that boundary parallel components may be contained in the curve systems. Thus our notion of curve system is still different from the classical approach.) We have decompositions:

$$\mathfrak{C}^a(F) = \bigcup_j \mathfrak{C}^a(F)\{j\}$$

by specifying the boundary pattern. Also note that $\mathfrak{C}(F) = \mathfrak{C}^a(F)\{0\}$ gives the set of curve systems only containing closed components considered in Sec. 1.

The Kauffman bracket module now also decomposes according to the grading. Our version of Kauffman bracket module is a generalization of the relative Kauffman bracket module [13]. We like to mention that the weak product defined above actually lifts to define a graded product structure on the Kauffman bracket module:

$$\mathfrak{K}^a(F)\{j\} \otimes \mathfrak{K}^a(F)\{j'\} \to \mathfrak{K}^a(F)\{j+j'\}$$

by placing a diagram D above a diagram D'. Because of isotopy invariance this is a well-defined product even in this relative case.

Question. Find the $SL(2,\mathsf{k})$-interpretation of the relative Kauffman bracket algebra?

The set-up described above suggests to assign to a curve system in $\mathfrak{C}^a(F)\{j\}$ with $\partial F \neq \emptyset$ a *regular* map on the space $Flat(F)$ of flat connection on a trivialized $SL(2,\mathsf{k})$-bundle over F with values in $\mathsf{k} \times SL(2,\mathsf{k})^{|j|}$ and $|j| := (1/2)(j_1 + \cdots + j_r)$ the number of arcs. The map is defined using the explicit boundary parametrizations, basepoint and orientation on each S^1 to give an ordering of the set of arc components of a diagram. In fact, given a flat connection, each arc components associates the holonomy along the arc. This defines the mapping to $SL(2,\mathsf{k})^{|j|}$. The unordered collection of closed components defines a product of functions given by calculating the traces of holonomies. The bracket relations induces an equivalence relation on this set of maps. Details will be discussed in [10].

The mapping class group acts on diagrams and on the free k-module spanned by diagrams preserving all gradings. Let ϕ_* be the k-endomorphism of degree $(0,0)$ induced by $\phi \in \mathcal{M}(F)$.

Let F (respectively F') be surfaces with $r \geq 1$ (respectively $r' \geq 1$) boundary components. Then we can define a new surface $F \cup F'$ by glueing the last boundary circles. The boundary parametrizations can be combined in some obvious way. Note that all isotopies and diffeomorphisms fix the boundary and can therefore be matched. We call $F \cup F'$ the *glueing of* F *and* F'. Note that $F \cup F'$ has $r + r' - 2$ boundary components. The glueing operation obviously induces glueing operations of diagrams in the following way:

$$\begin{aligned} &\mathcal{D}(F)[k]\{(j_1,\ldots,j_{r-1},\ell)\} \times \mathcal{D}(F_2)[k']\{(j'_1,\ldots,j'_{r-1},\ell)\} \\ &\qquad \to \mathcal{D}[k+k'](F \cup F')\{(j_1,\ldots,j_{r-1},j'_1,\ldots,j'_{r-1}\}. \end{aligned}$$

We let $(D,C) \cup (D',C')$ denote the result of glueing two diagrams. This is only defined when the number of boundary points in the last boundary components match. This operation is compatible with isotopy and defines $[D,C] \cup [D',C']$ for given $[D,C] \in \mathcal{D}(F)$, $[D',C'] \in \mathcal{D}(F')$ which are matching. The glueing operation extends linearly

$$\bigoplus_{\ell \geq 0} \mathsf{k}\mathcal{D}(F)\{(j,\ell\} \otimes \mathsf{k}\mathcal{D}(F')\{(j',\ell\} \to \mathsf{k}\mathcal{D}(F \cup F')\{(j,j')\}$$

using obvious multi-index notation.

We say that a diagram (D, C) or its isotopy class $[D, C]$ is a *weak product* of diagrams (D_1, C_1) and (D_2, C_2) on a surface F if D is given by superimposing D_1 and D_2 to form a diagram $D_1 \triangleright D_2$ with all crossings of the form D_1 over D_2. We will have $C = C_1 \cup C_2$ thus ignore all new crossings of D_1 with D_2. In general this requires modifications in the boundary using natural diffeomorphisms of the circle to be able to take the union in the boundary. Thus actually (D_i, C_i) will be modified by isotopy of F in a neighbourhood of the boundary circles. We will not formalize this construction at this point. In general superimposing diagrams is not a well-defined operation on isotopy classes of diagrams. Thus $[D_1, C_1] \triangleright [D_2, C_2]$ will be the notation for *any* weak product resulting from representatives (D_1, C_1) and (D_2, C_2). Similarly there are defined *strong products* by adding all crossings of D_1 with D_2 to the set of crossings C of D.

Now let $\mathcal{E}(F)[i]$ denote the set of k-endomorphisms of $\mathsf{k}\mathcal{D}(F)$ of bi-degree $(-i, 0)$, i.e. $\rho \in \mathcal{E}(F)[i]$ if $\rho(\mathcal{D}(F)[k]\{j\}) \subset \mathcal{D}(F)[k-i]\{j\}$ for all $k \geq 0$ and all $j \geq 0$. (Here we set $\mathcal{D}(F)[k] = 0$ for $k < 0$.) Then define the graded algebra

$$\mathcal{E}(F) := \bigoplus_{i \geq 0} \mathcal{E}(F)[i].$$

This is a subalgebra of the k-algebra of all k-endomorphisms of the k-module $\mathsf{k}\mathcal{D}(F)$.

