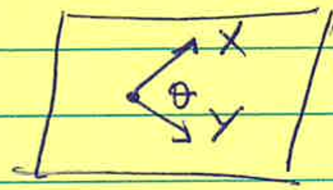
Lecture 3 Riemannian Metrics

Def If $f: M \to \mathbb{R}$ is a smooth positive function on Riemannian manifold (M, g) then $f \cdot g$ is a (smooth) metric on M called **conformal** to metric g.

↳ same angles.

$$\|X\|_g^2 = g(X, X)$$



T_pM

$$\theta = \arccos \frac{g_p(X, Y)}{\sqrt{g_p(X, X)\, g_p(Y, Y)}}$$

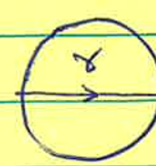
← Do not change if we replace g by $f \cdot g$.

Example on $B_1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$ consider the Riemannian metrics:

$$g_{Eucl} = dx^2 + dy^2$$

$$\left.\begin{aligned} g_{hyp} &= \frac{4}{(1 - x^2 - y^2)^2} (dx^2 + dy^2) \\ g_{sph} &= \frac{4}{(1 + x^2 + y^2)^2} (dx^2 + dy^2) \end{aligned}\right\} \text{Both are conformal but Not isometric to } g_{Euc}!$$

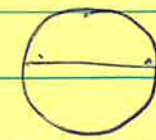
Ex Let $\gamma : (-1, 1) \to B_1$, $\gamma(t) = (t, 0)$ be "the diameter"



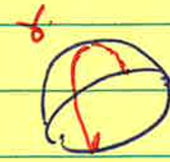
$\rightsquigarrow \dot{\gamma}(t) = (1, 0)$ so

$$L_{g_{Euc}}(\gamma) = \int_{-1}^{1} 1\, dt = 2$$

$$L_{g_{hyp}}(\gamma) = \int_{-1}^{1} \frac{2}{1-t^2}\, dt = \ln\left(\frac{1+t}{1-t}\right)\Big|_{-1}^{1} = +\infty$$



$$L_{g_{sph}} = \int_{-1}^{1} \frac{2}{1+t^2}\, dt = 2\arctan t\Big|_{-1}^{1} = \pi$$



Question How can we show that $g_{Euc}, g_{sph}, g_{hyp}$ are Not isometric

Just having different lengths For γ is Not Enough !

there could be some Diffeo ϕ so that

$$L_{g_{Euc}}(\gamma) = L_{g}(\phi \circ \gamma).$$

ONE way would be to compute the Area of B_1 with Respect to Each Metric. (volume)

$$\text{Area}(B_1, g_{Euc}) = \pi \quad (\text{Area of circle } \pi r^2)$$
$$\text{Area}(B_1, g_{hyp}) = \infty$$
$$\text{Area}(B_1, g_{sph}) = 2\pi$$

Another way would be to compute curvature :

$$\sec_{g_{Euc}} \equiv 0 \qquad \sec_{g_{hyp}} \equiv -1 \qquad \sec_{g_{sph}} \equiv 1$$

More about curvature AND volume later

Example surfaces in $\mathbb{R}^3$ (← beginning of History of Theory)

S is surface in $\mathbb{R}^3$ parametrized by (IMMERSION)

$$\iota : (u,v) \longrightarrow (f(u,v), g(u,v), h(u,v))$$

Then the induced Metric by Euclidean Metric g on S is

$$\iota^* g = (f_u^2 + g_u^2 + h_u^2)\, du^2 + (f_u f_v + g_u g_v + h_u h_v)(du \otimes dv + dv \otimes du)$$
$$+ (f_v^2 + g_v^2 + h_v^2)\, dv^2$$

so we can just write it as:

$$\iota^* g = E(u,v)\, du^2 + 2F(u,v)\, du \odot dv + G(u,v)\, dv^2$$

$\left(g\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\right) = F \right)$

Examples ① Cylinder $x = \cos\theta \quad y = \sin\theta \quad z = z$

$\hookrightarrow \iota^* g = d\theta^2 + dz^2$ ← Same as Euclidean Metric on $\mathbb{R}^2$.

(locally isometric but Not isometric)

② Cone $x = u\cos v \quad y = u\sin v \quad z = u$

$$\iota^* g = 2\, du^2 + 2u^2 dv^2 \underset{1/2}{\sim} du^2 + u^2 dv^2$$

(Not isometric to plane)

↑ Flat plane in polar coordinates

Basic Question Imagine we have SOME Metric given to us:

$$g = E(u,v)\, du^2 + 2F(u,v)\, du \odot dv + G(u,v)\, dv^2$$

Is this actually AN Euclidean Metric in SOME other coordinates? i.e are there $x = f(u,v)$, $y = g(u,v)$ such that

$$g = dx^2 + dy^2 \, ?$$

$$\Rightarrow (f_u^2 + g_u^2)\, du^2 + 2(f_u f_v + g_u g_v)\, du \odot dv + (f_v^2 + g_v^2)\, dv^2$$

$$= E(u,v)\, du^2 + 2F(u,v)\, du \odot dv + G(u,v)\, dv^2$$

So we get 3 PDE For unknown Functions f, g

$$\begin{cases} f_u^2 + g_u^2 = E(u,v) \\ f_u f_v + g_u g_v = F(u,v) \\ f_v^2 + g_v^2 = G(u,v) \end{cases}$$

Non linear overdetermined

3 Equations For 2 Functions $\rightarrow$ ONE constraint that Data will have to satisfy AND this should be where Geometry is!! More about this when we discuss curvature!

Example 1 linear transformation.

$$\phi(u,v) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \qquad (ad-bc \neq 0)$$

$\hookrightarrow$ $au + bv = x$ AND $cu + dv = y$

$$\leadsto \phi^* g_{Euc} = (a^2+c^2)\,du^2 + 2(ab+cd)\,du \odot dv + (b^2+d^2)\,dv^2$$

Exercise Show that Any constant Metric on $\mathbb{R}^2$ eg: $h = 2du^2 - du\,dv + 5dv^2$ is isometric to $g_{Euc} = dx^2 + dy^2$

(For the above h we use $\phi(u,v) = \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$)

But there are also Non linear diffeo of $\mathbb{R}^2$ such that

$$\phi(u,v) = \begin{pmatrix} \cos(u^2+v^2) & -\sin(u^2+v^2) \\ \sin(u^2+v^2) & \cos(u^2+v^2) \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} \leadsto \text{diffeo}$$

$$\Rightarrow \phi^* g_{Euc} = (1 + 4u^4 - 4uv + 4u^2v^2)\,du^2 + 4(u^2 - v^2 + 2u^3v + 2uv^3)\,du\,dv + (1 + 4v^4 + 4uv + 4u^2v^2)\,dv^2$$

This "ugly" Metric is isometric to g_{Euc}

wait For curvature