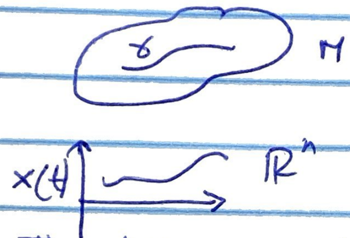
Lecture 9

Def A C^2 curve $\gamma:(a,b)\to(M,g)$ in a Riemannian Manifold is a geodesic if it's a geodesic for Levi-Civita connection of g i.e. $D^g_{\dot\gamma}\dot\gamma=0$.

Note The geodesic equation in a chart $\{x_1,\dots,x_n\}$ is given

$x=(x_1,\dots,x_n)$ by $\ddot x_k+\sum_{i,j}\dot x_i\dot x_j\Gamma^k_{ij}=0 \qquad k=1,\dots,n.$



and Γ^k_{ij} for the LC connection can be written as functions of g_{ij} and $\frac{\partial}{\partial x_\ell}g_{ij}$ so geodesics are determined by g.

Prop If γ is a geodesic in (M,g) then $g(\dot\gamma,\dot\gamma)^{1/2}$ is constant

Proof $\frac{d}{dt}g(\dot\gamma,\dot\gamma)\underset{\substack{\uparrow\\ \text{Metric compatibility}}}{=}2g(D^g_{\dot\gamma}\dot\gamma,\dot\gamma)=0 \Rightarrow g(\dot\gamma,\dot\gamma)$ constant along γ.

Def A vector field $X\in\mathfrak{X}(M)$ is Killing on (M,g) if its flow $\phi_t:M\to M$ is 1-parameter subgroup of $\mathrm{Isom}(M,g)$

Prop IF $X \in \mathfrak{X}(M)$ is A Killing v.F of (M, g) i.e $\mathcal{L}_X g = 0$ Then $g(X, \dot\gamma)$ is constant along Any geodesic γ.

Proof $\mathcal{L}_X g = 0 \Rightarrow X \cdot (g(Y, Z)) = g([X, Y], Z) + g(Y, [X, Z])$

$\Rightarrow X \cdot (g(Y, Z)) = g(D^g_X Y, Z) + g(Y, D^g_X Z) - g(D^g_Y X, Z) - g(Y, D^g_Z X)$

$g=0$ $\Rightarrow g(D^g_Y X, Z) = -g(D^g_Z X, Y) \Rightarrow (D^g X)$ is anti,

So $\frac{d}{dt} g(X, \dot\gamma) = g(D^g_{\dot\gamma} X, \dot\gamma) + g(X, D^g_{\dot\gamma} \dot\gamma) = 0.$

Note $\|X\|_g$ Need Not be constant along γ.

$$g(X, \dot\gamma) = \|X\|_g \underset{\substack{\uparrow \\ \text{constant}}}{\|\dot\gamma\|_g} \cos\theta$$

Also $\frac{X}{\|X\|_g}$ is Not in general Killing if $\|X\|_g$ Not constant.

Ex $(a,b) \times S^1, \; dr^2 + f(r)^2 d\theta^2)$

$X = \frac{\partial}{\partial\theta}$ is Killing $\Rightarrow \mathcal{L}_{\frac{\partial}{\partial\theta}} dr = d\mathcal{L}_{\frac{\partial}{\partial\theta}} r = 0$

$f(r)^2 d\theta^2) = f(r)^2 \mathcal{L}_{\frac{\partial}{\partial\theta}} d\theta^2 = f(r^2)\left[((d\mathcal{L}_{\frac{\partial}{\partial\theta}}\theta) \otimes \theta) + \theta \otimes d\mathcal{L}_{\frac{\partial}{\partial\theta}}\theta\right]$

$\| = f(r)$ Not constant

Canonical connection of a Riemannian submanifold

Let (N, h) be a Riemannian manifold
(M, g) submanifold (embedded) of N equipped with the induced Metric

$\phi(U \cap M) = \mathbb{R}^n \times \{0\} \subset \mathbb{R}^{n+p}$

$X(\phi^{-1}(x,y)) = \sum \tilde{X}^i(\phi^{-1}(x,0)) \frac{\partial}{\partial x_i}$

