MTH 42, Fall 2024

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Fourth Set of homework

The answers

1. Let A be an $m \times k$ matrix and B a $k \times n$ matrix. Prove that

$$\ker B \subseteq \ker A\,B.$$

Solution. We have

$$\begin{aligned} \mathbf{x} \in \ker B &\implies B\,\mathbf{x} = \mathbf{0} \\ &\implies A\,(B\,\mathbf{x}) = \mathbf{0} \\ &\implies (A\,B)\,\mathbf{x} = \mathbf{0} \\ &\implies \mathbf{x} \in \ker\,(A\,B)\,. \end{aligned}$$

Therefore,

$$\ker B \subseteq \ker A\,B.$$

□

2. Let A be a 4×3 matrix and B a 3×4 so that $A\,B$ is a square 4×4 matrix. Prove that $A\,B$ is not invertible.

Solution. We will prove that

$$\ker\,(A\,B) \neq \{\mathbf{0}\}\,,$$

and so $A\,B$ is not injective. To do that we will prove that $\ker B \neq \{\mathbf{0}\}$ and the result follows from Question 1.

By the *Rank-Nullity Theorem* (see Theorem 3.2.4 in the notes) we have

$$\operatorname{rank} B + \operatorname{null} B = 4,$$

which gives

$$\operatorname{null} B = 4 - \operatorname{rank} B. \tag{1}$$

But $\operatorname{rank} B \leq 3$ and therefore Equation (1) gives

$$\operatorname{null} B \geq 1,$$

that is

$$\dim\,(\ker B) \geq 1.$$

It follows that

$$\ker B \neq \{\mathbf{0}\}\,.$$

□

3. Find a basis and the dimension of the solution set of the following homogeneous system:

$$\begin{cases} x_1 - 3x_2 \qquad + \ x_4 + \ x_5 = 0 \\ 2x_1 - 6x_2 + 2x_3 + 4x_4 + 2x_5 = 0 \\ -3x_1 + 4x_2 \qquad\qquad + \ x_5 = 0 \\ \qquad x_2 + \ x_3 + \ x_4 \qquad = 0 \end{cases}$$

Answer. We get the reduced echelon form of the matrix of the system:

$$A = \begin{pmatrix} 1 & -3 & 0 & 1 & 1 \\ 2 & -6 & 2 & 4 & 2 \\ -3 & 4 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 & -1/3 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -4/3 \\ 0 & 0 & 0 & 1 & 4/3 \end{pmatrix}$$

The solution set is therefore

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = t \begin{pmatrix} 1/3 \\ 0 \\ 4/3 \\ -4/3 \\ 1 \end{pmatrix}.$$

Thus the solution set is the linear span $\langle (1/3, 0, 4/3, -4/3, 1) \rangle$, and therefore has dimension 1. □

4. For each of the following two transpose matrices:

$$A = \begin{pmatrix} 1 & 2 & 3 & 1 & 2 \\ 2 & 1 & 2 & 3 & 1 \\ 3 & 3 & 5 & 4 & 3 \\ 1 & -1 & -1 & 2 & -1 \end{pmatrix}, \quad A^* = \begin{pmatrix} 1 & 2 & 3 & 1 \\ 2 & 1 & 3 & -1 \\ 3 & 2 & 5 & -1 \\ 1 & 3 & 4 & 2 \\ 2 & 1 & 3 & -1 \end{pmatrix}.$$

(a) Find a basis for their ranges and state their rank.

Answer. We have

$$A \sim \begin{pmatrix} 1 & 0 & 1/3 & 5/3 & 0 \\ 0 & 1 & 4/3 & -1/3 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus a basis for $\mathcal{R}(A)$ is $\{(1, 2, 3, 1), (2, 1, 3, -1)\}$ and so $\operatorname{rank} A = 2$.
For A^* we have

$$A^* \sim \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus a basis for $\mathcal{R}(A^*)$ is $\{(1, 2, 3, 1, 2), (2, 1, 2, 3, 1)\}$, and so $\operatorname{rank} A^* = 2$. □

(b) Find a basis for their kernels and state their nullity.

