Lecture 5

Metrics on Lie groups

G Lie group (is smooth Manifold with A group structure such that map $G\times G \to G$ $(x,y) \to xy^{-1}$ is differentiable)

Then Left translations AND Right translations are diffeomorphism

$L_g : G \to G$ $L_g(x) = gx$ $R_g(x) = xg$.

$\mathcal{G} = T_eG$ Lie Algebra.

We can think of vectors $v \in \mathcal{G}$ as vector fields by defining left invariant vector fields. on G

$$V_g = (L_g)_*(v) \text{ For each } g \in G.$$

$$\text{where } (L_g)_* : \mathcal{G} \to T_gG.$$

Since L_g is A diffeomorphism we can compute push forward of V:

$$(L_h)_*(V)(g) = (L_h)_*\left(V_{(L_h^{-1}(g))}\right) = (L_h)_*\left(V_{h^{-1}g}\right) = (L_h)_*\left(L_{h^{-1}g}\right)_*(v)$$

chain Rule

$$= \left(L_{hh^{-1}g}\right)_*(v) = (L_g)_*(v) = V_g.$$

Recall $[X,Y]_x = \frac{d}{dt}\Big|_{t=0} (\phi^X_{-t})_* Y_{\phi^X_t(x)}$

Claim If X, Y are left invariant v.f then $[X,Y]$ is left inv.

Proof Recall that when ϕ is diffeomorphism

$$[\phi_* X, \phi_* Y] = \phi_* [X,Y]$$

If X, Y are left-inv there is $Z \in \mathfrak{g}$ such that

$$[X,Y]_g = (L_g)_*(\omega).$$

we abbreviate this $[u,v] = \omega$. where $u,v,\omega \in \mathfrak{g}$ ($X_g = (L_g)_*(u)$, $Y_g = (L_g)_*(v)$).

Def We say that a Riemannian metric g on G is left inv. if

$$g(u,v)_y = g\left((L_x)_{*y} u, (L_x)_{*y} v\right)_{L_x(y)}$$

for all $x, y \in G$ $u, v \in T_y G$ (so L_x is isometry)

Rmk we can define Right-inv Riemannian Metric, a Metric on G which is both left and Right is said to be bi-invariant.

Exercise A compact Lie group has a bi-invariant Riemannian metric.

Question How can we define left-inv Metric on G?

Answer Take an Arbitrary inner product $\langle , \rangle_e$ on $\mathfrak{g}$ and define

$$g(u,v)_x = \langle (L_{x^{-1}})_{*x}(u), (L_{x^{-1}})_{*x}(v) \rangle_e$$

$x \in G \quad u,v \in T_x G.$

Since L_x depends differentiably on x This construction produces a Riemannian Metric left-inv.

Idea Construct a Basis of left-inv vectors and Declare it to be orthonormal.

Example Consider upper half plane with group operation

$\mathbb{R}_{>0}$ group LINE $\leftarrow$ $(a,b)\cdot(c,d) = (a+bc, bd) \qquad b,d > 0.$

The identity element is $(0,1)$.

At $(0,1)$ we declare vectors $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$ to be orthonormal.

Compute left-translation :

we have $L_{(a,b)}(x,y) = (a+bx, by)$ so

$$(L_{(a,b)})_* \left(\frac{\partial}{\partial x}\Big|_{(0,1)}\right) = b\frac{\partial}{\partial x}\Big|_{(a,b)}$$

$$(L_{(a,b)})_* \left(\frac{\partial}{\partial y}\Big|_{(0,1)}\right) = b\frac{\partial}{\partial y}\Big|_{(a,b)}$$

Left-inv vector fields are

$$E_1(x,y) = (L_{(x,y)})_* \left(\frac{\partial}{\partial x}\Big|_{(0,1)}\right) = y \frac{\partial}{\partial x}\Big|_{(x,y)}$$

$$E_2(x,y) = (L_{(x,y)})_* \left(\frac{\partial}{\partial y}\Big|_{(0,1)}\right) = y \frac{\partial}{\partial y}\Big|_{(x,y)}$$

So $[E_1, E_2] = -y \frac{\partial}{\partial x} = -E_1$

so the Metric is : $g\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}\right) = \frac{1}{y^2}\langle E_1, E_1\rangle = \frac{1}{y^2}$

$$g\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) = \frac{1}{y^2}\langle E_1, E_2\rangle = 0$$

$$g\left(\frac{\partial}{\partial y}, \frac{\partial}{\partial y}\right) = \frac{1}{y^2}\langle E_2, E_2\rangle = \frac{1}{y^2}$$

$$\leadsto g = \frac{1}{y^2}(dx^2 + dy^2)$$

Exercise Compute The Right inv and compute compare it to the left inv (look different but they are isometric)

Lecture 5 cont'd.

Isometries

The isometry group of (M^n, g) is $\mathrm{Isom}(M^n, g) = \{\phi : M \underset{\text{diffeo}}{\to} M, \ \phi^* g = g\}$

Thm (Myers-Steenrod '39) $\mathrm{Isom}(M^n, g)$ is A lie group

Recall A lie group is A smooth manifold with A group structure such that group operations are smooth function.

Example $\mathrm{Isom}(\mathbb{R}^n, g_{Euc}) = O(n) \ltimes \mathbb{R}^n = \{\phi : \mathbb{R}^n \to \mathbb{R}^n$
$\phi(x) = Ax + B$
where $A \in O(n)$
$B \in \mathbb{R}^n\}$

$\mathrm{Isom}(S^n, g_{round}) = O(n+1)$

Riemannian submersion

Def A submersion $\pi : (M, g) \to (N, h)$ is A Riemannian submersion if $\forall p \in M$ the Restriction of $d\pi_p : T_pM \to T_{\pi(p)}N$ to $\mathcal{H}_p = (\mathrm{Ker}\, d\pi_p)^{\perp_g}$ is a linear isometry.

Question IF $\pi : (M, g) \to N$ is A submersion can we endow N with A Riemannian metric so that π becomes A Riemannian submersion

Answer In general no.
Need Horizontal spaces along each $\pi^{-1}(x)$ to be isometric.

write $g = g_{Hor} \oplus g_{ver}$ wrt $T_pM = \mathcal{H}_p \oplus \mathcal{V}_p$
where $\mathcal{V}_p = \ker d\pi_p$
$\mathcal{H}_p = \mathcal{V}_p^{\perp g}$

Prop Suppose $\pi : (M, g) \to N$ is A submersion with connected fibers $\pi^{-1}(x)$, $x \in N$
There exists a metric h on N s.t π is A Riemannian submersion iff $\mathcal{L}_V (g_{hor}) = 0$ $\forall\, V \in \mathcal{V}_p$ $p \in M$.

Proof See Textbook. LaFontaine

Recall $(\mathcal{L}_V g_{hor})(X, Y) = V \cdot (g_{Hor}(X, Y)) - g_{Hor}([V, X], Y)$
$- g_{Hor}(X, [V, Y])$

Corollary If $G \subset \mathrm{Isom}(M, g)$ acts on (M, g) such that M/G is a Manifold (e.g. freely) then there exists A Riemannian Metric $\breve{g}$ on M/G such that $\pi : (M, g) \to (M/G, \breve{g})$ is A Riemannian submersion.

Ex $S^1 \curvearrowright S^{2n+1} \subset \mathbb{C}^{n+1}$ by unit complex multiplication freely and isometrically so there is A Riemannian Metric called on $S^{2n+1}/S^1 = \mathbb{C}P^n$. $\pi : S^{2n+1}(1) \to \mathbb{C}P^n$ is A Riemanian submersion
Fubini Study Metric