MTH 42, Fall 2024

Nikos Apostolakis

Final Exam

1. Let $L\colon \mathbb{R}^4 \to \mathbb{R}^3$ be defined by
$$L(x, y, z, t) = (x - y + z + t, x + 2\,z - t, x + y + 3\,z - 3\,t).$$
   (a) Prove that L is linear.
   (b) Find a basis for the range of L.
   (c) Find a basis for the kernel of L.

2. Find A^{-1} where
$$A = \begin{pmatrix} 1 & 4 & 2 & 0 \\ 4 & 2 & 0 & 3 \\ 3 & 1 & 6 & 1 \\ 0 & 4 & 4 & 5 \end{pmatrix}$$
is a matrix in $\mathbb{Z}/7$.

3. Find all complex numbers z so that the vectors
$$\mathbf{v}_1 = (z, 0, 1), \quad \mathbf{v}_1 = (0, 1, z^3), \quad \mathbf{v}_3 = (1, 1, 1 + z)$$
do not form a basis of $\mathbb{C}^3$.

4. Consider $\mathbb{C}$ as a vector space over $\mathbb{R}$. For an arbitrary $z \in \mathbb{C}$ define
$$L_z\colon \mathbb{C} \to \mathbb{C}, \quad L_z(w) = z\,w.$$
   (a) Prove that L_z is a linear map.
   (b) Find the matrix of L_z with respect to the basis $\{1, i\}$ of $\mathbb{C}$.

5. Let $\mathbf{M}_2$ be the vector space of real matrices and let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$. Consider the function $L\colon \mathbf{M}_2 \to \mathbf{M}_2$ given by
$$L(X) = A\,X - X\,A.$$
   (a) Prove that L is linear map.
   (b) Find the matrix of L with respect to the standard basis of $\mathbf{M}_2$.
   (c) Find a basis for the kernel of L.

6. Let $B = \{42, 3\,x - 2, 5\,x^2 + x - 3, x^3 - 4\,x\}$. Prove that B is a basis of $\mathbf{P}_3$, the vector space of real polynomials of degree at most 3.

   Hint. Consider the isomorphism $L\colon \mathbf{P}_3 \to \mathbb{R}^4$ that sends the standard basis of $\mathbf{P}_3$ to the standard basis of $\mathbb{R}^4$, and work with $L(B)$.

7. Compute the following determinants.

A. $\begin{vmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \\ 8 & 9 & 1 \end{vmatrix}$ B. $\begin{vmatrix} 1 & 2 & 3 \\ 1 & 4 & 9 \\ 1 & 8 & 27 \end{vmatrix}$ C. $\begin{vmatrix} 1 & -2 & 3 & 4 \\ 4 & 1 & -2 & 3 \\ 3 & 4 & 1 & -2 \\ -2 & 3 & 4 & 1 \end{vmatrix}$ D. $\begin{vmatrix} 5 & 4 & 2 & 1 \\ 2 & 3 & 1 & -2 \\ -5 & -7 & -3 & 9 \\ 1 & -2 & -1 & 4 \end{vmatrix}$ E. $\begin{vmatrix} 6 & 2 & 1 & 0 & 5 \\ 2 & 2 & 1 & -2 & 1 \\ 1 & 1 & 2 & -2 & 3 \\ 3 & 0 & 2 & 3 & -1 \\ -1 & -1 & -3 & 4 & 2 \end{vmatrix}$

8. Let $x_1, x_2, \ldots, x_n$ be scalars, and consider the *Vandermonde matrix*:

$$V(x_1, \ldots, x_n) = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ x_1 & x_2 & x_3 & \cdots & x_n \\ x_1^2 & x_2^2 & x_3^2 & \cdots & x_n^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_1^{n-1} & x_2^{n-1} & x_3^{n-1} & \cdots & x_n^{n-1} \end{pmatrix}.$$

Prove that

$$\det V(x_1, \ldots, x_n) = \prod_{1 \le i < j \le n} (x_j - x_i).$$

Hint. For $n = 2$ the formula says

$$\begin{vmatrix} 1 & 1 \\ x_1 & x_2 \end{vmatrix} = x_2 - x_1$$

and it is clearly true.

