Review Midterm

Problem (3) $R_{(a,b)}(u,v) = (u+va, vb)$

$$F_1 = \left(R_{(u,v)}\right)_* \left(\frac{\partial}{\partial u}\Big|_{(0,1)}\right) = \frac{\partial}{\partial u}\Big|_{(u,v)}$$

$$\cancel{\begin{bmatrix} 1 & 0 \\ u & v \end{bmatrix}} \quad \begin{bmatrix} 1 & u \\ 0 & v \end{bmatrix}$$

$$F_2 = u\frac{\partial}{\partial u}\Big|_{(u,v)} + v\frac{\partial}{\partial v}\Big|_{(u,v)}$$

↳ $[F_1, F_2] = F_1$ (with left translations –)

↳ $g \Rightarrow g_{11} = \langle F_1, F_1 \rangle = 1$

$$g_{12} = g\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\right) = \left\langle F_1, \frac{1}{v}(F_2 - uF_1) \right\rangle = -\frac{u}{v}$$

$$g_{22} = g\left(\frac{\partial}{\partial v}, \frac{\partial}{\partial v}\right) = \left\langle \frac{1}{v}(F_2 - uF_1), \frac{1}{v}(F_2 - uF_1) \right\rangle = \frac{1+u^2}{v^2}$$

↳ $g = du^2 - \frac{u}{v}\, du \odot dv + \frac{1+u^2}{v^2}\, dv^2$

Notice that $(x,y)^{-1} = \left(-\frac{x}{y}, \frac{1}{y}\right)$ (Right inverse)

So the inverse map $= (u,v) = \left(\frac{-x}{y}, \frac{1}{y}\right)$

So

$$du^2 - \frac{u}{v}\, du \odot dv + \frac{1+u^2}{v^2}\, dv^2 = \left(-\frac{1}{y}dx + \frac{x}{y^2}dy\right)^2$$

$$+ x\left(-\frac{1}{y}dx + \frac{x}{y^2}dy\right) \odot \left(-\frac{1}{y^2}dy\right) + \left(\frac{y^2}{y^4} + \frac{x^2}{y^4}\right) dy^2$$

$$= \frac{1}{y^2}dx^2 - \cancel{\frac{x}{y^3}\, dx \odot dy} + \cancel{\frac{x^2}{y^4}\, dy^4} + \cancel{\frac{x}{y^3}\, dx \odot dy}$$

$$- \cancel{\frac{2x^2}{y^4}\, dy^2} + \left(\frac{1}{y^2} + \cancel{\frac{x^2}{y^4}}\right) dy^2$$

$$= \frac{dx^2 + dy^2}{y^2}$$ The inverse map is AN isometry

Problem 4 $\left(\frac{\partial}{\partial x}\right)^{\flat} = g\left(\frac{\partial}{\partial x}, \cdot\right) = x^2 dx + xy\, dy$

$g\left((dy)^{\#}, \cdot\right) = dy \Rightarrow$

$$g^{-1} = \begin{bmatrix} y^2 & -\frac{1}{\sqrt{2}}xy \\ -\frac{1}{\sqrt{2}}xy & x^2 \end{bmatrix} \cdot \frac{2}{x^2y^2} = \begin{bmatrix} \frac{2}{x^2} & -\frac{\sqrt{2}}{xy} \\ -\frac{\sqrt{2}}{xy} & \frac{2}{y^2} \end{bmatrix}$$

$$(dy)^{\#} = \sum_{i,j} g^{ij}\omega_j \frac{\partial}{\partial x_i} = g^{12}\frac{\partial}{\partial x} + g^{22}\frac{\partial}{\partial y}$$

$$\Rightarrow (dy)^{\#} = \frac{-\sqrt{2}}{xy}\frac{\partial}{\partial x} + \frac{2}{y^2}\frac{\partial}{\partial y}$$

$g(dx, dy) = dx\,(dy^{\#}) = dx\left(\frac{-\sqrt{2}}{xy}\frac{\partial}{\partial x} + \frac{2}{y^2}\frac{\partial}{\partial y}\right) = \frac{-\sqrt{2}}{xy}$

$g(dx\wedge dy, dx\wedge dy) = g(dx, dx)g(\cancel{dy}, dy) - g(dx, dy)$

$g(dx, dx) \Rightarrow (dx)^{\#} = g^{11}\frac{\partial}{\partial x} + g^{21}\frac{\partial}{\partial y} = \frac{2}{x^2}\frac{\partial}{\partial x} - \frac{\sqrt{2}}{xy}\frac{\partial}{\partial y}$

