Lecture I 01/27/25

Riemannian Metrics

Let M^n be a smooth manifold. TM Tangent bundle to M

A Riemannian metric g on a smooth manifold M is the assignment of an inner product g_p to T_pM for every $p \in M$ such that the function $M \to \mathbb{R}$ defined by

$$p \to g_p(X(p), Y(p)) \text{ is smooth}$$

Here $X, Y \in \mathfrak{X}(M)$ (smooth vector fields on M i.e. smooth sections of $TM \to M$)

In particular: since g_p inner product

Symmetry $g_p(X(p), Y(p)) = g_p(Y(p), X(p))$ we can actually omit p.

positive definite $g_p(X(p), X(p)) \geq 0$ and "$=0$" iff $X(p)=0$

Bilinear. $g_p(aX(p) + bY(p), Z(p)) = a\, g_p(X(p), Z(p)) + b\, g_p(Y(p), Z(p))$

Remark

Since g_p is bilinear the metric g can be seen as a smooth (symmetric) section of the vector bundle pointwise positive definite

$T^*M \otimes T^*M \to M$ where T^*M is cotangent bundle

$(T^*M \otimes T^*M = \bigcup_{p \in M} T_p^*M \otimes T_p^*M$ ← tensor product of vector spaces

Endowed with g we call (M^n, g) A Riemannian Manifold

In a chart $(x_1, \dots, x_n)$ with $T_pM = \text{span}\left\{\frac{\partial}{\partial x_1}(p), \dots, \frac{\partial}{\partial x_n}(p)\right\}$

we write $\quad g_p = \sum_{i,j} g_{ij}(p)\ dx_i(p) \otimes dx_j(p)$

(here $g_{ij} = g_{ji} = g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)$ are smooth Functions

and $\{dx_i(p)\}$ is dual basis (i.e basis of T_p^*M).

Examples

① If $M = \mathbb{R}^n$ The Euclidean Metric is The one obtained in Cartesian coordinates by

$$g = \sum_{i=1}^{n} dx_i \otimes dx_i \quad (\text{here } g_{ij} = \delta_{ij}).$$

in $\mathbb{R}^2$ we have $g = dx \otimes dx + dy \otimes dy = \underbrace{dx^2 + dy^2}_{\text{abbreviated to}}$

we can change coordinates (using $dx = \frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv$).
For Example in polar coordinates:

$$g = (\cos\theta\, dr - r\sin\theta\, d\theta)^2 + (\sin\theta\, dr + r\cos\theta\, d\theta)^2$$

$$\to g = dr^2 + r^2 d\theta^2 \quad (*)$$

For instance

$$g\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right) = 1 \text{ and } g\left(\frac{\partial}{\partial \theta}, \frac{\partial}{\partial \theta}\right) = r^2 \text{ and } g\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}\right) = 0$$

where $\frac{\partial}{\partial r} = \cos\theta \frac{\partial}{\partial x} + \sin\theta \frac{\partial}{\partial y}$ $\quad \frac{\partial}{\partial \theta} = -r\sin\theta \frac{\partial}{\partial x} + r\cos\theta \frac{\partial}{\partial y}$

② Given a Riemannian manifold (M^n, g) and $f: \underset{\substack{\uparrow \\ \text{smooth}}}{N} \to M$ an immersion (i.e. $Df_p: T_pN \to T_pM$ is injective $\forall p \in N$)

Then

$$h_p(X(p), Y(p)) := g_{f(p)}\big(Df_p(X(p)), Df_p(Y(p))\big)$$

defines a Riemannian metric on N.

③ S^1: we use angle coordinate θ

radius 1 ✓ (Think of $\theta: (0, 2\pi) \to S^1$ as a chart) Need 2 such charts to cover S^1

$S^1 \to \mathbb{R}^2$
$\theta \mapsto (\cos\theta, \sin\theta)$
immersed subm of $\mathbb{R}^2$

we have $g = d\theta \otimes d\theta = d\theta^2$

or back to $(*)$ and replaces $r=1$ ($dr = 0$ because r is constant)

$\downarrow$

g round metric induced by the Euclidean metric

S^1 is an immersed submanifold of $\mathbb{R}^2$ (injective immersion)

④ S^2: we use the parametrization

unit sphere $r=1$

$$\begin{cases} Z = \cos\theta \\ X = \cos\alpha \sin\theta \\ Y = \sin\theta \sin\alpha \end{cases}$$

so the induced metric on S^2 by the Euclidean is

compute g! $\quad g = d\theta^2 + \sin^2\theta \, d\alpha^2 \quad$ (generalized to S^n)