We will call the assignment of a graded subalgebra

$$F \mapsto \mathcal{R}(F) \subset \mathcal{E}(F)$$

a *Kauffman resolution functor* if it satisfies the following conditions (1)–(6). (This is formally a functor if we consider the category of surfaces as objects and morphisms between surfaces compatible with boundary parametrizations. In fact, morphisms will induce k-homomorphisms of diagram algebras and thus homomorphisms of their graded endomorphism algebras in a natural way.)

(1) Mapping class invariance. Given $\rho \in \mathcal{R}(F)$ and $\phi \in \mathcal{M}(F)$, then

$$\phi_* \circ \rho = \rho \circ \phi_*$$

(2) Glueing. Let surfaces F_1, F_2 be given (both with nonempty boundary) such that the glueing surface $F_1 \cup F_2 = F$ is defined. Then there exists a unique restriction homomorphism of degree 0:

$$\mathcal{R}(F) \ni \rho \mapsto (\rho|F_1) \in \mathcal{R}(F_1)$$

such that for all diagrams $[D, C]$ with $C \subset F_1$ such that D intersects the image of the distinguished circle in F transversely:

$$\rho[D, C] = (\rho|F_1)\,([D \cap F_1, C]) \cup [D \cap F_2, \emptyset].$$

(3) Weak product. For any two diagrams $[D', \emptyset], [D, C] \in \mathcal{D}(F)$ for which weak products are defined, we have:

$$\rho([D', \emptyset] \triangleright [D, C]) = [D', \emptyset] \triangleright \rho[D, C].$$

Here the right-hand side is interpreted as the linear combination of weak products (be aware that this operation is not a well-defined operation on isotopy classes) of some representative $(D', \emptyset)$ with the representatives of the terms in $\rho[D, C]$. The same identities are supposed to hold with the order of D', D switched.

Note that for $\rho \in \mathcal{R}(F)[i]$ and $[D, T] \in \mathcal{D}(F)[i]$ we have $\rho[D, T] \in \mathsf{k}\mathcal{D}(F)[0]$, so we write formally:

$$\rho[D, T] = [\overline{\rho}[D, T], \emptyset],$$

because the crossing information is empty for all terms. In fact, in general let D_i be a sequence of diagrams with the same set $C \subset F$ of crossings with respect to a choice of representative diagrams. Then

$$\sum \lambda_i [D_i, C] = \left[\sum \lambda_i D_i, C\right]$$

has a well-defined meaning.

(4) Generalized divergence. This property results from the idea that pairs (D, C) can be interpreted formally with D a function and C a set of variables of the function. Let F be a surface and $[D, C] \in \mathcal{D}(F)$, $\rho \in \mathcal{R}(F)[i]$. Then

$$\rho[D, C] = \sum_{T \subset C, |T| = i} [\overline{\rho}[D, T], C \backslash T].$$

In order to make sense of the right-hand expression, we need to justify that $C \backslash T$ is *naturally* a subset of all the diagrams in $\rho[D, T] \in \mathsf{k}\mathcal{D}(F)$. This is the essential technical step and follows from the glueing axiom (2) above applied to a splitting of F along a curve separating the crossings in T from the crossings in $C \backslash T$. Here we split the diagram into two diagrams $D_1 \subset F_1$ with all the crossings of T in D_1, and $D_2 \subset F_2$ containing the crossings of $C \backslash T$. Then the glueing axiom implies that complete diagram D_2 is glued back to the terms in $(\rho|F_1)[D_1, T]$ thus the result naturally contains $C \backslash T$.

(5) Skein relation. For each surface F, there exist two module epimomorphisms of degree -1:

$$\tilde{\zeta}, \tilde{\imath} \colon \mathcal{R}(F) \to \mathcal{R}(F)$$

satisfying

$$\tilde{\zeta}\tilde{\imath} = \tilde{\imath}\tilde{\zeta},$$

and such that $\tilde{\zeta}(\rho), \tilde{\imath}(\rho)$ are linearly independent for all $\rho \in \mathcal{R}(F)$. The homomorphisms $\tilde{\zeta}, \tilde{\imath}$ have to satisfy that for all $\rho \in \mathcal{R}(F)[i]$ and $[D, C] \in \mathcal{D}(F)[i]$ the following skein relation holds:

$$\rho[D_+, C] + (\tilde{\zeta}(\rho))[D_0, C\backslash\{+\}] + (\tilde{\iota}(\rho))[D_\infty, C\backslash\{+\}] = 0 \in \mathsf{k}\mathcal{D}(F),$$

where $+$ is a crossing of D in C, and D_0, D_∞ are the usual resolutions.

(6) Vacuum condition. For all F and $\rho \in \mathcal{R}[0]$ there exists a constant $\theta \in \mathsf{k}$ such that

$$\rho([\emptyset]) = \theta[\emptyset].$$

Moreover, for each $\theta \in \mathsf{k}$ there exists some $\rho \in \mathcal{R}(F)[0]$ with this property.

After this long technical preparation we can now state the main result of this section.

Theorem 1. *There exists at most one assignment $F \mapsto \mathcal{R}(F) \subset \mathcal{E}(F)$ satisfying properties* (1)–(6) *above. Moreover, under the assumption that the functor exists, the value of $\rho \in \mathcal{R}(F)[i]$ on a diagram $[D, C]$ with $|C| = i$ is determined by a state summation over Kauffman states on C with the coefficients in* k *determined only by the number of ∞-states of a state.*

Proof. It follows from the weak product property for $(D, C) = \emptyset$ that

$$\rho[D', \emptyset] = [D', \emptyset] \cdot \rho[\emptyset].$$

Because of the grading, ρ can possibly be nonzero on $[\emptyset]$ only for $\rho \in \mathcal{R}[0]$. Otherwise $\rho[\emptyset] = 0$ and thus ρ vanishes on $\mathcal{D}(F)[0]$.