$g(dy, dy) = g(dy^{\#}, dy^{\#})$

Problem 5

even permutation of $(1,2,3)$: $(1,2,3)$ $(3,1,2)$ $(2,3,1)$

odd permutation of $(1,2,3)$: $\cancel{(3,2,1)}$ $(2,1,3)$ $(3,2,1)$ $(1,3,2)$

permutation $\longrightarrow \Gamma^{i}_{JK} \to \cancel{\Gamma^{1}_{23}} = \cancel{\Gamma^{1}_{32}} = \cancel{\Gamma^{2}_{31}} = 1$ $\quad \cancel{\Gamma^{3}_{21} = -1}$

$\longrightarrow \cancel{\Gamma^{3}_{12}} = \cancel{\Gamma^{2}_{13}} = \cancel{\Gamma^{1}_{23}} = 1 \quad \cancel{\Gamma^{1}_{32}} = \cancel{-1}$

By linearity It's sufficient to consider $\frac{\partial}{\partial x_i} = X$

$$\langle \nabla_X Y, Z\rangle = \left\langle \nabla_{\frac{\partial}{\partial x_i}} \sum Y^J \frac{\partial}{\partial x_J}, Z\right\rangle$$

$$= \left\langle \sum_{J=1}^{3} \frac{\partial Y^J}{\partial x_i}\frac{\partial}{\partial x_J} + \sum_{J=1}^{3} Y^J \sum_{K=1}^{3} \Gamma^{K}_{iJ}\frac{\partial}{\partial x_K}, Z\right\rangle$$

$$= \sum_{K=1}^{3} Z^K\left(\frac{\partial Y^K}{\partial x_i} + \sum_{J=1}^{3} Y^J \Gamma^{K}_{iJ}\right)$$

same way $\langle Y, \nabla_X Z\rangle = \sum_{K=1}^{3} Y^K\left(\frac{\partial Z^K}{\partial x^i} + \sum_{J=1}^{3} Z^J \Gamma^{K}_{iJ}\right)$

So $\langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle = \sum_{K=1}^{3} Z^K \frac{\partial Y^K}{\partial x^i} + Y^K\frac{\partial Z^K}{\partial x^i} + \underbrace{\sum_{J,K=1}^{3}\left(Y^J Z^K + Z^J Y^K\right)\Gamma^{K}_{iJ}}_{=0}$

$$= \sum Z^K \frac{\partial Y^K}{\partial x_i} + Y^K\frac{\partial Z^K}{\partial x_i} = \frac{\partial}{\partial x_i}\langle Y, Z\rangle$$

b) The Equations For Geodesics

$$\ddot{x}_K + \underbrace{\sum_{i,J}^{3} \Gamma_{iJ}^{k} \dot{x}_i \dot{x}_J}_{=0} = 0 \quad \text{For } K=1,2,3$$

$\Rightarrow \ddot{x}_k = 0 \rightsquigarrow$ straight lines

c) Torsion

$$T = \sum_{i,J,K=1}^{3} \left(\Gamma_{iJ}^{K} - \Gamma_{Ji}^{K} \right) dx_i \otimes dx_J \otimes \frac{\partial}{\partial x_K}$$

$$= \sum \omega \left(\varepsilon_{KiJ} - \varepsilon_{KJi} \right) dx_i \otimes dx_J \otimes \frac{\partial}{\partial x_K}$$

$$= \sum_{i,J,K} 2 \omega \, \varepsilon_{iJK} \, dx_i \otimes dx_J \otimes \frac{\partial}{\partial x_K}$$

$\Rightarrow T = 0$ iff $\omega = 0$.

3) $\nabla = D^g$

$$2\langle f_* \nabla_X Y, f_* Z \rangle_{f(p)} = 2\langle \nabla_X Y, Z \rangle_p \overset{\text{Koszul}}{=} X_p \langle Y, Z \rangle + Y_p \langle Z, X \rangle - Z_p \langle X, Y \rangle - \langle [Y,Z], X \rangle_p - \langle [X,Z], Y \rangle_p - \langle [Y,X], Z \rangle_p$$

$(*)$

On other hand

$$2\langle D^h_{f_* X} f_* Y, f_* Z \rangle_{f(p)} \overset{\text{Koszul}}{=} (f_* X)_{f(p)} \langle f_* Y, f_* Z \rangle + \dots - \langle [f_* Y, f_* Z], f_* X \rangle_{f(p)}$$

$$= X_p \cdot \left(\langle f_* Y, f_* Z \rangle \circ f \right) - \langle f_* [Y,Z], f_* X \rangle_{f(p)} = (*)$$

b) $c: I \to M$ geodesic

$\tilde{c} = f \circ c : I \to N$

$\dot{\tilde{c}}(t) = \dfrac{d(f \circ c)}{dt} = (df)_{c(t)} \dot{c}$

we have $D^h_{\dot{\tilde{c}}} \dot{\tilde{c}} = D^h_{(df)_{c(t)} \dot{c}} \left((df)_{c(t)} \dot{c} \right) \overset{a)}{=} (df)_{c(t)} \left(D^g_{\dot{c}} \dot{c} \right) = 0.$