Answer. From the reduced echelon forms in Part (a) we have that a basis for the kernel of A is

$$\{(-1/3, -4/3, 1, 0, 0), (-5/3, 1/3, 0, 1, 0), (0, 1, 0, 0, 1)\}.$$

Thus $\operatorname{null} A = 3$.

Similarly, a basis for the kernel of A^* is

$$\{(-1, -1, 1, 0), (1, -1, 0, 1)\}.$$

Thus $\operatorname{null} A^* = 2$. □

5. Find the inverse of each of the following matrices:

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 2 & 4 \\ 1 & 3 & -3 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 1 & 3 & 1 & -2 \\ 1 & 4 & -2 & 4 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 2 & 0 & -2 \\ 0 & 2 & 1 & 4 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -8 \end{pmatrix}.$$

Answer. We have

$$\left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 2 & 2 & 4 & 0 & 1 & 0 \\ 1 & 3 & -3 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 9 & -3/2 & -5 \\ 0 & 1 & 0 & -5 & 1 & 3 \\ 0 & 0 & 1 & -2 & 1/2 & 1 \end{array}\right).$$

Therefore,

$$A^{-1} = \begin{pmatrix} 9 & -3/2 & -5 \\ -5 & 1 & 3 \\ -2 & 1/2 & 1 \end{pmatrix}.$$

Similarly,

$$B^{-1} = \begin{pmatrix} -10 & -20 & 4 & 7 \\ 3 & 6 & -1 & -2 \\ 5 & 8 & -2 & -3 \\ 2 & 3 & -1 & -1 \end{pmatrix}, \quad C^{-1} = \begin{pmatrix} 1 & -1 & 1/3 & -3/4 \\ 0 & 1/2 & -1/6 & 1/4 \\ 0 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & -1/8 \end{pmatrix}.$$

□

6. Let $\mathbf{C}$ be the following set of 2×2 matrices.

$$\mathbf{C} = \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} : a, b \in \mathbb{R} \right\}.$$

Prove the following:

(a) $\mathbf{C}$ is closed under matrix addition. That is,

$$A, B \in \mathbf{C} \implies A + B \in \mathbf{C}.$$

(b) $\mathbf{C}$ is closed under scalar multiplication. That is,

$$\lambda \in \mathbb{R},\ A \in \mathbf{C} \implies \lambda\, A \in \mathbf{C}.$$

In particular,

$$A \in \mathbf{C} \implies -A \in \mathbf{C}.$$

(c) $\mathbf{C}$ is closed under matrix multiplication. That is,

$$A, B \in \mathbf{C} \implies A\,B \in \mathbf{C}.$$

(d) All non-zero elements of $\mathbf{C}$ are invertible, and

$$A \in \mathbf{C},\ A \neq O \implies A^{-1} \in \mathbf{C}.$$

(e) Any two elements of $\mathbf{C}$ commute. That is,

$$A, B \in \mathbf{C} \implies A\,B = B\,A.$$

(f) Let

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Verify that

$$J^2 = -I.$$

(g) what is J^{503}?

(h) Every element of $\mathbf{C}$ can be uniquely expressed as a linear combination of I and J. In other words, if $A \in \mathbf{C}$ then there exist two unique real numbers a, b such that

$$A = a\,I + b\,J.$$

(i) Let $A \in \mathbf{C}$ with $A = a\,I + b\,J$ with at least one of a, b non-zero. Express A^{-1} as a linear combination of I and J.

(j) Prove that every non-zero element of $\mathbf{C}$ has exactly two *square roots*. That is, prove that if $A \neq O$ is an element of $\mathbf{C}$ then there are exactly two elements $B \in \mathbf{C}$ such that $B^2 = A$.

(k) Prove the that every quadratic equation

$$A\,X^2 + B\,X + C = O$$

where $A, B, C \in \mathbf{C}$ and $A \neq 0$, has two solutions (that may coincide) in $\mathbf{C}$.

