MTH 42, Fall 2024

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Second Set of homework

1. Solve the system
$$\begin{cases} 2x - \ \ 5y + 2z - 4s + 2t &= 4 \\ 3x - \ \ 7y + 2z - 5s + 4t &= 9 \\ 5x - 10y - 5z - 4s + 7t &= 22 \end{cases}$$
by first solving the corresponding homogeneous system and then finding a particular solution. Refer to Example 1.35 in Section 1.3.2 of the current set of notes.

2. Express the vector $\mathbf{c} = 3\,\mathbf{e}_1 - 2\,\mathbf{e}_2 - \mathbf{e}_3$ as a linear combination of the vectors
$$\begin{aligned} \mathbf{v}_1 &= \mathbf{e}_1 + 2\mathbf{e}_2 + 3\mathbf{e}_3 \\ \mathbf{v}_2 &= 2\mathbf{e}_1 + 3\mathbf{e}_2 + \mathbf{e}_3 \\ \mathbf{v}_3 &= 3\mathbf{e}_1 + \mathbf{e}_2 + 2\mathbf{e}_3. \end{aligned}$$

3. Express the vector $\mathbf{c} = 5\mathbf{e}_1 - \mathbf{e}_2 + 3\,\mathbf{e}_3$ as a linear combination of the vectors
$$\begin{aligned} \mathbf{v}_1 &= \mathbf{e}_1 - 2\mathbf{e}_2 + 3\mathbf{e}_3 \\ \mathbf{v}_2 &= 4\mathbf{e}_1 + \mathbf{e}_2 \\ \mathbf{v}_3 &= \mathbf{e}_1 - 11\mathbf{e}_2 + 15\mathbf{e}_3, \end{aligned}$$
in three different ways.

4. Find a vector $\mathbf{c}$ that cannot be expressed as a linear combination of the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ of the previous exercise.

5. Let
$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \qquad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}, \qquad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$
Let $\mathbf{y} = B\,\mathbf{x}$. Find the vector $\mathbf{z} = A\,\mathbf{y}$.

6. Find a 2×2 matrix A that interchanges $\mathbf{e}_1$ and $\mathbf{e}_2$, in other words such that
$$A\,\mathbf{e}_1 = \mathbf{e}_2 \quad \text{and} \quad A\,\mathbf{e}_2 = \mathbf{e}_1.$$

7. Prove that if
$$a_1\,b_2 - a_1\,b_3 + a_2\,b_3 - a_3\,b_2 + a_3\,b_1 - a_2\,b_1 \neq 0$$
then the system
$$\begin{cases} x + a_1 y + b_1 z &= c_1 \\ x + a_2 y + b_2 z &= c_2 \\ x + a_3 y + b_3 z &= c_2 \end{cases}$$
has a unique solution for all real numbers c_1, c_2, c_3.