For $n = 3$ the formula says

$$\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ x_1^2 & x_2^2 & x_3^2 \end{vmatrix} = (x_3 - x_2)(x_3 - x_1)(x_2 - x_1).$$

To see this, subtract x_3 times the second row from the third, and x_3 times the first row from the second, and then expand along the third column

$$\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ x_1^2 & x_2^2 & x_3^2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ x_1 - x_3 & x_2 - x_3 & 0 \\ x_1^2 - x_1 x_3 & x_2^2 - x_3 x_2 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ x_1 - x_3 & x_2 - x_3 & 0 \\ x_1 (x_1 - x_3) & x_2 (x_2 - x_3) & 0 \end{vmatrix}$$

$$= . \begin{vmatrix} x_1 - x_3 & x_2 - x_3 \\ x_1 (x_1 - x_3) & x_2 (x_2 - x_3) \end{vmatrix}$$

$$= .(x_1 - x_3)(x_2 - x_3) \begin{vmatrix} 1 & 1 \\ x_1 & x_2 \end{vmatrix}$$

$$= (x_3 - x_1)(x_3 - x_2)(x_2 - x_1).$$

In the last line we changed $x_1 - x_3$ and $x_2 - x_3$ to $x_3 - x_1$ and $x_3 - x_2$ respectively. Since we changed the sign of two factors the product is the same.

Now for $n = 4$ we can proceed similarly. Subtracting x_4 times the i-th row from the $(i+1)$-row for $i = 3, 2, 1$ we get

$$\det V(x_1, x_2, x_3, x_4) = \begin{vmatrix} 1 & 1 & 1 & 1 \\ x_1 - x_4 & x_2 - x_4 & x_3 - x_4 & 0 \\ x_1 (x_1 - x_4) & x_2 (x_2 - x_4) & x_3 (x_3 - x_4) & 0 \\ x_1^2 (x_1 - x_4) & x_2^2 (x_2 - x_4) & x_3^2 (x_3 - x_4) & 0 \end{vmatrix}$$

Expanding along the last column we get

$$\begin{aligned}\det V(x_1,x_2,x_3,x_4) &= -\begin{vmatrix} x_1-x_4 & x_2-x_4 & x_3-x_4 \\ x_1\,(x_1-x_4) & x_2\,(x_2-x_4) & x_3\,(x_3-x_4) \\ x_1^2\,(x_1-x_4) & x_2^2\,(x_2-x_4) & x_3^2\,(x_3-x_4)\end{vmatrix} \\ &= -(x_1-x_4)\,(x_2-x_4)\,(x_3-x_4)\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ x_1^2 & x_2^2 & x_3^2\end{vmatrix} \\ &= (x_4-x_1)\,(x_4-x_2)\,(x_4-x_3)\,\det V(x_1,x_2,x_3).\end{aligned}$$

and the result follows from the $n=3$ case. Notice that in the last line we have changed the sign of three factors, and thus we get an overall factor of -1 that cancels the negative sign from the expansion.

Prove the general case by induction. For the inductive step proceed as above and show that

$$\det V(x_1,\ldots,x_{n+1}) = (x_{n+1}-x_1)\,\cdots\,(x_{n+1}-x_n)\,\det V(x_1,\ldots,x_n).$$

9. For a positive integer n, let $x_1,\ldots,x_n$ be *pairwise distinct*[1] real numbers, and let $y_1,\ldots,y_n$ be any real numbers. Prove that there is a unique polynomial $p(x)$ of degree at most $n-1$ such that $p(x_i)=y_i$ for $i=1,\ldots,n$. That is prove that there exist unique real numbers $c_0,c_1,\ldots,c_{n-1}$ such that for $i=1,\ldots,n$ we have

$$c_0 + c_1\,x_i + c_2\,x_i^2 + \ldots + c_{n-1}\,x_i^{n-1} = y_i.$$

Hint. The coefficients of the polynomial satisfy a system of linear equations whose matrix is the transpose of the Vandermonde matrix of Question 8. Use the formula in that question to prove that the determinant of the matrix is non-zero.

10. Prove that a square matrix A is invertible if and only if 0 is not an eigenvalue of A.

11. Consider the matrix

$$A = \begin{pmatrix} 1 & 6 & -2 \\ -3 & 2 & 0 \\ 0 & 3 & -4 \end{pmatrix}.$$

   (a) Find $\mathrm{ch}_A(x)$, the characteristic polynomial of A.

   (b) Verify that $\mathrm{ch}_A(A) = O$.

   (c) Use Part (b) to find A^{-1}.
   Note. You have to use Part (b), if you find A^{-1} using a different method you won't receive any credit for this part of the question.

12. Consider the matrix

$$X = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 8 & -6 \\ 0 & 0 & 3 & -5 \\ 0 & 0 & 0 & 4 \end{pmatrix}.$$

   (a) Find a matrix P such that $P^{-1}\,X\,P$ is diagonal.

   (b) Use Part (a) to compute X^{-1}.

[1] i.e. $x_i \neq x_j$ for $i \neq j$.