Answer. Parts (a) through (e) are straightforward calculations, very similar to Example 52 in the notes. Let

$$A_1 = \begin{pmatrix} a_1 & -b_1 \\ b_1 & a_1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} a_2 & -b_2 \\ b_2 & a_2 \end{pmatrix},$$

be two elements of $\mathbf{C}$ and $\lambda \in \mathbb{R}$. We have

(a)

$$A_1 + A_2 = \begin{pmatrix} a_1 + a_2 & -b_1 - b_2 \\ b_1 + b_2 & a_1 + a_2 \end{pmatrix} = \begin{pmatrix} a_1 + a_2 & -(b_1 + b_2) \\ b_1 + b_2 & a_1 + a_2 \end{pmatrix} \in \mathbb{C}.$$

(b)

$$\lambda\,A_1 = \begin{pmatrix} \lambda\,a_1 & \lambda\,(-b_1) \\ \lambda\,b_1 & \lambda\,a_1 \end{pmatrix} = \begin{pmatrix} \lambda\,a_1 & -(\lambda\,b_1) \\ \lambda\,b_1 & \lambda\,a_1 \end{pmatrix} \in \mathbf{C}.$$

(c)

$$A_1\,A_2 = \begin{pmatrix} a_1\,a_2 + b_1\,b_2 & -\left(a_1\,b_2 + a_2\,b_1\right) \\ a_1\,b_2 + a_2\,b_1 & a_1\,a_2 + b_1\,b_2 \end{pmatrix} \in \mathbf{C}.$$

(d) Using the result of Example 43, we need to prove that if $A_1 \in \mathbf{C}$ is non-zero then the determinant of A is non-zero. But the determinant of A_1 is

$$a_1\,a_1 - (-b_1)\,b_1 = a_1^2 + b_1^2.$$

But if $A_1 \neq O$ then at least one of a_1, b_1 is non-zero and therefore the determinant $a_1^2 + b_1^2 \neq 0$ and thus A_1 is invertible.

Then using Equation (3.9) we have

$$A_1^{-1} = \frac{1}{a_1^2 + b_1^2}\begin{pmatrix} a_1 & b_1 \\ -b_1 & a_1 \end{pmatrix}.$$

Now

$$\begin{pmatrix} a_1 & b_1 \\ -b_1 & a_1 \end{pmatrix} = \begin{pmatrix} a_1 & -(-b_1) \\ -b_1 & a_1 \end{pmatrix} \in \mathbf{C}$$

and therefore by Part (b) it follows that A^{-1} in $\mathbf{C}$.

(e) This is a straightforward calculation.

(f) Again this is a straightforward calculation.

(g) Since $J^2 = -I$ we have $J^3 = -J$ and $J^4 = I$. It follows then that for $k, \ell \in \mathbb{Z}$ we have

$$J^{4\,k+\ell} = \left(J^4\right)^k\,J^\ell = I^k\,J^\ell = J^\ell.$$

Now $503 = 4 \cdot 125 + 3$ and therefore

$$J^{503} = J^3 = -J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

(h) We have

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}. = a\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = a\,I + b\,J.$$

This representation is unique because

$$a_1\,I + b_1\,J = a_2\,I + b_2\,J \implies \begin{pmatrix} a_1 & -b_1 \\ b_1 & a_1 \end{pmatrix} = \begin{pmatrix} a_2 & -b_2 \\ b_2 & a_2 \end{pmatrix},$$

and so $a_1 = a_2$ and $b_1 = b_2$.

