Lecture 6

Connections

How Do we differentiate vector fields with Respect to Each other in $\mathbb{R}^n$?

Vector fields $X, Y : \mathbb{R}^n \to \mathbb{R}^n$ are n-tuples of functions $a_i, b_j : \mathbb{R}^n \to \mathbb{R}$

$$X = a_1 \frac{\partial}{\partial x_1} + a_2 \frac{\partial}{\partial x_2} + \cdots + a_n \frac{\partial}{\partial x_n}$$

(Here we are identifying $T_p\mathbb{R}^n \cong \mathbb{R}^n$ possible because $\mathbb{R}^n$ is vectorspace)

$$Y = b_1 \frac{\partial}{\partial x_1} + b_2 \frac{\partial}{\partial x_2} + \cdots + b_n \frac{\partial}{\partial x_n}$$

And Each b_j can be differentiated at $p \in \mathbb{R}^n$ in direction $X(p)$

$$(X(b_j))_p = a_1(p) \frac{\partial b_j}{\partial x_1}(p) + \cdots + a_n \frac{\partial b_j}{\partial x_n}(p) = \sum_i a_i(p) \frac{\partial b_j}{\partial x_i}(p)$$

$$\underbrace{(dY)(X)}_{\nabla_X Y} = X(Y) = \sum_j \left(\sum_i a_i \frac{\partial b_j}{\partial x_i} \right) \frac{\partial}{\partial x_j}$$

On A Manifold V.F are sections of TM so $X, Y : M \to TM$

so $dY(p) : T_pM \to T_{Y(p)}TM$

so $dY(X)$ would land on $T(TM)$ Not TM

$= \mathbb{R}^n \to T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$ Then

$$dY : T\mathbb{R}^n \to T(T\mathbb{R}^n) \cong (\mathbb{R}^n \times \mathbb{R}^n) \times (\mathbb{R}^n \times \mathbb{R}^n)$$

$$(p, v) \mapsto \left((p, Y(p)), \underbrace{(Y(p), dY_p(v))}_{\text{Canonical Hori}} \right)$$

For $T(TM) \to TM$ is $M =$

$T_p(TM) = H_p^\nabla \oplus V$

$TE = H^\nabla \oplus \cancel{VE}$ ← Determine proj on V along H^∇

vector BUNDLE

Idea $\pi: E \to M$ $\quad \pi_*: TE \to TM$

Vertical tangent space V is $Ker(\pi_*)$

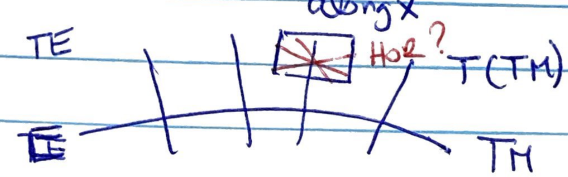
A connection is A choice of HOR dist.

Rmk On A Manifold we can choose coordinates Near a point and compute derivatives of v.f. in $\mathbb{R}^n$ but this would depend on the choice of coordinates (charts)

Can't define

$$dY(X)_p \text{ "}=\text{" } \lim_{\substack{q \to p \\ \text{along } X}} \frac{\overbrace{Y(q) - Y(p)}^{\text{Not in same vector space}}}{dist(p,q)}$$

Can't identify T_pM and T_qM in a canonical way.



Def A connection ∇ on A vector bundle $E \to M$ is A Map

$$\nabla : \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$$
$$(X, Y) \mapsto \nabla_X Y \qquad \text{st.}$$

1) $X \to \nabla_X Y$ is $C^\infty(M)$-linear

$$\nabla_{f_1 X_1 + f_2 X_2} Y = f_1 \nabla_{X_1} Y + f_2 \nabla_{X_2} Y \quad \forall f_i \in C^\infty$$

2) $Y \mapsto \nabla_X Y$ is $\mathbb{R}$-linear $\nabla_X (c_1 Y_1 + c_2 Y_2) = c_1 \nabla_X Y_1 + c_2 \nabla_X Y_2$ $\quad \forall c_i \in \mathbb{R}$.