Problem 7

1) $$g = d\theta^2 + \sin^2(\theta)\, d\alpha^2$$

$$g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix} \qquad g^{ij} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\sin^2\theta} \end{pmatrix}$$

ONLY NON Zero

$$\Gamma^\theta_{\varphi\varphi} = 1/2 \sum_{i=1}^{2} g^{\theta i} \left(\frac{\partial g_{\alpha i}}{\partial \alpha} + \frac{\partial g_{\alpha i}}{\partial \alpha} - \frac{\partial g_{\alpha\alpha}}{\partial x_i} \right)$$

$$= \frac{1}{2} g^{\theta\theta} \left(- \frac{\partial (\sin^2\theta)}{\partial \theta} \right) = - \sin\theta\cos\theta$$

$$\Gamma^\alpha_{\theta\alpha} = \Gamma^\alpha_{\alpha\theta} = \cot\theta \qquad \left(\Gamma^k_{ij} = 1/2 \sum_\ell g^{k\ell} \left(\frac{\partial}{\partial x_i} g_{\ell j} + \frac{\partial}{\partial x_j} g_{i\ell} - \frac{\partial}{\partial x_\ell} g_{ij} \right) \right.$$

2) Geodesic Equations $\left(\ddot{a}_k + \sum_{ij} \dot{a}_i \dot{a}_j \Gamma^k_{ij} \right.$

$$\ddot{\theta} + \sum_{i,j=1}^{2} \Gamma^\theta_{ij} \dot{x}^i \dot{x}^j = 0 \iff \ddot{\theta} - \sin\theta\cos\theta\, \dot{\varphi}^2 = 0$$

$$\ddot{\varphi} + \sum_{i,j=1}^{2} \Gamma^\alpha_{ij} \dot{x}^i \dot{x}^j = 0 \iff \ddot{\varphi} + 2\cot\theta\, \dot{\theta}\dot{\varphi} = 0$$

The curve $\gamma(t) = (\theta(t), \alpha(t)) = (\frac{\pi}{2}, t)$ is sol of The Equations $\Rightarrow$ Equator $\Rightarrow$ $\theta = \frac{\pi}{2}$ image of geodesic

3) Any Rotation about an Axis through origin in $\mathbb{R}^3$ is an isometry of $\mathbb{R}^3$ which preserves S^2
Since the Metric is induced by $\mathbb{R}^3 \Rightarrow$ isometry of S^2

Problem 8

①

$$g = \frac{1}{y^2}(dx^2+dy^2)$$

$$g^{ij} = \begin{pmatrix} y^2 & 0 \\ 0 & y^2 \end{pmatrix} \Rightarrow \Gamma^x_{xy} = \frac{1}{2}\sum g^{xi}\left(\frac{\partial g_{yi}}{\partial x} + \frac{\partial g_{xi}}{\partial y} - \frac{\partial g_{x}}{\partial x}\right.$$

$$= \frac{1}{2}g^{xx}\left(\frac{\partial}{\partial y}\left(\frac{1}{y^2}\right)\right) = -\frac{1}{y}$$

$$\Gamma^x_{yx} = -\Gamma^y_{xx} = \Gamma^y_{yy} = -\frac{1}{y}$$

②

$$\ddot{x} + \sum_{i,j=1}^{2} \Gamma^x_{ij}\dot{x}^i\dot{x}^j = 0 \Leftrightarrow \ddot{x} - \frac{2}{y}\dot{x}\dot{y} = 0$$

$$\ddot{y} + \sum_{i,j=1}^{2} \Gamma^y_{ij}\dot{x}^i\dot{x}^j = 0 \Leftrightarrow \ddot{y} + \frac{1}{y}\dot{x}^2 - \frac{1}{y}\dot{y}^2 = 0$$

$\alpha(t) = (0, e^t) \Rightarrow x=0 \quad y = e^t$

Since $\tanh^2 t + \frac{1}{\cosh t^2} = 1 \Rightarrow x^2+y^2=1$

$(\tanh t)' = 1 - (\tanh t)^2$ $\qquad \cosh^2 - \sin$

$(\tanh t)'' = 2\tanh t\,(1-\tanh t)^2$

$\left(\frac{1}{\cosh t}\right)' = \frac{-\sinh t}{\cosh^2 t}$ $\qquad \left(\frac{1}{\cosh t}\right)'' = \dots$