(i) From the calculation in Part (d) we have

$$(a\,I + b\,J)^{-1} = \frac{a}{a^2 + b^2}\,I - \frac{b}{a^2 + b^2}\,J.$$

(j) Following the hint assume that

$$B = \begin{pmatrix} x & -y \\ y & x \end{pmatrix} \in \mathbf{C}$$

satisfies $B^2 = A$.
Now
$$B^2 = \begin{pmatrix} x^2 - y^2 & -2\,x\,y \\ 2\,x\,y & x^2 - y^2 \end{pmatrix}$$
and so if $B^2 = A$ we have the system[1]
$$\begin{cases} x^2 - y^2 &= a \\ \quad 2\,x\,y &= b \end{cases} . \tag{2}$$
Now we consider two cases:

- **Case I:** $b \neq 0$. Then the second equation gives that $x \neq 0$ and $y \neq 0$ and
$$y = \frac{b}{2\,x}. \tag{3}$$
Substituting in the first equation then gives
$$x^2 - \frac{b^2}{4\,x^2} = a \iff 4\,x^4 - 4\,a\,x^2 - b^2 = 0.$$
Using the quadratic formula we have
$$x^2 = \frac{4\,a \pm \sqrt{16\,a^2 + 16\,b^2}}{8} = \frac{a \pm \sqrt{a^2 + b^2}}{2}$$
and since $a^2 + b^2 > a^2$ we have $\sqrt{a^2 + b^2} > a$ and therefore the
$$\frac{a - \sqrt{a^2 + b^2}}{2} < 0.$$
Since x is a real number, $x^2 \geq 0$ and therefore only the solution
$$x^2 = \frac{a + \sqrt{a^2 + b^2}}{2}$$
is acceptable. Thus we have
$$x = \pm\sqrt{\frac{a + \sqrt{a^2 + b^2}}{2}}.$$
Substituting into Equation (3) we get
$$y = \pm\frac{b}{\sqrt{a + \sqrt{a^2 + b^2}}}.$$
Thus the System (2) has two solutions and therefore A has two square roots.
- **Case II:** $b = 0$. Then from the second equation we get $x = 0$ or $y = 0$. Since $A \neq O$ we have $a \neq 0$ and from the first equation of System (2) we have that exactly one of the x, y is 0: If $a > 0$ then $y = 0$ and $x \neq 0$ and in that case we get solutions
$$x = \pm\sqrt{a}, \quad y = 0.$$
If $a < 0$ then $x = 0$ and $y \neq 0$ and in that case we get solutions
$$x = 0, \quad y = \pm\sqrt{-a}.$$

[1] Notice that this is not a linear system.

Remark 0.0.1. When $b = 0$, $A = a\, I$ a scalar matrix. Our conclusion is that for $a > 0$ the scalar matrix has two square roots $\pm\sqrt{a}\, I$, while if $a < 0$ then we have the square roots $\pm\sqrt{a}\, J$.

(k) For $Z \in \mathbf{C}$ let us denote by $\pm\sqrt{Z}$ the two square roots of Z whose existence was proven in Part (j). Then I claim that the equation

$$A\, X^2 + B\, X + C = O$$

has the solutions

$$X = \frac{1}{2}\left(-B \pm \sqrt{B^2 - 4\, A\, C}\right) A^{-1}.$$

Indeed, since by Part (e) any two elements of $\mathbf{C}$ commute we have

$$X^2 = \frac{1}{4}\left(B^2 \mp 2\, B\, \sqrt{B^2 - 4\, A\, C} + B^2 - 4\, A\, C\right) A^{-2}.$$

Now we calculate:

$$\begin{aligned}
A\, X^2 + B\, X &= A\left(\frac{1}{4}\left(B^2 \mp 2\, B\, \sqrt{B^2 - 4\, A\, C} + B^2 - 4\, A\, C\right) A^{-2}\right) \\
&\quad + B\left(\frac{1}{2}\left(-B \pm \sqrt{B^2 - 4\, A\, C}\right) A^{-1}\right) \\
&= \frac{1}{2}\, A^{-1}\left(B^2 \mp B\, \sqrt{B^2 - 4\, A\, C} - 2\, A\, C\right) \\
&\quad + \frac{1}{2}\, A^{-1}\left(B^2 \pm B\, \sqrt{B^2 - 4\, A\, C}\right) \\
&= \frac{1}{2} A^{-1}\, (-2\, A\, C) \\
&= -C.
\end{aligned}$$

□

7. For each of the following *permutation matrices* P:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Compute $P\, A$ and $A\, P$, where

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

Answer. The first matrix is the identity matrix and so $P\, A = A\, P = A$.

