Lecture 8

Def The torsion of a connection ∇ on TM is the $(1,2)$ tensor

Torsion free connection means $\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} - \nabla_{\frac{\partial}{\partial x_j}} \frac{\partial}{\partial x_i}$ $\leftarrow$

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y].$$

Def The connection ∇ is compatible with a Riemannian Metric g if $\nabla g = 0$ i.e $X\cdot(g(Y,Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$ $\forall X, Y, Z \in \mathfrak{X}(M)$

(we also say ∇ is a metric connection)

Thm Given a Riemannian Metric g on M $\exists!$ torsion free connection on TM compatible with g called Levi-Civita connection given by the "Koszul Formula"

$$g(\nabla^g_X Y, Z) = \tfrac{1}{2}\big(X\cdot g(Y,Z) + Y\cdot g(Z,X) - Z\cdot g(X,Y) - g([X,Z],Y) - g([Y,Z],X) - g([X,Y],Z)\big)$$

OR Equivalently whose Christoffel symbols are:

$$\Gamma_{ij}^{k} = \tfrac{1}{2} \sum_{\ell} g^{k\ell}\left(\frac{\partial}{\partial x_i} g_{\ell j} + \frac{\partial}{\partial x_j} g_{i\ell} - \frac{\partial}{\partial x_\ell} g_{ij}\right)$$

where $(g^{k\ell})$ is the inverse Matrix to $(g_{k\ell})$

Proof Suppose such A connection D^g Exists and compute

$$X\cdot g(Y,Z) = g(D^b_X Y, Z) + g(Y, D^b_X Z)$$
$$Y\cdot g(Z,X) = g(D^b_Y Z, X) + g(Z, D^b_Y X)$$
$$Z\cdot g(X,Y) = g(D^b_Z X, Y) + g(X, D^b_Z Y)$$

so $X\cdot g(Y,Z) + Y\cdot g(Z,X) - Z g(X,Y) = g([X,Z],Y) + g([Y,Z],X)$
$+ g(\nabla_X Y + \nabla_Y X, Z)$

Use $\nabla_X Y = \nabla_Y X + [X,Y]$ to Replace last term with $g([X,Y],Z)$
$+ 2g(D^b_Y X, Z)$
Solving for $g(D^b_Y X, Z)$ one obtains Koszul Formula

This proves uniqueness of ∇.

For The Existence simply define it by Koszul Formula

To compute Christoffel symbols set $Y = \frac{\partial}{\partial x_i}$ $X = \frac{\partial}{\partial x_j}$
$Z = \frac{\partial}{\partial x_l}$ so as $[X,Y] = [X,Z] = [Y,Z] = 0$

$$g\left(D^b_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_l}\right) = g(D^b_Y X, Z) = \frac{1}{2}\big(X\cdot g(Y,Z) + Y g(Z,X) -$$
$$Z g(X,Y)\big) = 1/2\left(\frac{\partial}{\partial x_j} g_{il} + \frac{\partial}{\partial x_i} g_{lj} - \frac{\partial}{\partial x_l} g_{ij}\right)$$

$$D_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = \sum \Gamma^k_{ij} \frac{\partial}{\partial x_k} \Rightarrow g\left(D^b_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_l}\right) = \sum_k \Gamma^k_{ij} g_{kl}$$

$$\Rightarrow g\left(D^{b}_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_l}\right) = \sum_k \Gamma_{ij}^k g_{kl}$$

Then $$\sum_l g^{lm} \frac{1}{2}\left(\frac{\partial}{\partial x_j} g_{il} + \frac{\partial}{\partial x_i} g_{lj} - \frac{\partial}{\partial x_l} g_{ij}\right) = \sum_{k,l} \Gamma_{ij}^k g_{kl} g^{lm}$$

$$= \sum \Gamma_{ij}^k \delta_{km}$$

$$= \Gamma_{ij}^m$$

so $$\Gamma_{ij}^k = \sum_l g^{lk} \frac{1}{2}\left(\frac{\partial}{\partial x_j} g_{il} + \frac{\partial}{\partial x_i} g_{lj} - \frac{\partial}{\partial x_l} g_{ij}\right)$$