Lecture 10

Def The Exponential map of (M,g) at $p\in M$ is

$$\exp_p : \Omega_p \subset T_pM \to M$$
$$\exp_p(v) = \gamma_v(1) \qquad (\text{run along the geodesic})$$

where $\gamma_v(t)$ is the (unique) geodesic in (M,g) with

$\begin{cases} \gamma_v(0) = p \\ \dot\gamma_v(0) = v \end{cases}$ and Ω_p is open subset of $v \in T_pM$ st $\gamma_v(t)$ is defined at least up to $t=1$

Note If $\gamma(t)$ is a geodesic then so is $\alpha(t) = \gamma(at+b)$ for any $a\neq 0$, $b\in\mathbb{R}$

affine Reparametrization

Proof $\dot\alpha(t) = a\,\dot\gamma(at+b)$

so $D^g_{\dot\alpha}\dot\alpha = D^g_{a\dot\gamma(at+b)}\, a\dot\gamma(at+b) = a^2\, D^g_{\dot\gamma}\dot\gamma = 0$

Note $\begin{cases} \gamma(t_0) = p \\ \dot\gamma(t_0) = v \end{cases} \overset{(*)}{\Longleftrightarrow} \begin{cases} \alpha(t_1) = p \\ \dot\alpha(t_1) = av \end{cases} \quad (t_0 = at_1 + b)$

Initial conditions are the same up to Rescaling the initial condition

By (*) $\gamma_{sv}(t) = \gamma_v(st)$ provided $|t|, |s|$ are sufficiently SMALL
$\uparrow$ ($\alpha_{av}(t) = \gamma_v(at)$)

Thus $\forall v \in T_pM \cong T_0(T_pM)$ $\quad$ ($\exp_p : \Omega \subset T_pM \to M$)

$$d(\exp_p)_0 v = \frac{d}{dt} \exp_p(tv)\Big|_{t=0} \underset{\text{def of exp}}{=} \frac{d}{dt} \gamma_{tv}(1)\Big|_{t=0}$$

$\gamma_{sv}(t) = \gamma_v(st) \rightarrow$ $= \frac{d}{dt}\gamma_v(t)\Big|_{t=0} = \dot{\gamma}_v(0) = v.$

ie $d(\exp_p)_0 = Id$. Thus by The inverse Function Theorem

$\exists$ open neighborhoods $U \ni 0$ in T_pM and $V \ni p$ in M s.t
$(\exp_p)|_U : U \to V$ is diffeomorphism.

This defines a local chart around $p \in M$ whose coordinates are called "NORMAL coordinates"
$\uparrow T_pM \cong \mathbb{R}^n$ by choosing an g-orthonormal basis

Properties of NORMAL coordinates:

$$\begin{array}{ccc} & & \mathbb{R}^n \\ & & \text{\rotatebox{90}{$\cong$}} \\ x = (x_1, \dots, x_n) : & V \longrightarrow & U \\ & \text{\rotatebox{90}{$\subset$}} & \subset T_pM \\ & M & \end{array} \quad \text{s.t } x^{-1} = (\exp_p)|_U$$

- $x(\gamma_v(t)) = tv$ $\forall v \in T_pM$ $|t|$ small (because $\exp_p(tv) = \gamma_{tv}(1) = \gamma_v(t)$ and $x^{-1} = (\exp_p)|_U$)

- $x(p) = 0$

- $g_{ij}(p) = \delta_{ij}$

we identify tangent space to T_pM at $v \in T_pM$ with T_pM itself we have by constuction :

$\partial_i|_p \longrightarrow d(\exp_p)_0 e_i = e_i$ where $\{e_i\}$ orthonormal basis of T_mM.

$\hookrightarrow g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)\Big|_m = \delta_{ij}$

- $\Gamma^K_{ij}(p) = 0$ because the geodesic γ_v associated to $v = \sum v^i e_i$ is given in local coordinates by $x(\gamma_v(t)) = tv\}$. The differential equation satisfied by geodesics gives for any i:

$x(t) = (tx^1, \dots, tx^n)$

$$\ddot{x}_i + \sum_{J,K} \Gamma^i_{JK}(\gamma_v(t))\, \dot{x}_J \dot{x}_K = 0$$

At $t=0 \Rightarrow \forall i$: $\sum_{J,K} \Gamma^i_{JK}(p)\, v^i v^k = 0$ for any $v \in T_pM$.

$\Rightarrow \Gamma^i_{JK}(p) = 0$.