If

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

then

$$PA = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}, \quad AP = \begin{pmatrix} a_{11} & a_{13} & a_{12} \\ a_{21} & a_{23} & a_{22} \\ a_{31} & a_{33} & a_{32} \end{pmatrix}.$$

If

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

then

$$PA = \begin{pmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \quad AP = \begin{pmatrix} a_{12} & a_{11} & a_{13} \\ a_{22} & a_{21} & a_{23} \\ a_{32} & a_{31} & a_{33} \end{pmatrix}.$$

If

$$P = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

then

$$PA = \begin{pmatrix} a_{31} & a_{32} & a_{33} \\ a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}, \quad AP = \begin{pmatrix} a_{12} & a_{13} & a_{11} \\ a_{22} & a_{23} & a_{21} \\ a_{32} & a_{33} & a_{31} \end{pmatrix}.$$

If

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

then

$$PA = \begin{pmatrix} a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{11} & a_{12} & a_{13} \end{pmatrix}, \quad AP = \begin{pmatrix} a_{13} & a_{12} & a_{11} \\ a_{23} & a_{22} & a_{21} \\ a_{33} & a_{32} & a_{31} \end{pmatrix}.$$

Finally, if

$$P = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

then

$$PA = \begin{pmatrix} a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \end{pmatrix}, \quad AP = \begin{pmatrix} a_{13} & a_{12} & a_{11} \\ a_{23} & a_{22} & a_{21} \\ a_{33} & a_{32} & a_{31} \end{pmatrix}.$$

□

8. Let

$$X = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad Y = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and let A be as in the previous question. Compute $X\,A$, $A\,X$, $Y\,A$, and $A\,Y$.

Answer. We have

$$X\,A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ -2\,a_{21} & -2\,a_{22} & -2\,a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \quad A\,X = \begin{pmatrix} a_{11} & -2\,a_{12} & a_{13} \\ a_{21} & -2\,a_{22} & a_{23} \\ a_{31} & -2\,a_{32} & a_{33} \end{pmatrix}.$$

And,

$$Y\,A = \begin{pmatrix} a_{11} + 3\,a_{31} & a_{12} + 3\,a_{32} & a_{13} + 3\,a_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \quad A\,Y = \begin{pmatrix} a_{11} & a_{12} & a_{11} + 3\,a_{13} \\ a_{21} & a_{22} & a_{23} + 3\,a_{21} \\ a_{31} & a_{32} & a_{33} + 3\,a_{31} \end{pmatrix}.$$

□

9. Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

(a) Compute A^n for $n = 0, 1, 2, 3, 4$.

(b) what pattern do you observe? Conjecture a formula for A^n based on that pattern.

(c) Prove your conjecture.

Answer. (a) We have

$$A^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A^1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad A^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \quad A^3 = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}, \quad A^4 = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}.$$

(b) We conjecture that

$$A^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}.$$

(c) We proceed by induction. For $n = 0$ our conjecture is true. Assume then that

$$A^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}.$$

Then,

$$A^{n+1} = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 \cdot 1 + n \cdot 0 & 1 \cdot 1 + n \cdot 1 \\ 0 \cdot 1 + 1 \cdot 0 & 0 \cdot 1 + 1 \cdot 1 \end{pmatrix} = \begin{pmatrix} 1 & n+1 \\ 0 & 1 \end{pmatrix},$$

and we established that our conjecture holds for $n + 1$ as well. By induction then, our conjecture is true for all $n \in \mathbb{N}$.

□

10. Let

$$A = \begin{pmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{pmatrix}.$$

(a) Find A^{42}.

