MTH 42, Fall 2024

Nikos Apostolakis

Fourth Set of homework

1. Let A be an $m \times k$ matrix and B a $k \times n$ matrix. Prove that
$$\ker B \subseteq \ker A\,B.$$

2. Let A be a 3×4 matrix and B a 4×3 so that $A\,B$ is a square 3×3 matrix. Prove that $A\,B$ is not invertible.
Hint. You might want to use Question 1.

3. Find a basis and the dimension of the solution set of the following homogeneous system:
$$\begin{cases} x_1 - 3x_2 + x_4 + x_5 = 0 \\ 2x_1 - 6x_2 + 2x_3 + 4x_4 + 2x_5 = 0 \\ -3x_1 + 4x_2 + x_5 = 0 \\ x_2 + x_3 + x_4 = 0 \end{cases}$$

4. For each of the following two "transpose" matrices:
$$A = \begin{pmatrix} 1 & 2 & 3 & 1 & 2 \\ 2 & 1 & 2 & 3 & 1 \\ 3 & 3 & 5 & 4 & 3 \\ 1 & -1 & -1 & 2 & -1 \end{pmatrix}, \quad A^* = \begin{pmatrix} 1 & 2 & 3 & 1 \\ 2 & 1 & 3 & -1 \\ 3 & 2 & 5 & -1 \\ 1 & 3 & 4 & 2 \\ 2 & 1 & 3 & -1 \end{pmatrix}.$$
   (a) Find a basis for their ranges and state their rank.
   (b) Find a basis for their kernels and state their nullity.

5. Find the inverse of each of the following matrices:
$$A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 2 & 4 \\ 1 & 3 & -3 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 1 & 3 & 1 & -2 \\ 1 & 4 & -2 & 4 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 2 & 0 & -2 \\ 0 & 2 & 1 & 4 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -8 \end{pmatrix}.$$

6. Let $\mathbf{C}$ be the following set of 2×2 matrices.
$$\mathbf{C} = \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} : a, b \in \mathbb{R} \right\}.$$
   Prove the following:
   (a) $\mathbf{C}$ is closed under matrix addition. That is,
$$A, B \in \mathbf{C} \implies A + B \in \mathbf{C}.$$

(b) $\mathbf{C}$ is closed under scalar multiplication. That is,

$$\lambda \in \mathbb{R},\ A \in \mathbf{C} \implies \lambda\, A \in \mathbf{C}.$$

In particular,

$$A \in \mathbf{C} \implies -A \in \mathbf{C}.$$

(c) $\mathbf{C}$ is closed under matrix multiplication. That is,

$$A, B \in \mathbf{C} \implies A\, B \in \mathbf{C}.$$

(d) All non-zero elements of $\mathbf{C}$ are invertible, and

$$A \in \mathbf{C},\ A \neq O \implies A^{-1} \in \mathbf{C}.$$

(e) Any two elements of $\mathbf{C}$ commute. That is,

$$A, B \in \mathbf{C} \implies A\, B = B\, A.$$

(f) Let

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Verify that

$$J^2 = -I.$$

(g) what is J^{503}?

(h) Every element of $\mathbf{C}$ can be uniquely expressed as a linear combination of I and J. In other words, if $A \in \mathbf{C}$ then there exist two unique real numbers a, b such that

$$A = a\, I + b\, J.$$

(i) Let $A \in \mathbf{C}$ with $A = a\, I + b\, J$ with at least one of a, b non-zero. Express A^{-1} as a linear combination of I and J.

(j) Prove that every non-zero element of $\mathbf{C}$ has exactly two *square roots*. That is, prove that if $A \neq O$ is an element of $\mathbf{C}$ then there are exactly two elements $B \in \mathbf{C}$ such that $B^2 = A$.
Hint. Set $B = \begin{pmatrix} x & -y \\ y & x \end{pmatrix}$ and see what $B^2 = A$ says for x, y.

(k) Prove the that every quadratic equation

$$A\, X^2 + B\, X + C = O$$

where $A, B, C \in \mathbf{C}$ has two solutions (that may coincide) in $\mathbf{C}$.
Hint. Think about the quadratic formula.

7. For each of the following *permutation matrices* P:

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix},\quad \begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix},\quad \begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix},\quad \begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix},\quad \begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix},\quad \begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}.$$

Compute $P\,A$ and $A\,P$, where

$$A = \begin{pmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{pmatrix}.$$

8. Let

$$X = \begin{pmatrix}1&0&0\\0&-2&0\\0&0&1\end{pmatrix},\quad Y = \begin{pmatrix}1&0&3\\0&1&0\\0&0&1\end{pmatrix},$$

and let A be as in the previous question. Compute $X\,A$, $A\,X$, $Y\,A$, and $A\,Y$.

9. Let $A = \begin{pmatrix}1&1\\0&1\end{pmatrix}$.

   (a) Compute A^n for $n = 0, 1, 2, 3, 4$.

   (b) what pattern do you observe? Conjecture a formula for A^n based on that pattern.

   (c) Prove your conjecture.

10. Let

$$A = \begin{pmatrix}2&-2&-4\\-1&3&4\\1&-2&-3\end{pmatrix},\quad B = \begin{pmatrix}1&1&3\\5&2&6\\-2&-1&-3\end{pmatrix}.$$

   (a) Find A^{42}.

   (b) Find B^{101}.

   Hint. It's not as hard as it may appear. For both matrices a simple pattern appears after you calculate the first few powers.

11. Let $p(x) = x^3 - 3\,x^2 + x - 3$ and let

$$A = \begin{pmatrix}5&0&13\\1&3&14\\-2&0&-5\end{pmatrix}.$$

Evaluate $p\,(A)$.

12. Let $A = \begin{pmatrix}5&2\\0&a\end{pmatrix}$. Find the real number a if A is a root of the polynomial $p(x) = x^2 - 7\,x + 10$.

13. Let

$$A = \begin{pmatrix}1&1&0\\2&0&-1\\3&-1&1\end{pmatrix},$$

and let $p(x) = x^3 - 2\,x^2 - 2\,x + 6$.

   (a) Verify that A is a root of $p(x)$.

(b) Express A^{-1} as a polynomial in A.

(c) Use Part (b) to find A^{-1}.

14. Find all 2×2 matrices $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ that commute with $B = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$.

15. Let A and B be two symmetric $n \times n$ matrices. Prove that $A\,B$ is symmetric if and only if A and B commute.

16. Let $A \in \mathbf{M}_n$. Prove that $A + A^*$ is symmetric, where A^* is the transpose of A.

17. A square matrix A is called *nilpotent* if $A^k = O$ for some positive integer k. Prove that if A is nilpotent then A is not invertible.

18. A square matrix A is called *idempotent* if $A^2 = A$. Find all the matrices that are both idempotent and invertible.

19. A square matrix is said to be *antisymmetric* if $A^* = -A$, in other words if for all i, j we have

$$a_{ji} = -a_{ij}.$$

Prove that if A and B are symmetric matrices then $A\,B - B\,A$ is antisymmetric.

20. Prove that all permutation matrices in Question 7 are orthogonal.