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Answers and solutions to Homework Sets I and II

1 First Homework

1. Solve each of the following systems:

   (a)
$$\begin{cases} x + 2y + 3z = 0 \\ 3x + y + 2z = 0 \\ 2x + 3y + z = 0 \end{cases}.$$

   Answer. This is a homogeneous system. A row echelon form of the coefficient matrix is
$$A \begin{pmatrix} 1 & 0 & 11 \\ 0 & 1 & 5 \\ 0 & 0 & 18 \end{pmatrix}.$$

   Since there are no free columns we conclude that the system has only the trivial solution. The solution set is therefore $\{(0,0,0)\}$. □

   (b)
$$\begin{cases} x - y + z = 0 \\ -x + 3y + z = 5 \\ 3x + y + 7z = 2 \end{cases}.$$

   Answer. The system is inconsistent. The solutions set is $\emptyset$. □

   (c)
$$\begin{cases} x_1 + 3x_2 - 2x_3 + 2x_5 = 0 \\ 2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 = -1 \\ 5x_3 + 10x_4 + 15x_6 = 5 \\ 2x_1 + 6x_2 + 8x_4 + 4x_5 + 18x_6 = 6 \end{cases}.$$

   Answer. The reduced echelon form (after discarding a zero row) of the augmented matrix is
$$\left(\begin{array}{cccccc|c} 1 & 3 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1/3 \end{array}\right).$$

   So the solution is
$$x_1 = -3\,s - 4\,t - 2\,w,\ x_2 = s,\ x_3 = -2\,t,\ x_4 = t,\ x_5 = w,\ x_6 = \frac{1}{3}.$$

   □

2. Find the real number k so that the following system is consistent

$$\begin{cases} x - 2y + 3z = 2 \\ x + \ y + \ \ z = k \\ 2x - \ y + 4z = k^2 \end{cases}.$$

Solution. We proceed to reducing the augmented matrix to echelon form.

$$\left(\begin{array}{ccc|c} 1 & -2 & 3 & 2 \\ 1 & 1 & 1 & k \\ 2 & -1 & 4 & k^2 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -2 & 3 & 2 \\ 0 & 3 & -2 & k-2 \\ 0 & 3 & -2 & k^2-4 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -2 & 3 & 2 \\ 0 & 3 & -2 & k-2 \\ 0 & 0 & 0 & k^2-k-2 \end{array}\right)$$

From the last row we see that the system is consistent if and only if

$$k^2 - k - 2 = 0 \iff k = -1 \text{ or } k = 2.$$

Thus the system is consistent only if $k = -1$ or $k = 2$. □

3. Find conditions on the real numbers a, b, c, if any, so that the system

$$\begin{cases} x + y = 0 \\ y + z = 0 \\ x - z = 0 \\ ax + by + cz = 0 \end{cases}$$

   (a) is inconsistent.
   (b) Has a unique solution.
   (c) Has more than one solution.

Answer. We can immediately answer the first part. This is a homogeneous system and is therefore consistent. Thus there exists no conditions on a, b, c that the system is inconsistent.

To answer parts (b) and (c) we proceed to reduce the matrix of the system to an echelon form.

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & -1 \\ a & b & c \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & b-a & c \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & b-a & c \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & b-a & c \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 \\ 0 & b-a & c \\ 0 & 0 & 1 \end{pmatrix}.$$

If $b \neq a$ then the system has a unique solution. If $b = a$ then the second column is free and thus the system has more than one solutions. □

4. Consider the 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where $a, b, c, d \in \mathbb{R}$.

   (a) Prove that if $ad - bc \neq 0$ then the reduced row echelon form of A is

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

(b) Prove that if $ad - bc \neq 0$ then the system

$$\begin{cases} ax + by &= k \\ cx + dy &= l \end{cases}$$

has a unique solution, for all real numbers k, l.

Answer. Look at Section 3 of the notes. □

5. Prove that there is a unique line passing through any two *distinct* points of the plane.

Solution. A line is a set of points in $\mathbb{R}^2$ whose coordinates (x, y)satisfy a linear equation of the form

$$a\,x + b\,y + c = 0 \tag{1}$$

where $a, b, c \in \mathbb{R}$ and at least one of a, b is non-zero. A non-zero multiple of Equation (1) defines the same line.