Now in general $\rho \in \mathcal{R}(F)[i]$ vanishes on $\mathcal{D}(F)[j]$ for $j < i$ because of the grading. Moreover, the divergence property determines ρ on $[D, C]$ with $|C| = j \rangle i$ from the values on diagrams in $\mathcal{D}(F)[i]$. The result then is proved by induction over i using the skein relation in combination with the glueing property. More precisely it follows from the skein relation that ρ is determined by $\tilde{\zeta}(P)$ and $\tilde{\iota}(P)$. For $i = 1$ it follows that $\tilde{\zeta}(P)$ (respectively $\tilde{\iota}(P)$) acts on $\mathcal{D}(F)[0]$ as multiplication by a constant in k. Thus due to the linear independence the degree 1 resolution is determined by two numbers $a_0, a_1 \in \mathsf{k}$. Of course in this case the coefficients of $\rho[D, C]$ are determined by the number 0 (respectively 1) of ∞-states. For the induction step from $i - 1$ to i, we first note that by induction hypothesis $\tilde{\zeta}(\rho)$ is determined by $i-1$ numbers giving the contribution of a state sum with i ∞-markers in a state summation over D_0. We will prove that $\tilde{\iota}(\rho)$ is determined by just one more coefficient. Now $\tilde{\iota}(\rho)$ is determined also by $i - 1$ numbers, and in fact from $\tilde{\zeta}\tilde{\iota}(\rho) = \tilde{\iota}\tilde{\zeta}(\rho)$ and $\tilde{\iota}\tilde{\iota}(\rho)$. Now the first contribution is already known from $\tilde{\zeta}(\rho)$. We can iterate the application of $\tilde{\iota}$ and $\tilde{\zeta}$ and use induction hypothesis to reduce to $\tilde{\iota}^{\,i}(\rho)$, which is of degree 0 and thus determined by a single coefficient. Since in the applications of $\tilde{\iota}$ we smooth a crossing each time it is obvious that this coefficient is determined by the number of ∞-states. $\square$

It is the main result of the next section that the algebra $\mathcal{R}(F)$ exists for each compact connected oriented surface and is naturally graded isomorphic to the polynomial algebra $\mathsf{k}[z, w]$.

Remarks. (a) It does not seem to be possible to characterize the resolution algebras $\mathcal{R}(F)$ without extending the axiomatic to surfaces with boundaries possibly containing arc components, even if we are finally only interested in closed surfaces. The crucial property is the glueing property which defines the locality of the operations. The glueing axiom is necessary just to formulate the crucial divergence property which *localizes* the operation of ρ of degree i on i-crossing diagrams. Similarly it seems difficult to develop the axiomatic characterization without the flexibility in the grading by crossing numbers.

(b) Suppose we consider the case $j = 0$, i.e. no arcs on a closed surface F. In this case the vacuum condition can be actually deduced from the other properties. Then we know that $\rho[\emptyset]$ is a k-linear combination of elements of $\mathcal{D}(F)[0]\{0\}$. Because F is closed, it follows easily from naturality applied to some Dehn twist ϕ of sufficient high order that only the empty diagram can appear in the linear combination. In fact $\phi_*[\emptyset] = [\emptyset]$ while ϕ can be chosen such that the finite linear combination of nonempty curve systems is *not* fixed under ϕ_*.

The involution τ given by changing crossings obviously extends to an involution of $\mathcal{D}(F)$ of degree $(0, 0)$ by changing the crossings of C but fixing the other crossings.

4. Existence of the Resolution Algebra

We will consider the sequence of brackets for $j \geq 1$, also denoted

$$\langle\ \rangle_j\colon\ \mathsf{k}\mathfrak{D}^a(F) \to \mathsf{k}\mathfrak{C}^a(F),$$

defining the Kauffman bracket as in Sec. 1. But we now work in the more general case of diagrams and skein modules possibly containing proper arcs.

In the following the grading of the polynomial algebra $\mathsf{k}[z, w]$ is given by the total degree. We like to point out that the variables z, w correspond to the state maps ι, ζ and its associated operator versions $\tilde{\iota}, \tilde{\zeta}$ from Sec. 3.

Theorem 2. *For each surface F there is a graded homomorphism of* k*-algebras*

$$\chi : \mathsf{k}[z, w] \to \mathcal{E}(F).$$

If k *is a ring of characteristic* 0 *then* χ *is injective and the image is a Kauffman resolution algebra of* F.

Proof. First let

$$\mathfrak{c}\colon\ \mathsf{k}[z, w] \to \mathsf{k}[z, w]$$

be the algebra homomorphism defined by mapping z to zw and w to w. The image of $\mathfrak{c}$ is the algebra of polynomials in $\mathsf{k}[z, w]$ of the form $\sum p_i(z)w^i$ with $\deg(p_i) \leq i$.

Note that the image of a homogeneous polynomial P of degree k is of the form $p(z)w^k$ with a polynomial p in z of degree $\leq k$. We will define χ_P in terms of the homogeneous components of $\mathfrak{c}(P(z,w))$. Given k we will first define χ_k on polynomials $p(z)$ of degree $\leq k$. Note that $p(z) = a_o + a_1 z + \cdots + a_k z^k$ is determined by the mapping

$$\mathfrak{p}\colon \{0, 1, \ldots, k\} \to \mathsf{k}$$

with $\mathfrak{p}(j) = a_j$.