Answer. We calculate:

$$A^2 = \begin{pmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{pmatrix} = A.$$

It follows that $A^n = A$ for all $n \geq 1$. Therefore

$$A^{42} = A = \begin{pmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{pmatrix}.$$

□

(b) Find B^{101}.

Answer. We calculate:

$$B^2 = \begin{pmatrix} 0 & 0 & 0 \\ 3 & 3 & 9 \\ -1 & -1 & -3 \end{pmatrix}, \quad B^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = O.$$

It follows that $B^n = O$ for $n \geq 3$. Therefore

$$B^{101} = O.$$

□

11. Let $p(x) = x^3 - 3\,x^2 + x - 3$ and let

$$A = \begin{pmatrix} 5 & 0 & 13 \\ 1 & 3 & 14 \\ -2 & 0 & -5 \end{pmatrix}.$$

Evaluate $p\,(A)$.

Answer. We have

$$p(A) = \begin{pmatrix} -9 & 2 & 1 \\ 7 & -12 & -2 \\ 5 & -5 & -8 \end{pmatrix}.$$

□

12. Let $A = \begin{pmatrix} 5 & 2 \\ 0 & a \end{pmatrix}$. Find the real number a if A is a root of the polynomial $p(x) = x^2 - 7\,x + 10$.

Answer. We have

$$A^2 = \begin{pmatrix} 25 & 2\,a + 10 \\ 0 & a^2 \end{pmatrix},$$

and so

$$\begin{aligned} p(A) &= A^2 - 7\,A + 10\,I \\ &= \begin{pmatrix} 25 & 2\,a+10 \\ 0 & a^2 \end{pmatrix} - 7\begin{pmatrix} 5 & 2 \\ 0 & a \end{pmatrix} + 10\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 25 & 2\,a+10 \\ 0 & a^2 \end{pmatrix} + \begin{pmatrix} -35 & -14 \\ 0 & -7\,a \end{pmatrix} + \begin{pmatrix} 10 & 0 \\ 0 & 10 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 2\,a-4 \\ 0 & a^2-7\,a+10 \end{pmatrix}. \end{aligned}$$

So if A is a root of $p(x)$ we have

$$\begin{pmatrix} 0 & 2\,a-4 \\ 0 & a^2-7\,a+10 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

So we must have

$$2\,a-4=0, \text{ and } a^2-7\,a+10=0.$$

Thus $a=2$. □

13. Let

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & -1 \\ 3 & -1 & 1 \end{pmatrix},$$

and let $p(x) = x^3 - 2\,x^2 - 2\,x + 6$.

(a) Verify that A is a root of $p(x)$.

(b) Express A^{-1} as a polynomial in A.

(c) Use Part (b) to find A^{-1}.

Answer. (a) We calculate

$$A^2 = \begin{pmatrix} 3 & 1 & -1 \\ -1 & 3 & -1 \\ 4 & 2 & 2 \end{pmatrix}, \quad A^3 = \begin{pmatrix} 2 & 4 & -2 \\ 2 & 0 & -4 \\ 14 & 2 & 0 \end{pmatrix}.$$

And then with straightforward calculations we verify that

$$A^3 - 2\,A^2 - 2\,A + 6\,I = O.$$

(b) We have

$$\begin{aligned} A^3 - 2\,A^2 - 2\,A + 6\,I = O &\implies A^3 - 2\,A^2 - 2\,A = -6\,I \\ &\implies A\left(A^2 - 2\,A - 2\,I\right) = -6\,I \\ &\implies A\left(-\frac{1}{6}\left(A^2 - 2\,A - 2\,I\right)\right) = I. \end{aligned}$$

Therefore,

$$A^{-1} = -\frac{1}{6}\left(A^2 - 2\,A - 2\,I\right).$$

(c) We have

$$\begin{aligned} A^{-1} &= -\frac{1}{6}\left(A^2 - 2\,A - 2\,I\right) \\ &= -\frac{1}{6}\left(\begin{pmatrix} 3 & 1 & -1 \\ -1 & 3 & -1 \\ 4 & 2 & 2 \end{pmatrix} - 2\begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & -1 \\ 3 & -1 & 1 \end{pmatrix} - 2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right) \\ &= -\frac{1}{6}\begin{pmatrix} -1 & -1 & -1 \\ -5 & 1 & 1 \\ -2 & 2 & -2 \end{pmatrix} \\ &= \frac{1}{6}\begin{pmatrix} 1 & 1 & 1 \\ 5 & -1 & -1 \\ 2 & -2 & 2 \end{pmatrix}. \end{aligned}$$

□

14. Find all 2×2 matrices $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ that commute with $B = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$.

