9/18/25

- One-to-one function, inverse functions

warm up:

Question For each of the following functions determine if they are 1-1 and if so find their inverse.

a) $f(x) = 3x - 5$

b) $g(x) = x^3 - 5$

c) $h(x) = x^2 + 1$

d) $f(x) = \dfrac{x+1}{2x-3}$

e) $g(x) = x^2$ with domain $(-\infty, 0]$

$$\boxed{\begin{array}{l} y = f(x) \\ x = f^{-1}(y) \end{array}}$$

Answer: To check if a fn is 1-1 we try to find the inverse. (solve for x).

a) $y = 3x - 5 \iff \dfrac{y+5}{3} = \dfrac{3x}{3}$

$\iff \boxed{\dfrac{y+5}{3} = x}$ $f^{-1}(y) = \dfrac{y+5}{3}$

$\boxed{f^{-1}(x) = \dfrac{x+5}{3}}$ Yes 1-1

b) $y = x^3 - 5 \iff y + 5 = x^3$ Yes 1-1

$\iff \sqrt[3]{y+5} = x$

$g^{-1}(y) = \sqrt[3]{y+5}$

or $g^{-1}(x) = \sqrt[3]{x+5}$

c) $y = x^2+1 \iff y-1 = x^2$

$\iff \pm\sqrt{y-1} = x$ → NOT UNIQUE SOLUTION

So $\boxed{\text{NOT } 1-1}$

d) $y = \dfrac{x+1}{2x-3} \iff y(2x-3) = x+1$

$\iff 2yx - 3y = x+1$

$\iff 2yx - x = 3y+1$

$\iff \dfrac{x(\cancel{2y-1})}{\cancel{2y-1}} = \dfrac{3y+1}{2y-1}$

$\iff x = \dfrac{3y+1}{2y-1}$

YES 1-1

$h^{-1}(y) = \dfrac{3y+1}{2y-1}$

or $\boxed{h^{-1}(x) = \dfrac{3x+1}{2x-1}}$

e) $y = x^2 \iff -\sqrt{y} = x$

domain $x \le 0$

$(-\infty, 0]$

In general

$y = x^2 \iff x = \pm\sqrt{y}$

one negative, one positive

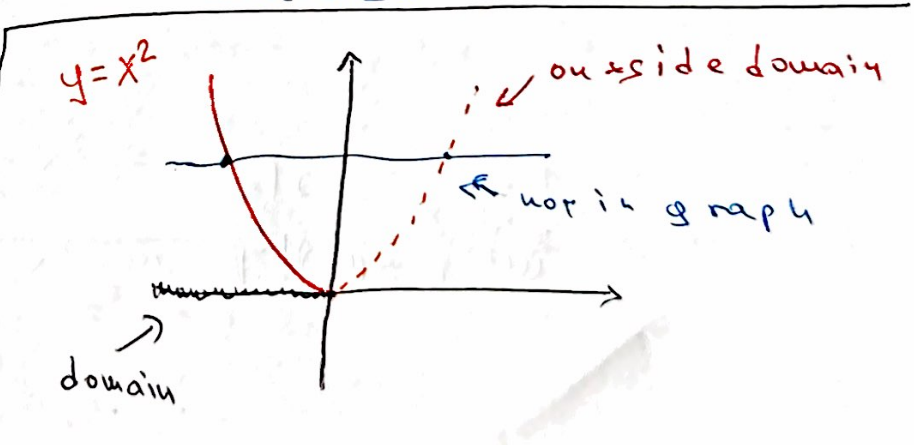
ONLY NEGATIVE x are allowed

YES INVERTIBLE

$g^{-1}(y) = -\sqrt{y}$

OR

$g^{-1}(x) = -\sqrt{x}$



Range of f = Domain of f^{-1}

Domain of f = Range of f^{-1}

Example Let $f(x) = \dfrac{3x-1}{x+2}$ $\to \neq 0$

Find domain and range.