Now let $(D, C) \in \mathcal{D}(F)$. A k-state on C is a choice of k-element subset T of C and a mapping $\sigma\colon T \to \{0, \infty\}$. We denote the set of all k-states on (D, C) by $\Sigma_k(C)$. Then define $\chi_k\colon \mathsf{k}[z] \to \mathcal{E}(F)$ by

$$\chi_k(p)[D, C] := (-1)^k \sum_{\sigma \in \Sigma_k(C)} \mathfrak{p}(\iota(\sigma))[D(\sigma), C(\sigma)],$$

where $D(\sigma)$ is the diagram which results from D by smoothing the crossings in the domain T of σ as determined by σ, and $C(\sigma) = C \backslash T$ for each state $\sigma\colon T \to \{0, \infty\}$. For each natural number k let $\pi_k\colon \mathsf{k}[z, w] \to \mathsf{k}[z]$ be the map that sends a polynomial $P(z, w)$, considered as an element of $\mathsf{k}[z][w]$, to its kth coefficient. Now for a general polynomial $P(z, w) \in \mathsf{k}[z, w]$ define χ by

$$\chi = \sum_{j=0}^{k} \chi_j \circ \pi_j \circ \mathfrak{c}.$$

Consider two homogeneous polynomials $P, Q \in \mathsf{k}[z, w]$ of degree j (respectively k). Let $\mathfrak{p}$ (respectively $\mathfrak{q}$) be the corresponding function $\{0, 1, \ldots, k\} \to \mathsf{k}$. Let $\mathfrak{r}\colon \{0, 1, \ldots, j + k\} \to \mathsf{k}$ be the function determined by the polynomial PQ. Note that if $\mathfrak{c}(P) = p(z)w^j$ and $\mathfrak{c}(Q) = q(z)w^k$ with $\deg(p(z) \leq j$ and $\deg(q(z)) \leq k$, then

$$\mathfrak{c}(PQ) = \mathfrak{c}(P)\mathfrak{c}(Q) = p(z)q(z)w^{j+k}.$$

Thus

$$\mathfrak{r}(i) = \sum_{i = i_1 + i_2, i_1 \leq j, i_2 \leq k} \mathfrak{p}(i_1)\mathfrak{q}(i_2).$$

It follows from the definition that

$$\chi_P \chi_Q[D, C] = (-1)^{j+k} \sum_{\sigma \in \Sigma_j(C(\tau))} \mathfrak{p}(\iota(\sigma)) \sum_{\tau \in \Sigma_k(C)} \mathfrak{q}(\iota(\tau))[(D(\tau))(\sigma), C((\tau)(\sigma))],$$

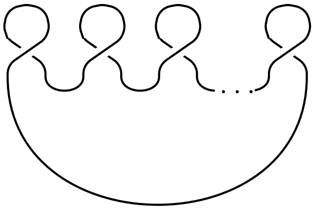
which is equal to the state sum

$$(-1)^{j+k} \sum_{\eta \in \Sigma_{j+k}(C(\tau))} \mathfrak{r}(\iota(\eta))[D(\eta), C(\eta)].$$

It is obvious from the definition that χ_P is compatible with the action of $\mathcal{M}$. It is also clear that the image satisfies (1)–(6) of Sec. 3. It remains to show injectivity for rings k of characteristic 0. We only have to show that $P \neq 0$ implies that $\chi_P \in \mathcal{E}(F)$ is not the trivial endomorphism. Let

$$P = P_i + P_{i+1} + \cdots + P_N$$

be the decomposition of P into homogeneous components with $P_i \neq 0$. Consider the following i-crossing diagram of a circle on a disk in F.



We let (D, C) be this diagram with C the set of all crossings of D. Then $\chi(P_j)$ is trivial on $\mathcal{D}(F)[i]$ for $j > i$ because there are no j-element subsets of the set of crossings. The smoothing according to some i-state with $\sigma^{-1}(\infty) = \ell$ is an $\ell + 1$-component diagram in the disk in F. There are precisely $\binom{i}{\ell}$ states of this form, and all give rise to the same diagram. Thus each nontrivial coefficient a_ℓ in P will contribute a coefficient $\binom{i}{\ell} a_\ell \neq 0$ in $\chi_P[D, C]$, which does not cancel with any other contribution. □

In the following we only consider (D, C) with C the set of all crossings of D. In this case we only write D both for a representative diagram and its isotopy class.

Examples. (a) For a constant polynomial $P = a_0 \in \mathsf{k} \subset \mathsf{k}[z, \zeta]$ the sum is over the single 0-element subset of the set of crossings of D and contributes $a_0 D$ because $|\sigma^{-1}(\infty)| = 0$. Thus χ_{a_0} is just multiplication by a_0. (b) For $k = 1$ and $P = w - z$ thus $\mathfrak{c}(P) = (1 - z)w$, we know that $\mathfrak{p}(0) = 1$ and $\mathfrak{p}(1) = -1$. Thus

$$\chi_P(D) = - \sum_{\text{crossings } p \text{ of } D} (D_{p,0} - D_{p,\infty}),$$

which is just the Poisson bracket defined in Sec. 1.

(c) If D is a diagram with k crossings and P is a homogeneous polynomial of degree $> k$, then $\chi_P(D) = 0$. Thus if diagram resolutions are considered to operate like differential operators on functions, this behavior very much suggests k-crossing diagrams to correspond to polynomial functions of degree k. We say that each element in $\mathcal{R}(F)$ has *finite support* (with respect to the [ ]-grading respectively number of crossings if restricted to real diagrams).