3) Leibniz Rule $\quad \nabla_X (fY) = f \nabla_X Y + X(f)\, Y.$

Note Using partitions of unity we can show that Every vector bundle $E \to M$ can be endowed with A connection ($dY(X) = \nabla_X Y$ is A connection on $T\mathbb{R}^n \to \mathbb{R}^n$.

Let's specialize to $E = TM$ so $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$

On a chart using coordinate vectors $\left\{ \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right\}$

$$\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = \sum_{k=1}^{n} \Gamma_{ij}^{k} \frac{\partial}{\partial x_k}$$

(Note that ~~$\Gamma_{ij}^k = \Gamma_{ji}^k$ because $\left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right] = 0$ No Torsion~~)

called "Christoffel symbols" of ∇.

Then if $X = \sum a_i \frac{\partial}{\partial x_i}$ and $Y = \sum b_j \frac{\partial}{\partial x_j}$ on U

we have $\nabla_X Y = \nabla_{\sum_i a_i \frac{\partial}{\partial x_i}} \left(\sum_j b_j \frac{\partial}{\partial x_j} \right) = \sum_i a_i \nabla_{\frac{\partial}{\partial x_i}} \left(\sum b_j \frac{\partial}{\partial x_j} \right)$

$$= \sum_{i,j} a_i b_j \underbrace{\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j}}_{\sum_k \Gamma_{ij}^k \frac{\partial}{\partial x_k}} + \underbrace{\sum_{i,j} a_i \frac{\partial b_j}{\partial x_i} \frac{\partial}{\partial x_j}}_{X(b_j) \frac{\partial}{\partial x_j}}$$

as in $\mathbb{R}^n$

$$= \sum_{i,k} \Big[\underbrace{a_i \frac{\partial b_k}{\partial x_i}}_{\text{This is "}X(Y)\text{" in coordinates of } \nabla \text{ in } \mathbb{R}^n} + \underbrace{\sum a_i b_j \Gamma_{ij}^k}_{\text{this is connection to make coordinates invariant.}} \Big] \frac{\partial}{\partial x_k}$$

(in $\mathbb{R}^n$, the usual ∇ has $\Gamma_{ij}^k = 0$ so $\nabla_X Y = X(Y)$)

<u>Note ①</u> $(\nabla_X Y)_p$ only depends on $a_i(p)$ and b_j near p.

<u>Note ②</u> From ∇ we defined Γ_{ij}^k

Conversely given Γ_{ij}^k we can define

A unique connection with these prescribed

Christoffel symbols (in charts by Formula above

Lecture 7

IF ∇ AND $\tilde{\nabla}$ are connections on TM with Christoffel symbols Γ and $\tilde{\Gamma}$ then
$A = \nabla - \tilde{\nabla}$ is a $(1,2)$-tensor $\quad A^k_{ij} = \Gamma^k_{ij} - \tilde{\Gamma}^k_{ij}$
where $A \in \Gamma(T^*M \otimes T^*M \otimes TM)$
The space of connections on TM is therefore an Affine space $\{ \underbrace{\nabla}_{\text{Fixed connection}} + A \}$

Proof

Check A is C^∞-Linear.

$$A_X(fY) = \nabla_X(fY) - \tilde{\nabla}_X(fY) = f(A_X Y)$$

Remark From A connection ∇ on TM we can define induced connections on T^*M or Any tensor power $\otimes^p TM \otimes \otimes^q T^*M$ in particular on p-Forms

1-Form ω : $\quad (\nabla_X \omega)(Y) = X(\omega(Y)) - \omega(\nabla_X Y) \quad \forall X, Y$