Ans Domain Need $x + 2 \neq 0 \iff \boxed{x \neq -2}$

In interval notation $(-\infty, -2) \cup (-2, \infty)$

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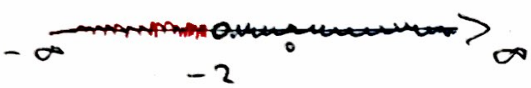


To find range we find f^{-1} and its domain.

$$y = \frac{3x-1}{x+2} \iff y(x+2) = 3x-1$$

$$\iff yx + 2y = 3x - 1$$

$$\iff yx - 3x = -2y - 1$$

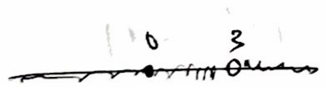
$$\iff x(y-3) = -2y - 1$$

$$\iff x = \frac{-2y-1}{y-3}$$

So $f^{-1}(y) = \dfrac{-2y-1}{y-3}$ Domain of f^{-1}:

Need $y - 3 \neq 0$

$\iff y \neq 3$



$(-\infty, 3) \cup (3, \infty)$

So Range of f is $(-\infty, 3) \cup (3, \infty)$

Composition of functions

f, g

$f \circ g(x) = f(g(x))$

$g \circ f(x) = g(f(x))$

$\boxed{(a \pm b)^2 = a^2 \pm 2ab + b^2}$

Ex. 1 $f(x) = x^2 - 3$, $g(x) = 2x + 1$

$$f \circ g(x) = (2x+1)^2 - 3$$
$$= 4x^2 + 4x + 1 - 3$$
$$= \boxed{4x^2 + 4x - 2}$$

$$g \circ f(x) = 2(x^2 - 3) + 1$$
$$= 2x^2 - 6 + 1$$
$$= \boxed{2x^2 - 5}$$

Composition of f^ns is $\boxed{\text{not commutative}}$ (in general).

Ex 2 $f(x) = 3x$, $g(x) = 5x$.

$$f \circ g(x) = 3 \cdot (5x) = \boxed{15x}$$

$$g \circ f(x) = 5(3x) = \boxed{15x}$$

In this case $f \circ g = g \circ f$. ← This is rare

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Ex 3 Given $f(3) = 5$, $g(5) = 7$, $f(7) = 3$

Find $f \circ g(5) = f(g(5))$
$$= f(7)$$
$$= 3$$

$$g \circ f(3) = g(f(3))$$
$$= g(5)$$
$$= \boxed{7}$$

$5 \to \boxed{g} \to 7 \to \boxed{f} \to 3$

$$\boxed{f \circ f^{-1}(x) = x \quad \text{and} \quad f^{-1} \circ f(x) = x}$$

<u>identity</u> fn $\boxed{f(x) = x}$

<u>Ex.</u> $f(x) = 3x$, $f^{-1}(x) = \dfrac{x}{3}$

~~$f \circ f^{-1}(x)$~~ (

$$f \circ f^{-1}(x) = f\left(f^{-1}(x)\right)$$
$$= \cancel{3}\left(\frac{x}{\cancel{3}}\right)$$
$$= x$$

$$f^{-1} \circ f(x) = \frac{(3x)}{3}$$
$$= x$$

Show
Verify

<u>Ex</u> <u>Prove</u> that $f(x) = 3x - 5$ and $g(x) = \dfrac{x+5}{3}$ is a pair of inverse functions.

<u>Sol</u> Verify two things

$f \circ g(x) = x$ $\qquad$ $g \circ f(x) = x$

$$f \circ g(x) = f\left(g(x)\right)$$
$$= \cancel{3}\left(\frac{x+5}{\cancel{3}}\right) - 5$$
$$= (x + \cancel{5}) - \cancel{5}$$
$$= x \checkmark$$

$$g \circ f(x) = g\left(f(x)\right)$$
$$= \frac{(3x - \cancel{5}) + \cancel{5}}{3}$$
$$= \frac{\cancel{3}x}{\cancel{3}}$$
$$= x \checkmark$$

Done!