As immersed submanifold $\iota: S^2 \to \mathbb{R}^3$

$$\iota(\theta, \alpha) = (\sin\theta\cos\alpha, \sin\theta\sin\alpha, \cos\theta) \qquad \left(\begin{matrix} D\iota \\ \| \\ \iota_* = [\,\vdots\,] \end{matrix}\right.$$

$$\iota_*\left(\frac{\partial}{\partial\theta}\right) = \cos\theta\cos\alpha \frac{\partial}{\partial x} + \cos\theta \sin\alpha \frac{\partial}{\partial y} - \sin\theta \frac{\partial}{\partial z}$$

$$\iota_*\left(\frac{\partial}{\partial\alpha}\right) = -\sin\theta\sin\alpha \frac{\partial}{\partial x} + \sin\theta\cos\alpha \frac{\partial}{\partial y} \to g\left(\frac{\partial}{\partial\theta}, \frac{\partial}{\partial\theta}\right) = 1 \text{ etc}$$

Back to S^1: $g = d\theta^2$
we can also have
$h = f(\theta)^2 d\theta^2$ another Metric where
$f: S^1 \to \mathbb{R}$ is positive
eg. $f(\theta) = 2 + \sin\theta$.

Are The circles (S^1, g) and (S^1, h) the same?

Def The Riemannian Manifolds (M^n, g) and (N^n, h) are isometric if there is a diffeomorphism $\phi: (M^n, g) \to (N^n, h)$ (ϕ bijection, ϕ and ϕ^{-1} are differentiable) such that $\forall p \in M$ $\forall X, Y \in \mathfrak{X}(M)$

$$h_{\phi(p)}\left(Df_p(X(p)), Df_p(Y(p))\right) = g_p(X(p), Y(p))$$

we say $g = \phi^* h$

Such ϕ is called an isometry!

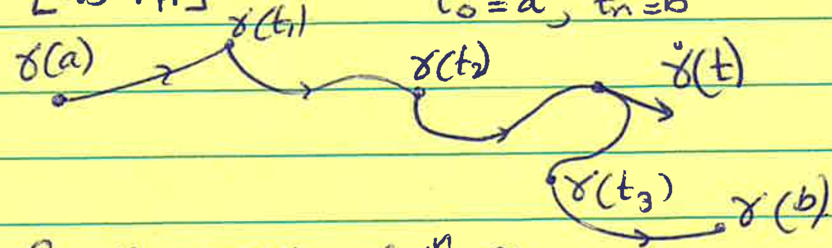
Idea Roughly Isometry is a Transformation that preserves distance between points (preserves shape, size)

→ Two Riemannian Manifolds are "The same" if They are isometric.

→ Isometries preserves distances, volume, curvature, scalar curvature: Anything that depends on Metric

How Do we Measure distances on Riemannian Manifold?

Length Let $\gamma : [a,b] \longrightarrow (M^n, g)$ be A piecewise C^1 curve
i.e $\gamma /_{[t_i, t_{i+1}]}$ is C^1 $(i = 0, \dots, n)$
$t_0 = a$, $t_n = b$



The length of γ in (M^n, g) is

$$L_g(\gamma) = \int_a^b g(\dot{\gamma}(t), \dot{\gamma}(t))^{1/2}\, dt$$

Exercise ① Show that L_g is invariant UNDER Reparametrization
i.e if $\eta : [a,b] \to [c,d]$ is A diffeomorphism then

$$L_g(\gamma) = L_g(\gamma \circ \eta)$$

Exercise ② Suppose that $\phi : (M^n, g) \longrightarrow (M^n, h)$ is AN isometry AND $\gamma : [a,b] \to M$ is A piecewise C^1 curve
show that $L_g(\gamma) = L_h(\phi \circ \gamma)$

$\gamma : [0, 2\pi] \longrightarrow S^1$

$\gamma(\theta) = (\cos\theta, \sin\theta)$ $\quad \dot{\gamma}(\theta) = \frac{\partial}{\partial\theta}$

$$L_g(\gamma) = \int_0^{2\pi} g\left(\frac{\partial}{\partial\theta}, \frac{\partial}{\partial\theta}\right)^{1/2} d\theta = 2\pi$$

$$L_h(\gamma) = \int_0^{2\pi} h\left(\frac{\partial}{\partial\theta}, \frac{\partial}{\partial\theta}\right)^{1/2} d\theta = \int_0^{2\pi} 2 + \cos\theta \, d\theta = 4\pi$$

$\to$ (S^1, g) AND (S^1, h) Not isometric