$T \in \Gamma((T^*M)^{\otimes r} \otimes (TM)^{\otimes s}) \qquad \nabla T \in \Gamma((T^*M)^{\otimes r+1} \otimes (TM)^{\otimes s})$

$$\ldots, Y_r, \omega_1, \ldots, \omega_s) = X \cdot (T(Y_1, \ldots, Y_r, \omega_1, \ldots, \omega_s)$$
$$- \sum_{k=1}^{r} T(Y_1, \ldots, \nabla_X Y_k, \ldots, Y_r, \omega_1, \ldots, \omega_s)$$
$$- \sum_{k=1}^{s} T(Y_1, \ldots, Y_r, \omega_1, \ldots, \nabla_X \omega_k, \ldots, \omega_s)$$

RMK $f^*E = \{(m,e) \in M \times E \mid f(m) = \pi(e)\}$
$\subseteq M \times E$

Def Given a connection ∇ on $E \xrightarrow{\pi} N$ and a smooth map $f : M \to N$ the pullback connection $f^*\nabla$ on $f^*E \to M$ is determined by

$$(f^*\nabla)_X (f^* s) = f^* \left(\nabla_{df(X)} s \right) \qquad s \in \Gamma(E)$$

Important Example

Let $\gamma : (a,b) \to M$ be a smooth curve in M and $\gamma^* TM$ be pull back bundle

The vector bundle over (a,b) whose sections $V \in \Gamma(\gamma^* TM)$ are vector fields along γ.

$$V : (a,b) \to TM \quad \text{st.} \quad V(t) \in T_{\gamma(t)} M \quad \forall\, t \in (a,b)$$

Examples ① $\dot\gamma(t) \in \Gamma(\gamma^* TM)$ is a vector field along γ which may not be the Restriction of an ambient V.f.

② Restrict an Ambient v.f to γ

Given a connection on TM we use the pullback connection $\gamma^*\nabla$ to define the covariant derivative of V.f along γ:

$$V \in \Gamma(\gamma^* TM) \rightsquigarrow \frac{D}{dt} V := (\gamma^*\nabla)_{\partial/\partial t} V \in \Gamma(\gamma^* TM)$$

satisfying following properties:

- $\frac{D}{dt}(c_1 V_1 + c_2 V_2) = c_1 \frac{D}{dt} V_1 + c_2 \frac{D}{dt} V_2 \quad \forall c_1, c_2 \in \mathbb{R}$
- $\frac{D}{dt}(f V) = f' V + f \cdot \frac{D}{dt} V \quad \forall f \in C^\infty([a,b] \to \mathbb{R})$
- If V is the Restriction of An ambient V.f $\tilde{V}$ on M then $\frac{D}{dt} V = \nabla_{\dot\gamma(t)} \tilde{V}$ $\left((\gamma^* \nabla)_{\frac{\partial}{\partial t}} V = \nabla_{\gamma_*(\frac{\partial}{\partial t})} \tilde{V} \right)$ where $V = \gamma^* \tilde{V}$

In coordinates with $\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_J} = \sum_K \Gamma_{iJ}^K \frac{\partial}{\partial x_K}$ we have

that A V.f along γ say $V \in \Gamma(\gamma^* TM)$ can be written

as: $\quad V(t) = \sum_{J=1}^n v_J(t) \left.\frac{\partial}{\partial x_J}\right|_{\gamma(t)}$ For some Fuctions $v_J : (a,b) \to \mathbb{R}$

$$\frac{D}{dt} V(t) = \sum_J v_J'(t) \left.\frac{\partial}{\partial x_J}\right|_{\gamma(t)} + v_J(t) \underbrace{\frac{D}{dt}\left(\left.\frac{\partial}{\partial x_J}\right|_{\gamma(t)} \right)}_{\nabla_{\dot\gamma(t)} \frac{\partial}{\partial x_J}}$$

because $\left.\frac{\partial}{\partial x_J}\right|_{\gamma(t)}$ is Restriction of $\frac{\partial}{\partial x_J}$

$$= \sum_J v_J'(t) \left.\frac{\partial}{\partial x_J}\right|_{\gamma(t)} + \sum_{i,J,K} a_i'(t) v_J(t) \Gamma_{iJ}^K(\gamma(t)) \left.\frac{\partial}{\partial x_K}\right|_{\gamma(t)}$$