(d) We can apply Theorem 2 to the ring $\mathsf{k}[t, t^{-1}]$ itself. Define for $k \geq 0$ the sequence of polynomials

$$P[k](z, w) = t^k w^k + t^{k-2} w^{k-1} z + \cdots + t^{-k} z^k \in \mathsf{k}[t, t^{-1}][z, w].$$

Then for each diagram D with k crossings

$$\chi_{P[k]}(D) = \langle D \rangle' \in \mathsf{k}[t, t^{-1}]\mathfrak{D}^a(F)[0].$$

Then there is a natural homomorphism

$$\mathsf{k}[t, t^{-1}]\mathfrak{D}^a(F)[0] \to \mathsf{k}[t, t^{-1}]\mathfrak{C}^a(F)$$

defined by mapping the curve system γ to $(-t^2 - t^{-2})^{\mu(\gamma)}\gamma_0$, where μ is the number of trivial components in γ and γ_0 is the result of discarding the trivial components from γ. The definition of $\langle D \rangle$ thus is separated into two steps. Similarly we will separate the calculation of $\langle D \rangle_j$ for all $j \geq 0$. The Kauffman bracket respectively its extension can be considered as an operator with *infinite* support:

$$\mathcal{D}(F) \to \mathcal{D}(F)[0]$$

of the form

$$\hat{P} = \sum_{k=0}^{\infty} \chi_{P[k]} \circ \Pi_k \in \mathcal{E}(F),$$

where

$$\Pi_k \colon \mathcal{D}(F) \to \mathcal{D}(F)[k]$$

is the projection onto the kth grading module. It maps $\mathcal{D}(F)[k] \subset \mathcal{D}(F)[k]$ by the identity and mapping all other $\mathcal{D}(F)[j]$ trivially. Thus $\Pi_k \in \mathcal{E}(F)[0]$ but $\Pi_k \notin \mathcal{R}(F)$. Theorem 2 can be considered as an *operator expansion* of the $\mathsf{k}[t, t^{-1}]$-operator with infinite support in terms of finite support k-operators.

The following result is immediate from the definition of χ and generalizes the skein relation from Sec. 3.

Theorem 3. *For each homogeneous polynomial P and diagram D with usual Kauffman triple D_+, D_0, D_∞, the following $SL(2, \mathbb{C})$-skein relation holds*:

$$\chi_P(D_+ - D_0 - D_\infty) + \chi_{(P - a_k z^k)w^{-1}}(D_0) + \chi_{(P - a_0 w^k)z^{-1}}(D_\infty) = 0.$$

In order to be able to find combinatorial expressions of

$$\mathfrak{D}^a(F) \ni D \to \langle D \rangle_j \in \mathsf{k}\mathfrak{C}^a(F)$$

in terms of our algebra $\mathcal{R}(F)$, we need to define certain projection homomorphisms into $\mathsf{k}\mathfrak{C}^a(F)$.

Let $\varphi\colon\ \mathbb{N} = \{0,1,2,\ldots\} \to \mathsf{k}$ be any map. Define

$$\varphi_*\colon\ \mathsf{k}\mathcal{D}(F) \to \mathsf{k}\mathcal{D}(F)$$

by

$$\varphi_*[D,C] = (-1)^{|C|} \sum_{\text{states } \sigma \text{ on } D} \varphi(\mu(\sigma))[D(\sigma),\emptyset],$$

where as before $|C|$ is the number of crossings of D, $\mu(\sigma)$ is the number of trivial components of the smoothing of D using σ, and $D(\sigma)$ is the diagram resulting from discarding the trivial components from this smoothing.

If applied to $(D,C) \in \mathfrak{D}^a(F)$, then

$$\varphi_*[D,C] \in \mathsf{k}\mathfrak{C}^a(F).$$

For the function $\varphi(i) = (-2)^i$, we have $\phi_*(D) = \langle D\rangle_0$.

Also note that $\varphi_*(P(D)) = \langle D\rangle_1$ if $P = w - z$.

Suppose that k has characteristic 0. Our basic projections are defined from the sequence of maps

$$\varphi_j\colon\ \mathbb{N} \to \mathbb{Q} \subset \mathsf{k}$$

defined by $\varphi_j(i)$ is the coefficient of h^{2j} in the expansion of $(-t^2-t^{-2})^i$ with $t = e^h$. Then

$$\varphi_j(i) = (-1)^i \frac{2^{2j}}{(2j)!} \sum_{k=0}^{i} \binom{i}{k} (2k-i)^{2j}.$$

In particular

$$\varphi_0(i) = (-1)^i \sum_{k=0}^{i} \binom{i}{k} = (-2)^i,$$

and

$$\varphi_1(i) = -(-2)^{i+1} i.$$

Remark. It is important to observe that $\varphi_*[D,C]$ does *not* depend on the over-undercrossing information of the crossings in C for any map φ.

5. Combinatorial Expansion

We are now ready to state the main result of the paper. We assume that k is a ring of characteristic zero and identify $\mathbb{Q} \subset \mathsf{k}$.

Theorem (technical version). *For each $k \geq 0$ there exists a polynomial*

$$P_k \in \mathbb{Q}[z, w]$$

of degree k (but not homogeneous for $k > 1$) such that

$$\langle D \rangle_k = \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} (\varphi_j)_* \circ \chi(P_{k-2j})(D) \in \mathbb{Q}\mathfrak{C}^a(F) \subset \mathsf{k}\mathfrak{C}^a(F)$$

for all $[D] \in \mathfrak{D}^a(F)$. Moreover the homogeneous degree k component of P_k is given by

$$w^k - zw^{k-1} + z^2 w^{k-2} - \cdots + (-1)^k z^k.$$

The terms with $j > 0$ are the loop correction terms. They play an important tole in Sec. 6.

Corollary. *Let α, β be two simple essential loops on F and $\alpha \cdot \beta$ be a diagram of α over β. Then for all $k \geq 0$:*

$$\lambda_k(\alpha, \beta) = \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} (\varphi_j)_* \chi(P_{k-2j})(\alpha \cdot \beta).$$

The corollary also holds more generally for products for α, β possibly proper arcs.