Solution. We have

$$A\,B = \begin{pmatrix} a & 2\,a+b \\ c & 2\,c+d \end{pmatrix}, \quad B\,A = \begin{pmatrix} a+2\,c & b+2\,d \\ c & d \end{pmatrix}.$$

So we have the system

$$\begin{cases} a & = a + 2c \\ 2a + b & = b + 2d \\ c & = c \\ 2c + d & = d \end{cases}.$$

The first equation gives $c = 0$ and the second $a = d$. Thus the centralizer of B is

$$\left\{\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} : a, b \in \mathbb{R}\right\}.$$

□

15. Let A and B be two symmetric $n \times n$ matrices. Prove that $A\,B$ is symmetric if and only if A and B commute.

Solution. Since A and B are symmetric we have $A^* = A$ and $B^* = B$ and so

$$(A\,B)^* = B^*\,A^* = B\,A.$$

Thus

$$(A\,B)^* = A\,B \iff B\,A = A\,B.$$

In words, $A\,B$ is symmetric if and only if A and B commute. □

16. Let $A \in \mathbf{M}_n$. Prove that $A + A^*$ is symmetric, where A^* is the transpose of A.

Solution. We have

$$(A + A^*)^* = A^* + (A^*)^* = A^* + A = A + A^*.$$

□

17. A square matrix A is called *nilpotent* if $A^k = O$ for some positive integer k. Prove that if A is nilpotent then A is not invertible.

Solution. Assume,to get a contradiction, that A is invertible. Then A^k is also invertible and $= (A^{-1})^k$. But then

$$A^k = O \implies A^k \left(A^k\right)^{-1} = O \left(A^k\right)^{-1}$$

$$\implies I = O,$$

a contradiction. Therefore A is not invertible. □

18. A square matrix A is called *idempotent* if $A^2 = A$. Find all the matrices that are both idempotent and invertible.

Solution. If we multiply both sides of the equation

$$A^2 = A$$

with A^{-1} we get

$$A = I.$$

Thus the only idempotent and invertible matrix is the identity matrix I. □

19. A square matrix is said to be *antisymmetric* if $A^* = -A$, in other words if for all i, j we have

$$a_{ji} = -a_{ij}.$$

Prove that if A and B are symmetric matrices then $A\,B - B\,A$ is antisymmetric.

Solution. We have

$$(A\,B - B\,A)^* = (A\,B)^* - (B\,A)^* = B^*\,A^* - A^*\,B^*.$$

Now if A and B are symmetric we have

$$B^*\,A^* - A^*\,B^* = B\,A - A\,B = -\left(A\,B - B\,A\right).$$

So if A and B are symmetric matrices then

$$(A\,B - B\,A)^* = -\left(A\,B - B\,A\right),$$

and so $A\,B - B\,A$ is antisymmetric. □

20. Prove that all permutation matrices in Question 7 are orthogonal.

Solution. Let P be any permutation matrix. Then the columns of P are obtained from the standard basis of $\mathbb{R}^3$ by applying a permutation. Now for the standard basis of $\mathbb{R}^3$ we have

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}, \quad i, j = 1, 2, 3.$$

Therefore, if $\mathbf{p}_i$, $i = 1, 2, 3$ are the columns of P we have

$$\mathbf{p}_i \cdot \mathbf{p}_j = \delta_{ij}, \quad i, j = 1, 2, 3.$$

By Proposition (8) we have then that P is orthogonal. □