$$= \sum_K \underbrace{\left(v_K' + \sum_{i,J} a_i' v_J \Gamma_{iJ}^K \right)}_{\text{coefficients of } \frac{D}{dt}V \text{ as A V.f along } \gamma} \left.\frac{\partial}{\partial x_K}\right|_{\gamma(t)}$$

write $\gamma(t)$ in coordinates $\gamma(t) = (a_i(t))$ so $\dot\gamma(t) = \sum a_i'(t) \left.\frac{\partial}{\partial x_i}\right|_{\gamma(t)}$

and hence

$$\nabla_{\dot\gamma(t)} \frac{\partial}{\partial x_J} = \sum a_i'(t) \nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_J} = \sum a_i'(t) \Gamma_{iJ}^K \frac{\partial}{\partial x_K}$$

covariant derivative along γ

A v.f. V along γ is parallel if $\frac{D}{dt}V \equiv 0$
(similarly for any tensor along γ)

Given a (piecewise) smooth curve $\gamma:(a,b)\to M$
a connection on TM and a vector $v \in T_{\gamma(t_0)}M$
for $t_0 \in (a,b)$ there exists a unique parallel v.f
$V:(a,b)\to TM$ along γ s.t $V(t_0)=v$
Moreover $V(t)$ depends smoothly on the initial data v.

Existence uniqueness and smooth dependence on initial data
for 1st order linear ODE system: (Picard Lindelöf)

$$\begin{cases} V_K'(t) + \sum_{i,J} a_i'(t)\, v_J(t)\, \Gamma_{iJ}^K(\gamma(t)) = 0 & K=1,\dots,n \\ V_K(t_0) = v_K \end{cases}$$

coefficient of $\dot\gamma$

The above $V(t)$ is called parallel transport wrt ∇
of $v \in T_{\gamma(t_0)}M$ along γ.

In $\mathbb{R}^n$ wrt $\nabla_X Y = X(Y)$ parallel transport is constant
and independent of path.

Def Let M be a smooth manifold and ∇ be connection on TM. A curve $\gamma : (a,b) \to M$ is a ∇-geodesic in M if $\nabla_{\dot\gamma}\dot\gamma = 0$

In coordinates if $V = \dot\gamma(t) = \sum a_i'(t) \frac{\partial}{\partial x_i}\Big|_{\gamma(t)}$ then

$$\nabla_{\dot\gamma}\dot\gamma = \sum_k \Big(a_k''(t) + \sum_{i,j} a_i'(t)\, a_j'(t)\, \Gamma_{ij}^k(\gamma(t)) \Big) \frac{\partial}{\partial x_k}\Big|_{\gamma(t)}.$$

so $\gamma(t) = (a_1(t), \dots, a_n(t))$ is ∇-geodesic iff.

2nd ORDER quasilinear ODE.

$$a_k'' + \sum_{i,j} a_i' a_j' \,\Gamma_{ij}^k = 0 \qquad k = 1, \dots, n$$

($\nabla_{\dot\gamma}\dot\gamma$ is "the acceleration")

→ **Ex** sphere $\gamma(t)^T\gamma(t) = 1 \Rightarrow \gamma^T\dot\gamma + \dot\gamma^T\gamma = 0$ so $\dot\gamma$ is not tangent
or $\gamma^T b = 0 \Rightarrow \gamma^T [\dot b + \gamma\dot\gamma^T b] = 0$

Prop Given $p \in M$ and $v \in T_pM$ and a connection ∇ on TM $\exists!$ maximal (domain can't be extended) ∇-geodesic $\gamma : I \to M$ with $t_0 \in I$ and $\gamma(t_0) = p$, $\dot\gamma(t_0) = v$ which depends smoothly on $(p, v) \in TM$.

Proof Existence uniqueness and smooth dependence for second orders ODE.

~~Note~~