The polynomials P_k are given for small k by

$$\begin{aligned}
P_0 &= 1, \\
P_1 &= w - z, \\
P_2 &= (w^2 - zw + z^2) + \frac{1}{2}(w + z), \\
P_3 &= (w^3 - zw^2 + z^2 w - z^3) + (w^2 - z^2) - \frac{1}{6}(w - z),
\end{aligned}$$

A polynomial $P(z, w)$ is called *symmetric* respectively *anti-symmetric* if $P(w, z) = P(z, w)$ respectively $P(w, z) = -P(z, w)$.

Proposition. *The polynomials P_k are symmetric for k even and anti-symmetric for k odd.*

Proof. This follows from the remark at the end of Sec. 2 together with the obvious fact that if $\bar{P}(z, w) = P(w, z)$ then $\chi_{\bar{P}}(D) = \chi_P(\tau(D))$. □

Remarks. (a) Note that $\mathsf{k}[z, w]$ has a module splitting in symmetric and anti-symmetric polynomials. Of course, as an algebra it is generated by $z - w$ and $z + w$. While the first polynomial corresponds to the Goldman Poisson bracket, the symmetric generator does not define an algebraic structure on $\mathcal{C}(F)$. Note that the commutator $[\alpha, \beta] = \alpha \star \beta - \beta \star \alpha$ expands in terms of only anti-symmetric resolution operations (i.e. coming from anti-symmetric polynomials via χ). But this module is not spanned by $z - w$ alone. The important point is that the symmetry defines an additional $\mathbb{Z}_2$-grading on the algebra. Thus our result shows that essentially the coefficients λ_j can all be deduced from first order operations. But the existence of higher order operations seems to be related to the associativity of the $\star$-product. Compare [3] and also the recent work of Abouzaid on the Fukaya category of higher genus surfaces [1].

(b) The homogeneous component of maximum degree in each order is reminiscent of the natural star products of Gutt and Rawnsley [9].

Proof of the theorem. The main idea of the proof is already contained in [5]. We discuss a state model computation for $(\varphi_j)_* \circ \chi_P$ and P a homogeneous polynomial of degree k. Note that states for the computation here consist of pairs consisting of a state on a k-element subset and a state on the remaining set of crossings. The set of those states maps onto the set of Kauffman states. In fact, many states for $(\varphi_j)_* \circ \chi_P$ will contribute to the same Kauffman state. Recall that a Kauffman state has $\zeta(\sigma)$ 0-states and $\iota(\sigma)$ ∞-states. Recall that the polynomial P is determined by the sequence of coefficients $a_0, \ldots, a_k$ giving the weights associated to states on k-element subsets where a_j is the weight corresponding to a state with j ∞-markers. Now there are $\binom{\iota(\sigma)}{j}\binom{\zeta(\sigma)}{k-j}$ different states, which will all give rise to the same Kauffman state and will be have weight a_j. The idea is to work within a Kauffman state and expand using the functions ζ, ι and μ on states as variables.

In the calculation of $\langle\ \rangle$ the term of order h^k is calculated from the expansions of $e^{h(\zeta-\iota)}$ and the expansion of $(-e^{2h} - e^{-2h})^\mu$ by collecting the terms whose degree adds up to k. We will consider that summand with order k in $e^{h(\zeta-\iota)}$ and order 0 in $(-e^{2h} - e^{-2h})^\mu$. Note that this means that the contribution from the trivial components will give multiplication by 2^μ precisely as in the definition of $(\varphi_0)_*$. Then it is easy to see that the other terms are calculated from the polynomials $(\varphi_j)_* \circ P_{k-2j}$. Note that

$$e^{h(\zeta-\iota)} = \sum_{k=0}^{\infty} \frac{1}{k!}(\zeta - \iota)^k h^k,$$

so in order k we have to calculate

$$\frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}(-1)^j \zeta^{k-j}\iota^j.$$

Consider the jth term in this sum with coefficient

$$c_j = (-1)^j \frac{\iota^j \zeta^{k-j}}{j!(k-j)!}.$$

This has to be compared with the term

$$\binom{\iota}{j}\binom{\zeta}{k-j},$$

which is equal to

$$\frac{1}{j!(k-j)!}\frac{\iota!}{(\iota-j)!}\frac{\zeta!}{(\zeta-(k-j))!}$$

or

$$\frac{1}{j!(k-j)!}\iota(\iota-1)\cdots(\iota-j+1)\zeta(\zeta-1)\cdots(\zeta-(k-j)+1).$$

This is in homogeneous order k in ζ and ι precisely $(-1)^j c_j$. The result now follows by choosing the coefficients of P_k as in the theorem. Then the highest homogeneous terms coincide and we have expanded $\langle\ \rangle_k$ in terms of the degree k-term $P_k^{(k)}$ of P_k as given above and lower order terms. These lower order terms of P_k are necessary to compensate for the additional contributions of $(\varphi_0)_* \circ P_k^{(k)}$. □

The proof of the theorem shows that the explicit calculation of the polynomials while easy in principle, is in fact a tedious exercise in binomial combinatorics. It should be very interesting to have an inductive way of calculation which then could be considered as a combinatorial *Baker–Campell–Hausdorff* expansion, hopefully related with geometric structures on the representation variety, see also [11].