Let (x_1, y_1) and (x_2, y_2) two points, to find all lines that pass through these two points we solve the system

$$\begin{cases} a\,x_1 + b\,y_1 + c &= 0 \\ a\,x_2 + b\,y_2 + c &= 0 \end{cases}$$

for a, b, c.

Set $\Delta x = x_2 - x_1$, and $\Delta y = y_2 - y_1$. Since the points are distinct at least one of Δx, Δy is non-zero. Without loss of generality we assume that $\Delta x \neq 0$. Subtracting the two equations we get

$$a\,\Delta x + b\,\Delta y = 0 \implies a = -\frac{\Delta y}{\Delta x}\,b.$$

Substituting in the first equation we get

$$-\frac{x_1\,\Delta y}{\Delta x}\,b + y_1\,b + c = 0 \implies c = \left(\frac{x_1(y_2 - y_1) - y_1(x_2 - x_1)}{x_2 - x_1}\right) b \implies c = \frac{x_1 y_2 - x_2 y_1}{x_2 - x_1}\,b$$

So the solution is

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = t \begin{pmatrix} \dfrac{x_1\,y_2 - x_2\,y_1}{x_2 - x_1} \\ 1 \\ \dfrac{y_2 - y_1}{x_2 - x_1} \end{pmatrix}, \quad t \in \mathbb{R}.$$

Thus all equations of the form (1) that are satisfied by the coordinates of both points are multiples of the same equation and therefore determine the same line. □

6. Find the cubic polynomial

$$p(x) = a\,x^3 + b\,x^2 + c\,x + d$$

given that $p(1) = 0$, $p(2) = 3$, $p(-1) = -6$, and $p(-2) = -21$.

Answer. Substituting the given values we get the system

$$\begin{cases} a + b + c + d = 0 \\ 8a + 4b + 2c + d = 3 \\ -a + b - c + d = -6 \\ -8a + 4b - 2c + d = -21 \end{cases}.$$

Passing to the augmented matrix we have

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 8 & 4 & 2 & 1 & 3 \\ -1 & 1 & -1 & 1 & -6 \\ -8 & 4 & -2 & 1 & -21 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right).$$

Therefore the polynomial is

$$p(x) = x^3 - 2x^2 + 2x - 1.$$

□

7. Look at Examples 1.9 and 1.10, there is a geometric reason why in Example 1.10 the polynomial we got was not quadratic. The graph of a quadratic polynomial is a parabola so in these examples we were trying to find a parabola that passes through three distinct points. But the points in Example 1.10 are *colinear* and so there is no parabola that passes through all three of them.

   (a) Prove that given any three *distinct* real numbers x_1, x_2, x_3 and any three real numbers y_1, y_2, y_3 we can always find a polynomial $p(x) = a\,x^2 + b\,x + c$ such that $p(x_1) = y_1$, $p(x_2) = y_2$, and $p(x_3) = y_3$.

   Solution. Let $a, b, c \in \mathbb{R}$ be the coefficients of p. Then (a, b, c) is a solution of the system

$$\begin{cases} c + x_1\,b + x_1^2\,a &= y_1 \\ c + x_2\,b + x_2^2\,a &= y_2 \\ c + x_3\,b + x_3^2\,a &= y_3 \end{cases}$$

   The augmented matrix of the system is

$$\left(\begin{array}{ccc|c} 1 & x_1 & x_1^2 & y_1 \\ 1 & x_2 & x_2^2 & y_2 \\ 1 & x_3 & x_3^2 & y_3 \end{array}\right).$$

   Subtracting the first row from the other two we get

$$\left(\begin{array}{ccc|c} 1 & x_1 & x_1^2 & y_1 \\ 0 & x_2 - x_1 & x_2^2 - x_1^2 & y_2 - y_1 \\ 0 & x_3 - x_1 & x_3^2 - x_1^2 & y_3 - y_1 \end{array}\right).$$

   Since x_1, x_2 and x_3 are distinct, $x_2 - x_1$ and $x_3 - x_1$ are non-zero. We can also assume that $x_1 \neq 0$, for if it is zero then x_2 is non-zero and we just rename our numbers. So we can divide each row by its leading entry to get[1]

[1] Remember "Difference of Squares"?

$$\left(\begin{array}{ccc|c} 1 & 1 & x_1 & y_1/x_1 \\ 0 & 1 & x_2+x_1 & (y_2-y_1)/(x_2-x_1) \\ 0 & 1 & x_3+x_1 & (y_3-y_1)/(x_3-x_1) \end{array}\right).$$

