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Answers and solutions to Third Homework and Take Home Exam

1. Let S_1 and S_2 be two subsets of $\mathbb{R}^n$ with $S_1 \subseteq S_2$. Prove
   (a) If S_1 is spanning then S_2 is also spanning.
   (b) If S_1 is linearly dependent then S_2 is also linearly dependent.
   (c) If S_2 is linearly independent then S_1 is also linearly independent.

   Solution. Note that (b) and (c) are logically equivalent: (c) is the contrapositive of (b). So we'll prove (a) and (b). Both follow from the fact that a linear combination of elements of S_1 is also a linear combination of elements of S_2.

   (a) If S_1 is spanning then every vector $\mathbf{v} \in \mathbb{R}^n$ can be expressed as a linear combination of elements of S_1, and hence as a linear combination of elements of S_2. Therefore S_2 is spanning.
   (b) If S_1 is linearly dependent then $\mathbf{0}$ can be expressed as a non-trivial linear combination of elements of S_1, and hence as a non-trivial linear combination of elements of S_2. Therefore S_2 is linearly dependent.

   □

2. Decide whether each of the following subsets is a vector subspace of the given standard real vector space.
   (a) $\{(x, 3x + y, 0, y - z) : x, y, z \in \mathbb{R}\} \subseteq \mathbb{R}^4$.
   (b) $\{(x, y, z) \in \mathbb{R}^3 : x, y, z \in \mathbb{R}$ and $3\,x - 4\,y = 11\,z\}$.
   (c) The set of points in $\mathbb{R}^2$ that lie in the parabola $y = x^2$.
   (d) The set of points in $\mathbb{R}^3$ that lie in the plane with equation $2x - 3y + 4z = 0$.
   (e) The set of points in $\mathbb{R}^3$ that lie in the plane with equation $2x - 3y + 4z = 8$.
   (f) $\{(x, 2, 3x + 4y, y - z) : x, y, z \in \mathbb{R}\} \subseteq \mathbb{R}^4$.
   (g) $\{(3\,w, 2\,z - 5\,t, x - 4\,y + 5\,t, -2x + z - 3t + 4w) : x, y, z, w, t \in \mathbb{R}\} \subseteq \mathbb{R}^4$.

   Answer. (a) Yes, this is a vector subspace of $\mathbb{R}^4$. It is nonempty since it contains the zero vector. Simple calculations show that

$$\begin{aligned}&\lambda\,(x_1, 3\,x_1 + y_1, 0, y_1 - z_1) + \mu\,(x_2, 3\,x_2 + y_2, 0, y_2 - z_2)\\ &= (\lambda\,x_1 + \mu\,x_2, 3\;(\lambda\,x_1 + \mu\,x_2) + (y_1 + y_2), 0, (y_1 + y_2) - (z_1 + z_2))\end{aligned}$$

   So both conditions of Theorem 4.5 hold.

   Alternatively we can show that this subset is the linear span of the vectors

$$(1, 3, 0, 0), (0, 1, 0, 1), (0, 0, 0, -1).$$

(b) Yes, this subset is a vector subspace of $\mathbb{R}^3$. Perhaps the easiest way to see this is to note that this subset is the solution set of the homogeneous linear equation $3\,x - 4\,y - 11\,z = 0$.

(c) No this is not a vector subspace. For example it contains $\mathbf{v} = (1,1)$ but not $2\,\mathbf{v} = (2,2)$.

(d) Yes, it's the solution set of a homogeneous linear equation.

(e) No. This subset does not contain the zero vector.

(f) No. This subset does not contain the zero vector.

(g) Yes. We can either use Theorem 4.5 or note that this subset is the linear span of the vectors

$$(0,0,1,-2), (0,0,-4,0), (0,2,0,1), (0,-5,5,4), (3,0,0,4).$$

□

3. For those subsets in Question 2 that are subspaces find a basis and the dimension.

Answer. For (a) we have the spanning set

$$S = \{(1,3,0,0), (0,1,0,1), (0,0,0,-1).\}.$$

The reduced echelon form is:

$$\begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

There are no free columns so these vectors are linearly independent, and form a basis. Therefore the dimension is 3.

