October 23

Functions

Ex.
Is the following relation a fn?

a) $y + 3x^2 = 5x$ b) $x^2 + y^2 = 4$

Sol We should be able to solve for y uniquely

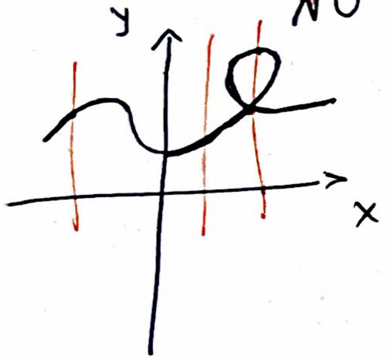
a) $y + 3x^2 = 5x \iff \boxed{y = 5x - 3x^2}$ YES

b) $x^2 + y^2 = 4 \iff y^2 = 4 - x^2 \iff y = \pm\sqrt{4 - x^2}$ NO

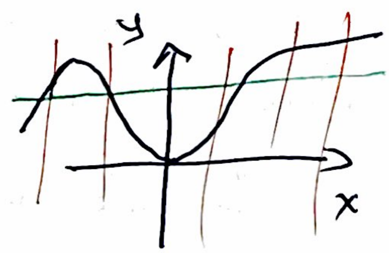
not unique

— o —

NO



YES



Is this (one-one) (invertible) NO

Ex 2 Is the following function 1-1. If yes find its inverse.

a) $f(x) = \dfrac{x}{3 + x}$ b) $f(x) = x^2 - 1$

Sol Solve $y = f(x)$ for x.

a) $y = \frac{x}{3+x} \iff y(3+x) = x$

$\iff 3y + yx = x$

$\iff 3y = x - yx$

$\iff 3y = x(1-y)$

$\iff \boxed{x = \frac{3y}{1-y}}$

Yes 1-1

$f^{-1}(y) = \frac{3y}{1-y}$

$\boxed{f^{-1}(x) = \frac{3x}{1-x}}$

b) $y = x^2 - 1 \iff x^2 = y+1$

$\iff x = \pm\sqrt{y+1} \longrightarrow$ Not invertible

Ex 3 Find domain and range of $f(x) = \frac{x}{3+x}$.

Ans. Domain $3 + x \neq 0 \iff x \neq -3$

$\boxed{(-\infty,-3) \cup (-3,\infty)}$

$-\infty$ ——o—— $+\infty$
-3

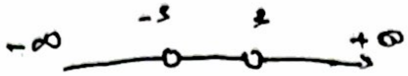
Range of f = Domain of f^{-1}

$f^{-1}(y) = \frac{3y}{1-y}$ Need $1-y \neq 0 \iff y \neq 1$

$\boxed{(-\infty,1) \cup (1,\infty)}$

Ex. Find domain: a) $f(x) = \dfrac{3x+1}{x^2-9}$

$x \neq -3, 3$

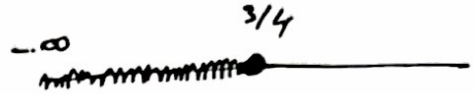


$\boxed{(-\infty, -3) \cup (-3, 3) \cup (3, \infty)}$

b) $g(x) = \sqrt{3-4x}$

Need $3 - 4x \geq 0 \iff 3 \geq 4x$

$\iff \boxed{\dfrac{3}{4} \geq x}$



$\left(-\infty, \dfrac{3}{4}\right]$

Example Prove that $f(x) = \dfrac{x}{3+x}$, $g(x) = \dfrac{3x}{1-x}$

are inverse functions, using compositions.

Ans. we have to show:

$f(g(x)) = x$ $\qquad\qquad g(f(x)) = x$

$$f(g(x)) = \frac{\left(\frac{3x}{1-x}\right)}{3 + \left(\frac{3x}{1-x}\right)}$$

$$= \frac{3x}{3(1-x) + 3x}$$

$$= \frac{3x}{3 - \cancel{3x} + \cancel{3x}}$$

$$= \frac{3x}{3}$$

$$= \boxed{x} \checkmark$$

$$g(f(x)) = \frac{3\left(\frac{x}{3+x}\right)}{1 - \left(\frac{x}{3+x}\right)}$$

$$= \frac{3x}{3 + \cancel{x} - \cancel{x}}$$

$$= \frac{3x}{3}$$

$$= \boxed{x} \checkmark$$

Ex a) Graph $f(x) = x^2 - 7x + \boxed{12}$

Sol

Vertex (h, k)