Now subtract the second row from the third to get

$$\left(\begin{array}{ccc|c} 1 & 1 & x_1 & y_1/x_1 \\ 0 & 1 & x_2+x_1 & (y2-y1)/(x_2-x_1) \\ 0 & 0 & x_3-x_2 & (y_3-y_1)/(x_3-x_1)-(y_2-y_1)/(x_2-x_1) \end{array}\right).$$

Since, $x_3 - x_2 \neq 0$ we conclude that the system has a unique solution. □

(b) The polynomial in part (a) is quadratic (i.e. $a \neq 0$) if and only if the points (x_1, y_1), (x_2, y_2), and (x_3, y_3) are not colinear.

Solution. From the echelon form from part one we see that $a = 0$ if and only if

$$\frac{y_3 - y_1}{x_3 - x_1} = \frac{y_2 - y_1}{x_2 - x_1}. \tag{2}$$

The fraction on the LHS of Equation (2) is the slope of the line through the points (x_1, y_1) and (x_3, y_3) and the one on the RHS is the slope of the line through (x_1, y_1) and (x_2, y_2). Since these lines share the point (x_1, y_1) they are the same line if and only if they have the same slope. Now (x_1, y_2), (x_2, y_2), and (x_3, y_3) are colinear if and only if these lines are the same line, and we conclude that $a = 0$ if and only if the three points are colinear. □

2 Second Homework

1. Solve the system

$$\begin{cases} 2x - 5y + 2z - 4s + 2t &= 4 \\ 3x - 7y + 2z - 5s + 4t &= 9 \\ 5x - 10y - 5z - 4s + 7t &= 22 \end{cases}$$

by first solving the corresponding homogeneous system and then finding a particular solution. Refer to Example 1.35 in Section 1.3.2 of the current set of notes.

Answer. The corresponding homogeneous system is

$$\begin{cases} 2x - 5y + 2z - 4s + 2t &= 0 \\ 3x - 7y + 2z - 5s + 4t &= 0 \\ 5x - 10y - 5z - 4s + 7t &= 0 \end{cases}$$

We find the reduced echelon for of its matrix:

$$\begin{pmatrix} 2 & -5 & 2 & -4 & 2 \\ 3 & -7 & 2 & -5 & 4 \\ 5 & -10 & -5 & -4 & 7 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & \frac{11}{5} & \frac{42}{5} \\ 0 & 1 & 0 & \frac{8}{5} & \frac{16}{5} \\ 0 & 0 & 1 & -\frac{1}{5} & \frac{3}{5} \end{pmatrix}.$$

The general solution of the homogeneous system is therefore

$$\begin{pmatrix} x \\ y \\ z \\ s \\ t \end{pmatrix} = \frac{a}{5} \begin{pmatrix} 11 \\ 8 \\ -1 \\ 5 \\ 0 \end{pmatrix} + \frac{b}{5} \begin{pmatrix} 42 \\ 16 \\ 3 \\ 0 \\ 5 \end{pmatrix}.$$

To find a particular solution of the original system we try to guess: we substitute values to some of the variables and solve for the others until we find a solution that works. If we put $z = s = 0$ and $t = 1$ we find that $x = 11$ and $y = 4$ works for all equations. So $(11, 4, 0, 1, 0)$ is a particular solution and so the general solution of the original system is

$$\begin{pmatrix} x \\ y \\ z \\ s \\ t \end{pmatrix} = \frac{a}{5} \begin{pmatrix} 11 \\ 8 \\ -1 \\ 5 \\ 0 \end{pmatrix} + \frac{b}{5} \begin{pmatrix} 42 \\ 16 \\ 3 \\ 0 \\ 5 \end{pmatrix} + \begin{pmatrix} 11 \\ 4 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

□

2. Express the vector $\mathbf{c} = 3\,\mathbf{e}_1 - 2\,\mathbf{e}_2 - \mathbf{e}_3$ as a linear combination of the vectors

$$\begin{aligned} \mathbf{v}_1 &= \mathbf{e}_1 + 2\,\mathbf{e}_2 + 3\,\mathbf{e}_3 \\ \mathbf{v}_2 &= 2\,\mathbf{e}_1 + 3\,\mathbf{e}_2 + \mathbf{e}_3 \\ \mathbf{v}_3 &= 3\,\mathbf{e}_1 + \mathbf{e}_2 + 2\,\mathbf{e}_3. \end{aligned}$$