For (b) we first find a spanning set. We have that the solution is

$$(x,y,z) = \left(\frac{1}{3}\,(4t + 11s)\,, t, s\right) = t\,(\frac{4}{3}, 1, 0) + s\,(\frac{11}{3}, 0, 1).$$

The set $\left\{(\frac{4}{3}, 1, 0), (\frac{11}{3}, 0, 1)\right\}$ is thus spanning and since it's linear independent if forms a basis. The dimension is therefore 2.

(d) is similar to (c). There are two free variables, y, z, and we again get a 2-dimensional subspace. A basis is

$$\left\{\left(\frac{3}{2}, 1, 0\right), (2,0,1)\right\}.$$

(g) is similar to (a). The basic columns are the first, second, third, and fifth. So a basis is

$$\{(0,0,1,-2), (0,0,-4,0), (0,2,0,1), (3,0,0,4)\},$$

and the dimension is 4. □

4. Which of the following subsets of $\mathbb{R}^3$ are a basis?
   (a) $\{(1,2,3),(3,2,1)\}$.
   (b) $\{(1,1,2),(1,-2,0),(2,0,1)\}$
   (c) $\{(1,2,3),(3,1,2),(2,3,1)\}$.
   (d) $\{(1,2,3),(1,1,0),(0,3,1),(1,0,0)\}$.

   Answer. A basis of $\mathbb{R}^3$ contains exactly 3 vectors, so (a) and (d) are not bases. Both (b) and (c) are bases since the reduced echelon form of the matrices with columns those vectors is the identity matrix. □

5. Let
$$B=\{(1,1,1,1,1),(0,1,1,1,1),(0,0,1,1,1),(0,0,0,1,1),(0,0,0,0,1)\}.$$
   (a) Prove that B is a basis of $\mathbb{R}^5$.
   (b) Express the elements of the standard basis of $\mathbb{R}^5$ as linear combinations of elements of B.

   Answer. As indicated in the hint, if we succeed in completing part (b), that is if we express every vector of the standard basis as a linear combination of elements of B then it follows that B is a basis[1].

   Let $\mathbf{v}_i$, $i=1,\ldots,5$ be the elements of B in the order given. We augment the matrix with columns the vectors of B with the five vectors of the standard basis and find its reduced echelon form.
$$\left(\begin{array}{ccccc|ccccc}1&0&0&0&0&1&0&0&0&0\\1&1&0&0&0&0&1&0&0&0\\1&1&1&0&0&0&0&1&0&0\\1&1&1&1&0&0&0&0&1&0\\1&1&1&1&1&0&0&0&0&1\end{array}\right)\sim\left(\begin{array}{ccccc|ccccc}1&0&0&0&0&1&0&0&0&0\\0&1&0&0&0&-1&1&0&0&0\\0&0&1&0&0&0&-1&1&0&0\\0&0&0&1&0&0&0&-1&1&0\\0&0&0&0&1&0&0&0&-1&1\end{array}\right).$$
   Therefore
$$\begin{aligned}\mathbf{e}_1&=\mathbf{v}_1-\mathbf{v}_2\\\mathbf{e}_2&=\mathbf{v}_2-\mathbf{v}_3\\\mathbf{e}_3&=\mathbf{v}_3-\mathbf{v}_4\\\mathbf{e}_4&=\mathbf{v}_4-\mathbf{v}_5\\\mathbf{e}_5&=\mathbf{v}_5.\end{aligned}$$
□

6. Let $T\colon\ \mathbb{R}^3\to\mathbb{R}^4$ be defined by
$$T(x,y,z)=(x+2\,y+z,x+y,y-3\,z,4\,x-3\,y+2\,z).$$
   (a) Prove that T is linear.
   (b) Find the matrix of T.

[1] Why?

Answer. (a) We need to check that for $\lambda_i \in \mathbb{R}$ and $\mathbf{v}_i = (x_i, y_i, z_i) \in \mathbb{R}^3$, where $i = 1, 2$ we have:

$$T(\lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2) == \lambda_1 T \mathbf{v}_1 + \lambda_2 T \mathbf{v}_2.$$

This is a straightforward calculation.