6. Differentiability of the Deformation

Let A be a k-algebra. Let $D \subset \mathrm{End}_{\mathsf{k}}(A)$ be a filtered subalgebra, i.e. a sequence of sub modules

$$D_0 \subset D_1 \subset \cdots \subset D_p \subset \cdots \subset \mathrm{End}_{\mathsf{k}}(A)$$

such that the restriction of the multiplication of A satisfies

$$D_i \cdot D_j \subset D_{i+j}$$

for all non-negative integers i, j. For $i \geq 0$, elements of $D_i \backslash D_{i-1}$ are called D-operators of order i. Let

$$D := \bigcup_{i\geq 0} D_i.$$

Recall that a $\star$-product on a k-algebra A is a $\mathsf{k}[[h]]$-bilinear map

$$A[[h]] \otimes A[[h]] \to A[[h]],$$

thus is determined by

$$A \otimes A \to A[[h]],$$

and thus by a sequence of k-bilinear homomorphisms

$$\lambda_k\colon\ A \otimes A \to A$$

for $k \geq 0$.

Definition. A $\star$-product on A is called *D-differentiable* if for all $k \geq 0$ the restrictions of the corresponding sequence of k-bilinear homomorphisms

$$\lambda_k\colon\ A \otimes A \to A$$

to each variable are D-operators of order $\leq k$.

Note that for $\star$-products with λ_k symmetric or anti-symmetric for each k, it suffices to consider the restriction to the second (or first) variable.

The above definition generalizes the usual definition of differentiability of deformations of algebras using the filtered subalgebra $\mathcal{D} = \mathcal{D}(A)$ of differential operators defined as follows, see [12]: Let $\mathcal{D}(A)_0 := A$ acting by multiplication of A on A and inductively for $p \geq 1$

$$\mathcal{D}(A)_p := \{f \in \mathrm{End}_{\mathsf{k}}(A) | fa - af \in \mathcal{D}(A)_{p-1} \text{ for all } a \in A\}.$$

In our situation we have $A = \mathsf{k}\mathfrak{C}(F)$ equipped with the $\star$-product induced by the Kauffman bracket in $F \times I$. Then following *combinatorial* filtration is naturally defined in this case. We will write φ_i for $(\varphi_i)_*$ to simplify notation. Let D_p be the set of those $f \in \mathrm{End}_{\mathsf{k}}(A)$, that can be written as linear combinations of homomorphisms

$$\beta \mapsto \varphi_{r_\ell}\chi_{Q_\ell}(\alpha_{\ell-1} \triangleright \cdots (\alpha_2 \triangleright \varphi_{r_1}\chi_{Q_1}(\alpha_0 \triangleright \varphi_0\chi_{Q_0}\beta)\cdots)$$

with $2r_0 + q_0 + 2r_1 + q_1 + 2r_2 + q_2 + \cdots + 2r_\ell + q_\ell \leq p$ for some elements $\alpha_i \in \mathfrak{C}(F)$ for $i = 0, \ldots, \ell - 1$ and polynomials $Q_i \in \mathsf{k}[z, w]$ of homogeneous degree q_i for $i = 0, 1, \ldots, \ell$ and $\ell \geq 0$.

This obviously defines a filtered subalgebra of $\mathrm{End}_{\mathsf{k}}(A)$. It is easy to see that D_0 consists of endomorphisms defined by

$$\beta \mapsto a\beta$$

for some $a \in \mathsf{k}\mathfrak{C}(F)$. This follows because χ_{Q_0} is defined by multiplication with a constant in k. Note that $D_0 = \mathcal{D}_0$.

The theorem of Sec. 5 implies:

Theorem. *The $\star$-bracket on $\mathsf{k}\mathfrak{C}(F)$ defined by the Kauffman bracket is D-differentiable with respect to the filtered subalgebra D defined above.*

Note that in this case the restriction of λ_k to the second variable for fixed α is given by

$$\varphi_0\chi_{P_k} + \varphi_1\chi_{P_{k-2}} + \cdots$$

with the polynomials P_j from the theorem in Sec. 5.

Finally we will show that the restriction of λ_2 to one of the variables is *not* a usual differential operator of order ≤ 2. In order to see this recall that

$$\lambda_2(\alpha, \beta) = \varphi_0\chi_{P_2}(\alpha \triangleright \beta) + \varphi_1(\alpha \triangleright \beta),$$

because $P_0 = 1$. Even though *only* the sum of the two terms is a well-defined pairing $\mathsf{k}\mathfrak{C}(F) \otimes \mathsf{k}\mathfrak{C}(F) \to \mathsf{k}\mathfrak{C}(F)$, it can still be checked whether the differentiability formula holds separately for each term. But be aware that the value of each term depends on the choice of diagram $\alpha \triangleright \beta$.

We want to check whether

$$\beta \mapsto \lambda_2(\alpha\beta, \gamma) - \alpha\lambda_2(\beta, \gamma)$$

is an operator of order ≤ 1. Let $P := P_2$. The term $\varphi_0\chi_P$ is a differential operator of order ≤ 2. Consider

$$\Delta := \beta \mapsto \varphi_0\chi_P[\alpha \triangleright \beta \triangleright \gamma, (\alpha \cap \gamma) \cup (\beta \cap \gamma)] - \varphi_0\left(\alpha \cdot \chi_P[\beta \triangleright \gamma, \beta \cap \gamma]\right).$$

Note that χ_P is defined by state summations over pairs of crossings, so the above difference is determined by those states with at least one of the two crossings on α. The application of φ_0 does not change the formula since both α and each term in $\chi_P[\beta \triangleright \gamma, \beta \cap \gamma]$ have no crossings, and

$$\varphi_0(\alpha \triangleright \beta) = \varphi_0(\alpha)\varphi_0(\beta)$$

for all $\alpha, \beta \in \mathfrak{C}(F)$. Note that such a multiplicative property does *not* hold for the higher order projections φ_i with $i > 0$. It follows that for all β'

$$\Delta(\beta\beta') - \beta\Delta\beta'$$

involves only smoothings of pairs of crossings with one crossing in α and one in β and therefore is a multiple of β'. Thus the first term of λ_2 has the differentiability property of a differential operator of order ≤ 2.