Answer. The coefficients x, y, z will be solutions of the system

$$\begin{cases} x + 2y + 3z = 3 \\ 2x + 3y + z = -2 \\ 3x + y + 2z = -1 \end{cases}.$$

Working with the augmented matrix we get

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 3 \\ 2 & 3 & 1 & -2 \\ 3 & 1 & 2 & -1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & -\frac{4}{3} \\ 0 & 1 & 0 & -\frac{1}{3} \\ 0 & 0 & 1 & \frac{5}{3} \end{array}\right).$$

So,

$$\mathbf{c} = -\frac{4}{3}\,\mathbf{v}_1 - \frac{1}{3}\,\mathbf{v}_2 + \frac{5}{3}\,\mathbf{v}_3.$$

□

3. Express the vector $\mathbf{c} = 5\,\mathbf{e}_1 - \mathbf{e}_2 + 3\,\mathbf{e}_3$ as a linear combination of the vectors

$$\begin{aligned}\mathbf{v}_1 &= \mathbf{e}_1 - 2\,\mathbf{e}_2 + 3\,\mathbf{e}_3\\ \mathbf{v}_2 &= 4\,\mathbf{e}_1 + \mathbf{e}_2\\ \mathbf{v}_3 &= \mathbf{e}_1 - 11\,\mathbf{e}_2 + 15\,\mathbf{e}_3,\end{aligned}$$

in three different ways.

Answer. The augmented matrix of the system we get reduces to

$$\left(\begin{array}{ccc|c} 1 & 0 & 5 & 1\\ 0 & 1 & -1 & 1\end{array}\right).$$

and the solution is

$$(x, y, z) = t\,(-5, 1, 1) + (1, 1, 0).$$

Setting arbitrarily, $t = 0, \pm 1$, we get three different solutions:

$$\begin{aligned}\mathbf{c} &= \mathbf{v}_1 + \mathbf{v}_2\\ &= -4\,\mathbf{v}_1 + 2\,\mathbf{v}_2 + \mathbf{v}_3\\ &= 6\,\mathbf{v}_1 - \mathbf{v}_3.\end{aligned}$$

□

4. Find a vector $\mathbf{c}$ that cannot be expressed as a linear combination of the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ of the previous exercise.

Solution. From the previous question we know that reduced echelon form of the matrix A with columns $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ has a zero row. If $\mathbf{c}$ is such that the matrix A augmented by $\mathbf{c}$ has at that point transformed to a matrix that has a non-zero entry in that row, the system is inconsistent and therefore $\mathbf{c}$ cannot be expressed as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.

Let's apply the Gauss-Jordan procedure then until we get the zero row.

$$\begin{pmatrix} 1 & 4 & 1\\ -2 & 1 & -11\\ 3 & 0 & 15\end{pmatrix} \sim \begin{pmatrix} 1 & 4 & 1\\ 0 & 9 & -9\\ 0 & -12 & 12\end{pmatrix} \sim \begin{pmatrix} 1 & 4 & 1\\ 0 & 1 & -1\\ 0 & -1 & 1\end{pmatrix} \sim \begin{pmatrix} 1 & 4 & 1\\ 0 & 1 & -1\\ 0 & 0 & 0\end{pmatrix}.$$

We applied, in order, the following row operations:

1. Added 2 times the first row to the second.
2. Added -3 times the first row to the third.
3. Divided the second row by 3.
4. Divided the third row by 4.
5. Added the second row to the third.

If we start with any vector $\mathbf{c}'$ with non-zero third coordinate and apply the *reverse* of the above operations, *in reverse order*, we will get a vector $\mathbf{c}$ that is not in the span of $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. For example starting with $\mathbf{c}' = (1, 3, -2)$ we get

$$(1,3,-2) \sim (1,3,-5) \sim \left(1,3,-\frac{5}{4}\right) \sim \left(1,1,-\frac{5}{4}\right) \sim \left(1,1,\frac{7}{4}\right) \sim \left(1,-1,\frac{7}{4}\right).$$

So, one such $\mathbf{c}$ is

$$\mathbf{c} = \mathbf{e}_1 - \mathbf{e}_2 + \frac{7}{4}\mathbf{e}_3.$$

□

Remark 2.0.1. In the above solution I chose a random vector to illustrate the idea. However there is a much easier choice, I could have chosen $\mathbf{c}' = (0, 0, 4)$. Then only the fourth of the operations affect $\mathbf{c}'$ and as a result I would get $\mathbf{c} = \mathbf{e}_3$.

