Lecture 4

Volume Form

Def The volume form $Vol_g \in \Omega^n(M^n)$ induced by a Riemannian metric on an orientable mfld is given in local coordinates $(x_1, \dots, x_n)$ by

$$Vol_g = \sqrt{\det(g_{ij})}\, dx_1 \wedge \dots \wedge dx_n$$

The (total) volume of (M^n, g) is $Vol(M, g) = \int_M vol_g$.

Example $B_1 = \{x^2+y^2<1\}$

Area $= \text{Volume}\left(B_1, \frac{4}{(1-x^2-y^2)^2}(dx^2 + dy^2)\right)$

Use Polar coordinates! $\iint \frac{4}{(1-x^2-y^2)^2} = \infty$!

Exercise

Area $= \text{Volume}\left(B_1, \frac{4}{(1+x^2+y^2)^2}(dx^2+dy^2)\right) = 2\pi$

Distance the distance between p, q wrt Metric g is

$$\cancel{\mathrm{dist}_g(p} \quad d(p,q) = \inf \{ L_g(\gamma) \mid \gamma : [a,b] \to M$$

piecewise smooth

$\gamma(a) = p$

$\gamma(b) = q$

Prop (M^n, g) is A Metric space.

(The Metric topology agrees with Manifold topology)

Proof $d(p,q) = d(q,p)$ is immediate

$d(p,q) \geq 0$ is immediate

Triangle inequality : $\gamma_1 : [a,b] \to M \quad \gamma_2 : [a', b'] \to M$

with $\gamma_1(b) = \gamma_2(a')$

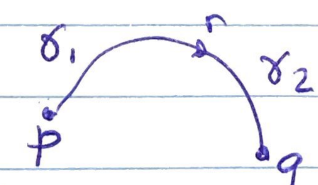
Define concatenation $\gamma_1 \# \gamma_2 : [0, b+b'-a-a'] \to M$ by

$$\gamma_1 \# \gamma_2 (t) = \begin{cases} \gamma_1(t+a) & \text{For } 0 \leq t \leq b-a \\ \gamma_2(t+a-b+a') & \text{For } b-a \leq t \leq b+b'-a-a' \end{cases}$$

Thus $\gamma_1 \# \gamma_2$ is A curve with length $L_g(\gamma_1) + L_g(\gamma_2)$.

ven $p, q, r \in M$, For Any ε we can choose γ_1, γ_2

h that



$$L_g(\gamma_1) < d(p,r) + \varepsilon \quad \text{AND} \quad L_g(\gamma_2) < d(r,q) + \varepsilon$$

$$\Rightarrow d(p,q) \leq L_g(\gamma_1 \# \gamma_2) = L_g(\gamma_1) + L_g(\gamma_2) < d(p,r) + d(r,q) + 2\varepsilon$$

Let $\varepsilon \to 0$!

We still Need to check $d(p,q)=0$ only when $p=q$
(This can be done using chart)
to say that y is outside A Ball around x)

divergence of vector Field

Def Suppose M is Riemannian manifold with Riemannian volume Form μ. Suppose X is A vector field. The divergence of X is A Function $\operatorname{div} X$ defined by

$$d(\iota_X \mu) = (\operatorname{div} X)\mu.$$

where the inner product $\iota_X \mu$ is $(n-1)$-Form defined by

$$(\iota_X \mu)(Y_1, \dots, Y_{n-1}) = \mu(X, Y_1, \dots, Y_{n-1})$$

By Stokes thm:

$$\int_M (\operatorname{div} X)\mu = \int_{\partial M} \iota_X \mu.$$

in particular if $\partial M \neq 0$ then $\int_M (\operatorname{div} X)\mu = 0.$

Product Manifolds

$M = M_1 \times M_2 \qquad TM = TM_1 \oplus TM_2$

IF (M_i, g_i) are RIEMANNIAN Manifolds then
$M = M_1 \times M_2$ can be Endowed with product metric $g_1 \oplus g_2$
i.e

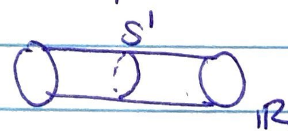
$\forall p = (p_1, p_2) \in M \quad g_p(v, w) = (g_1)_{p_1}(v_1, w_1) + (g_2)_{p_2}(v_2, w_2) \quad \forall\ v = (v_1, v_2)$, $w = (w_1, w_2) \in T_pM.$

Given a positive Function $f: M_1 \to \mathbb{R}$ ONE may endow M with A "warped product" metric

$$g^f = g_1 + f g_2$$

Example on $\mathbb{R} \times S^1 \ni (r, \theta)$ we have the product (cylinder)

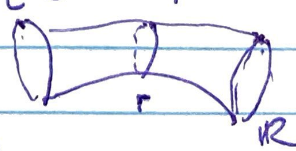
metric $dr^2 + d\theta^2$



OR The warped product Metric

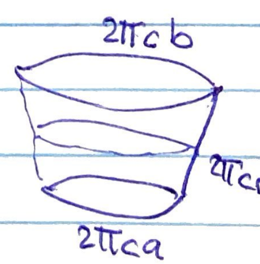
$$dr^2 + f(r)^2 d\theta^2$$

where the circle $S^1_{r_0} = \{(r_0, \theta) \mid \theta \in S^1\}$ has length $2\pi f(r_0)$



Setting $f(r) = cr$ for SOME $c > 0$

we have A CONE Metric $dr^2 + c^2 r^2 d\theta^2$ ON $[a, b] \times S^1$



Letting $a \searrow 0$ AND $b \nearrow +\infty$ we get

$$g^c = dr^2 + c^2 r^2 d\theta^2 \text{ on } (0,+\infty)\times S^1 \cong \mathbb{R}^2\setminus\{0\}$$

Question For what values of $c>0$ does g^c extend smoothly to $\mathbb{R}^2$?

Answer Around $0\in\mathbb{R}^2$ we have the diff

$$\phi(r,\theta) = (x = r\cos\theta, y = r\sin\theta) \quad \text{for } r>0$$

AND so

$$\begin{pmatrix} dx \\ dy \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}}_{\text{rotation with angle } \theta} \begin{pmatrix} dr \\ r\,d\theta \end{pmatrix}$$

$$\rightsquigarrow \begin{pmatrix} dr \\ r\,d\theta \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}}_{\text{angle } (-\theta) \text{ rotation.}} \begin{pmatrix} dx \\ dy \end{pmatrix}$$

$$\Rightarrow dr = \cos\theta\, dx + \sin\theta\, dy = \frac{x}{\sqrt{x^2+y^2}}dx + \frac{y}{\sqrt{x^2+y^2}}dy$$

$$d\theta = \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$$

$$\Rightarrow dr^2 = \frac{1}{x^2+y^2}\left(x^2dx^2 + 2xy\,dx\,dy + y^2dy^2\right)$$

$$d\theta^2 = \frac{1}{(x^2+y^2)^2}\left(y^2dx^2 - 2xy\,dx\,dy + x^2dy^2\right)$$

$$\hookrightarrow (\phi^{-1})^* g^c \underset{\substack{\uparrow \\ \text{compute}}}{=} \frac{x^2+c^2y^2}{x^2+y^2}dx^2 + \frac{2xy(1-c^2)}{x^2+y^2} + \frac{y^2+c^2x^2}{x^2+y^2}dy^2$$