We now study the terms derived from the second contribution $\beta \mapsto \varphi_1(\alpha \triangleright \beta)$ thus whether

$$\delta\colon\ \beta \mapsto \varphi_1(\alpha \triangleright \beta \triangleright \gamma) - \alpha\varphi_1(\beta \triangleright \gamma)$$

is an operator of order ≤ 1 for all α, γ. This is the case if

$$\beta' \mapsto \delta(\beta\beta') - \beta\delta(\beta')$$

is an operator of order 0 thus given by multiplying β' by some element of $\mathsf{k}\mathfrak{C}(F)$. If we let $\gamma = \emptyset$ and write the condition explicitly we get

$$\beta' \mapsto \varphi_1(\alpha \triangleright \beta \triangleright \beta') - \alpha\varphi_1(\beta \triangleright \beta') - \beta\varphi_1(\alpha \triangleright \beta') - \beta\alpha\varphi_1(\beta').$$

Thus $\beta' = \emptyset$ maps to some element $\phi_1(\alpha \triangleright \beta) \in \mathsf{k}\mathfrak{C}(F)$.

Now consider the situation of α, β two curves with no crossings but both α and β such that each state smoothing on $\beta \triangleright \beta'$ and $\alpha \triangleright \beta'$ does not involve an inessential component while there exists a smoothing of $\alpha \triangleright \beta \triangleright \beta'$ involving an inessential component.

In this case the differentiability condition is equivalent to

$$\varphi_1(\alpha \triangleright \beta \triangleright \beta') = \alpha\varphi_1(\beta \triangleright \beta') + \beta\varphi_1(\alpha \triangleright \beta').$$

Then $\varphi_1(\beta') = \varphi_1(\alpha \triangleright \beta) = \varphi_1(\beta \triangleright \beta') = \varphi_1(\alpha \triangleright \beta') = 0$ while $\varphi_1(\alpha \triangleright \beta \triangleright \beta') \neq 0$. Therefore the second term does not satisfy the differentiability condition. It follows that λ_2 restricted to one of the variables is not a differential operator of order ≤ 2.

Remark. The arguments above generalize to show that the top term $\varphi_0\chi_{P_k}$ of λ_k satisfies the condition of being a differentiable operator of order $\leq k$ in each variable. Note that this assertion is not precise in this form since we are discussing homomorphisms from the module of diagrams into the algebra $\mathsf{k}\mathfrak{C}(F)$. In fact, the operation $\triangleright$ used above is not well-defined but just a notation for a collection of all diagrams. What we mean that the differentiability formula holds if we calculate $\varphi_k\chi_{P_k}$ on any diagram $\alpha \triangleright \beta$, and multiplication in $\mathsf{k}\mathfrak{C}(F)$ is *lifted* to $\mathsf{k}\mathfrak{D}(F)$ in this way.

It seems a very difficult problem to compare the combinatorial filtered subalgebra D with the filtered subalgebra of differential operators $\mathcal{D}$. But this problem is at the heart of relating the combinatorial deformation with the geometric deformations of the character variety.

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References

[1] M. Abouzaid, On the Fukaya categories of higher genus surfaces, arXiv:math.SG/0606598.
[2] J. Andersen, Mattes and Reshitikin, Quantization of the algebra of chord diagrams, *Math. Proc. Cambridge Phil. Soc.* **124** (1998) 451–467.
[3] M. Chas and D. Sullivan, String topology, preprint CUNY, arXiv:math.GT/9911159, to appear in *Ann. Math.*
[4] D. Bullock, Rings of $SL(2,\mathbb{C})$-characters and the Kauffman bracket skein module, *Comment. Math. Helv.* **72**(4) (1997) 521–542.
[5] D. Bullock, C. Frohman and J. Kania-Bartoszynska, Understanding the Kauffman bracket skein module, *J. Knot Theory Ramifications* **8**(3) (1999) 265–277.
[6] C. Frohman and R. Gelca, Skein modules and the noncommutative torus, *Trans. Amer. Math. Soc.*
[7] B. Goldman, Invariant functions on Lie groups and Hamiltonian flows of surface group representations, *Invent. Math* **85** (1986) 263–302.
[8] B. Goldman, The complex-symplectic geometry of $SL(2,\mathbb{C})$-characters of surfaces in Algebraic groups and arithmetic, Tata Institute of Fundamental Research Mumbai (2004), pp. 375–407.
[9] S. Gutt and C. Rawnsley, Natural star products on symplectic manifolds and quantum moment maps, arXiv:math.SG/0304498.
[10] U. Kaiser, Relative Kauffman bracket algebras, in preparation.
[11] V. Kurlin, Exponential Baker-Campbell-Hausdorff formula and applications to formal vector fields, arXiv:math.QA/0606330.
[12] J. C. McConnell and J. C. Robbson, *Noncommutative Noetherian Rings*, Pure and Applied Mathematics (Wiley Interscience, 1987).

[13] J. Przytycki, Skein modules of 3-manifolds, *Bull. Pol. Acad. Sci.* **39** (1991) 91–100.

[14] J. Przytycki and A. Sikora, On skein algebras and $SL(2,\mathbb{C})$-character varieties, *Topology* **39**(1) (2000) 155–148.

[15] A. Sikora, SL_n-character varieties as spaces of graphs, *Trans. Amer. Math. Soc.* **353** (2001) 2773–2804.

[16] H. Burszyn and S. Waldmann, Deformation quantization of hermitian vector bundles, arXiv:math.QA/000917012.