Note that since $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is not spanning we know that at least one vector from the standard basis (or any basis) of $\mathbb{R}^3$ is not in $\langle \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \rangle$[2].

5. Let

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \qquad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}, \qquad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Let $\mathbf{y} = B\,\mathbf{x}$. Find the vector $\mathbf{z} = A\,\mathbf{y}$.

Solution. We first find $\mathbf{y}$:

$$\mathbf{y} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_{11}\,x_1 + b_{12}\,x_2 \\ b_{21}\,x_1 + b_{22}\,x_2 \end{pmatrix}.$$

Now $\mathbf{z}$:

$$\mathbf{z} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11}\,x_1 + b_{12}\,x_2 \\ b_{21}\,x_1 + b_{22}\,x_2 \end{pmatrix} = \begin{pmatrix} a_{11}\,(b_{11}\,x_1 + b_{12}\,x_2) + a_{12}\,(b_{21}\,x_1 + b_{22}\,x_2) \\ a_{21}\,(b_{11}\,x_1 + b_{12}\,x_2) + a_{22}\,(b_{21}\,x_1 + b_{22}\,x_2) \end{pmatrix}.$$

We now factor x_1 and x_2 to get

$$\mathbf{z} = \begin{pmatrix} (a_{11}\,b_{11} + a_{12}\,b_{21})\;x_1 + (a_{11}\,b_{12} + a_{12}\,b_{22})\;x_2 \\ (a_{21}\,b_{11} + a_{22}\,b_{21})\;x_1 + (a_{21}\,b_{12} + a_{22}\,b_{22})\;x_2 \end{pmatrix}.$$

□

6. Find a 2×2 matrix A that interchanges $\mathbf{e}_1$ and $\mathbf{e}_2$, in other words such that

$$A\,\mathbf{e}_1 = \mathbf{e}_2 \quad \text{and} \quad A\,\mathbf{e}_2 = \mathbf{e}_1.$$

Solution. If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then

$$A\,\mathbf{e}_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \cdot 1 + b \cdot 0 \\ c \cdot 1 + d \cdot 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix},$$

[2] Why?

and

$$A\,\mathbf{e}_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} a \cdot 0 + b \cdot 1 \\ c \cdot 0 + d \cdot 1 \end{pmatrix} = \begin{pmatrix} b \\ d \end{pmatrix}.$$

So we have the following two vector equations

$$\begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} b \\ d \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Thus the matrix is

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

□

7. Prove that if

$$a_1\,b_2 - a_1\,b_3 + a_2\,b_3 - a_3\,b_2 + a_3\,b_1 - a_2\,b_1 \neq 0$$

then the system

$$\begin{cases} x + a_1 y + b_1 z &= c_1 \\ x + a_2 y + b_2 z &= c_2 \\ x + a_3 y + b_3 z &= c_3 \end{cases}$$

has a unique solution for all real numbers c_1, c_2, c_3.

Answer. We proceed to get an echelon form of the augmented matrix of the system.

$$\left(\begin{array}{ccc|c} 1 & a_1 & b_1 & c_1 \\ 1 & a_2 & b_2 & c_2 \\ 1 & a_3 & b_3 & c_3 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & a_1 & b_1 & c_1 \\ 0 & a_2 - a_1 & b_2 - b_1 & c_2 - c_1 \\ 0 & a_3 - a_1 & b_3 - b_1 & c_3 - c_1 \end{array}\right).$$

Now notice that the condition given implies that at least one of $a_2 - a_1$, $a_3 - a_1$ is non-zero. For, if both were zero then we would have $a_1 = a_2 = a_3$ and the LHS of the condition would be zero. Assume then that $a_2 - a_1 \neq 0$ so we can divide the second row by it. Do that and then add $(a_1 - a_3)$ times the second row to the third. The second entry of the third row will then be 0 while the third is

$$b_3 - b_1 + \frac{(a_1 - a_3)(b_2 - b_1)}{a_2 - a_1}.$$

Combining and expanding this will give a fraction with numerator the LHS of the given inequality. Thus the third entry of the third row is non-zero. It follows that the system has a unique solution. □