(b) We have

$$T \mathbf{e}_1 = (1, 1, 0, 4), \quad T \mathbf{e}_2 = (2, 1, -1, -3), \quad T \mathbf{e}_3 = (1, 0, -3, 2).$$

So the matrix is

$$T = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 0 \\ 0 & -1 & -3 \\ 4 & -3 & 2 \end{pmatrix}.$$

□

7. Let

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix}.$$

(a) Prove that the columns of A form a basis of $\mathbb{R}^3$.

(b) Express each of the vectors in the standard basis of $\mathbb{R}^3$ as linear combinations of the columns of A.

(c) Let T be the linear function that sends the i-th column to the i-th row of A. That is if $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$ are the columns of A then T is defined by

$$T \mathbf{a}_1 = (1, 2, 3), \quad T \mathbf{a}_2 = (0, 2, 1), \quad T \mathbf{a}_3 = (1, 0, 3).$$

Find the matrix of T.

Answer. (a) The reduced echelon form of A is the identity matrix.

(b) We have

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 1 & 0 & 3 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -3 & -2 \\ 0 & 1 & 0 & 1/2 & 0 & -1/2 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right).$$

And so, letting $\mathbf{a}_i$ be the columns of A, we have:

$$\begin{aligned} \mathbf{e}_1 &= 3\,\mathbf{a}_1 + \frac{1}{2}\,\mathbf{a}_2 - \mathbf{a}_3 \\ \mathbf{e}_2 &= -3\,\mathbf{a}_1 + \mathbf{a}_3 \\ \mathbf{e}_3 &= -2\,\mathbf{a}_1 - \frac{1}{2}\,\mathbf{a}_2 + \mathbf{a}_3. \end{aligned}$$

(c) We need to find $T \mathbf{e}_i$ for $i = 1, 2, 3$. We will use the linearity of T and the expressions of $\mathbf{e}_i$ as linear combinations of $\mathbf{a}_i$ from Part (b).

We have:

$$
\begin{aligned}
T\,\mathbf{e}_1 &= T\left(3\,\mathbf{a}_1 + \frac{1}{2}\,\mathbf{a}_2 - \mathbf{a}_3\right)\\
&= 3\,T\,\mathbf{a}_1 + \frac{1}{2}\,T\,\mathbf{a}_2 - T\,\mathbf{a}_3\\
&= (3,6,9) + \left(0,1,\frac{1}{2}\right) - (1,0,3)\\
&= \left(2,7,\frac{13}{2}\right).
\end{aligned}
$$

Similarly,

$$T\,\mathbf{e}_2 = (-2,-6,-6), \quad T\,\mathbf{e}_3 = \left(-1,-5,-\frac{7}{2}\right).$$

Thus T is induced by the matrix

$$T = \begin{pmatrix} 2 & -2 & -1 \\ 7 & -6 & -5 \\ \frac{13}{2} & -6 & -\frac{7}{2} \end{pmatrix}.$$

□

8. Find a polynomial of degree at most 4 that satisfies the following conditions

$$p(0) = -5, \quad p(-1) = -10, \quad p(1) = 0, \quad p(2) = 29, \quad p(-2) = -15.$$

Proof. The polynomial is

$$p(x) = x^4 + 2\,x^3 - x^2 + 3\,x - 5.$$

□

9. Let V be the subspace of $\mathbb{R}^5$ spanned by the vectors

$$
\begin{aligned}
\mathbf{v}_1 &= (1,2,-1,3,4), & \mathbf{v}_2 &= (2,4,-2,6,8),\\
\mathbf{v}_3 &= (1,3,2,2,6), & \mathbf{v}_4 &= (1,4,5,1,8),\\
\mathbf{v}_5 &= (2,7,3,3,9), & \mathbf{v}_6 &= (4,9,-1,11,18).
\end{aligned}
$$

Find a basis and the dimension of V.

Answer. We have:

$$
\begin{pmatrix} 1 & 2 & 0 & -1 & 0 & 3 \\ 0 & 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 0 & -1 & 0 & 3 \\ 0 & 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}
$$

So $\{\mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_5\}$ is a basis, and the dimension of the subspace is 3. □