$h = -\dfrac{b}{2a}$

$k = f(h) = -\dfrac{D}{4a}$

$D = b^2 - 4ac$

x-intercepts

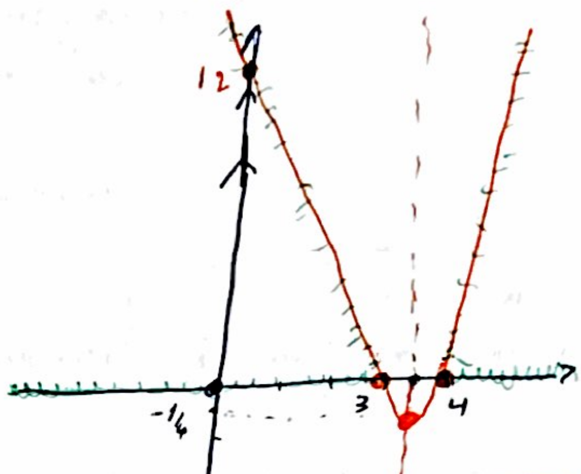
$x = \dfrac{-b \pm \sqrt{D}}{2a}$

$= \dfrac{7 \pm \sqrt{1}}{2} = \begin{cases} \frac{8}{2} = 4 \\ \frac{6}{2} = 3 \end{cases}$

$D = 49 - 4 \cdot 1 \cdot 12$
$= 49 - 48$
$= 1$

$h = \dfrac{7}{2}$

$k = -\dfrac{1}{4}$



$\boxed{x = 7/2}$ axis symmetry

y-intercept

$y = 12$

point symmetric y-intercept $(7, 12)$

b) Solve $x^2 - 7x + 12 > 0$

Ans $\boxed{(-\infty, 3) \cup (4, \infty)}$

Example. a) Sketch a rough graph of

$$f(x) = (2x+1)(x-2)^2(x+2)$$

$1 \quad 4 \quad 2$

$2x \quad x^2 \quad x$

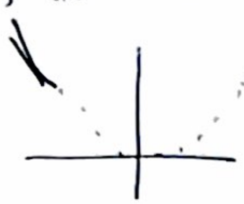
Sol. Leading term

$2x^4$

End behavior

$x \to \pm\infty$

$f(x) \sim 2x^4$

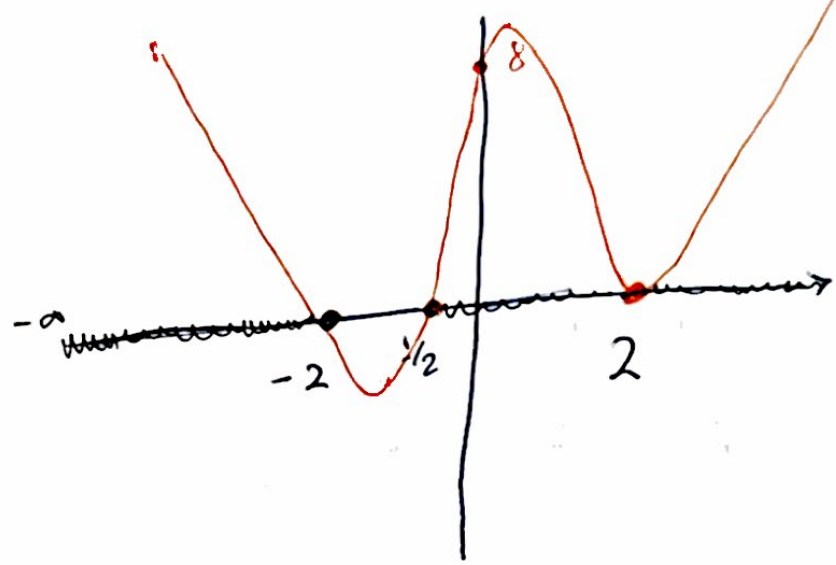


Roots (x-intercepts)

$2x+1=0 \iff x=-1/2$ simple

$x-2=0 \iff x=2$ double

$x+2=0 \iff x=-2$ simple



y-intercept

$y = 1 \cdot 4 \cdot 2$
$= 8$

b) Solve

$(2x+1)(x-2)^2(x+2) \geq 0$

sol $(-\infty,-2] \cup [-1/2,\infty)$

c) Solve $f(x) < 0$

$(-2,-1/2)$

Example $\quad$ Factor $\quad f(x) = x^3 + 2x^2 - x - 2$

Sol. $\quad$ Candidate roots $\quad \pm 1, \pm 2$

Divisors of -2

$x = 1$ root

$x - 1$ factor

$$
\begin{array}{r|rrrr}
1 & 1 & 2 & -1 & -2 \\
  &   & 1 & 3 & 2 \\
\hline
  & \underbrace{1 \quad 3 \quad 2}_{\text{degree } 2} & & & \boxed{0}
\end{array}
$$

So $\quad f(x) = (x-1)(x^2 + 3x + 2)$

$\qquad = (x-1)(x+1)(x+2)$ ✓