Fact $\tilde{X} \in \mathfrak{X}(M)$ smooth v.F on M ~~then $\exists$~~. Let $p \in M$ then $\exists$ open U of N containing p ~~extending~~ and $X \in \mathfrak{X}(U)$ extending $\tilde{X}$ on U. (i.e. $\forall x \in U \cap M$ $\tilde{X}_x = X_x$) See Gallot p72

Let $Z \in \mathfrak{X}(N)$ and $\tilde{Z} = Z|_M : M \to TN$ be Restriction of Z to M (Note that $\tilde{Z}$ is Not Necessarily in $\mathfrak{X}(M)$)

$\forall p \in M$ $\qquad \tilde{Z}_p = \tilde{Z}_p^T + \tilde{Z}_p^\perp$ $\qquad$ ~~h~~ orthogonal decomposition

$T_pM \ni$ tangential part $\qquad$ Normal part

Thm Let (N, h) be a Riemannian manifold and (M, g) be a submanifold of N with the induced Metric. Then the operator:

$\tilde{X}, \tilde{Y} \in \mathfrak{X}(M)$
X, Y their Extensions to $U \subset N$.

$$\tilde{\nabla} : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$$

given by

$$\left(\tilde{\nabla}_{\tilde{X}} \tilde{Y}\right)_p = \left(\nabla^g_X Y\right)_p^T$$

is the Levi-Civita connection of

Idea Take $\tilde{X}, \tilde{Y}, \tilde{Z}$ on $M \longrightarrow$ take $\tilde{\tilde{X}}, \tilde{\tilde{Y}}, \tilde{\tilde{Z}}$ local extensions
Compare Koszul formal
From the formula of the extensions we have
$[X, Y]_m = [\tilde{X}, \tilde{Y}]_m$ and $\tilde{g}(\tilde{X}, \tilde{Y}) = h(X, Y)$ for any $m \in U \cap M$

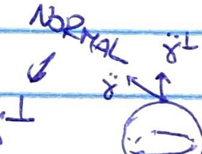
$$\Rightarrow \tilde{g}(\tilde{D}_{\tilde{X}} \tilde{Y}, \tilde{Z}) = h(D^b_X Y, Z) \text{ on } U \cap M.$$

Remark

The operator $B(X,Y) = (D_X Y)^{\perp}$ $X, Y \in \mathfrak{X}(M)$ is called second fundamental form.
(1^{st} Fundamental Form for surfaces in $\mathbb{R}^3$ is the Metric)

Example

Let $\gamma : I \to S^n$ is a C^2 curve on sphere



$$\tilde{\nabla}_{\dot{\gamma}} \dot{\gamma} = (D_{\dot{\gamma}} \dot{\gamma})^{T} = \left(\frac{D}{dt} \dot{\gamma}\right)^{T} = (\ddot{\gamma})^{T} = \ddot{\gamma} - \ddot{\gamma}^{\perp}$$

in $\mathbb{R}^n$

$$= \ddot{\gamma} - \langle \ddot{\gamma}, \gamma \rangle \gamma \quad (*)$$

γ is a geodesic on S^n iff

$$\ddot{\gamma} = \langle \ddot{\gamma}, \gamma \rangle \gamma$$

(acceleration is tangential to γ)
$\Rightarrow$ NORMAL to TS^n

Define the curve $\gamma = \gamma_{(p,X)}$ by

$$\gamma(t) = \cos(|X| t) \cdot p + \sin(|X| t) \cdot \frac{X}{|X|}$$

(if $|X| \neq 0$
Rmk if $X = 0$ $\gamma(t) = p$

$\Rightarrow \gamma(0) = p$ and $\dot{\gamma}(0) = X$ γ satisfies Equation $(*)$
By uniqueness geodesics are Great circles

In general Geodesics of A Riemannian submanifold $M \subset \mathbb{R}^{n+k}$ are curves with NORMAL acceleration v.f.

indeed $\gamma''(t) = \underset{\substack{\uparrow \\ \text{in } \mathbb{R}^{n+k}}}{\frac{D^g}{dt}} \gamma'(t) = D^g_{\dot\gamma} \dot\gamma \Rightarrow$

Since $\tilde{D}^g_{\dot\gamma} \dot\gamma = (D^g_{\dot\gamma} \dot\gamma)^\perp = 0$ we get the conclusion.

($D^g_{\dot\gamma} \dot\gamma$ is NORMAL)