lectur2

Riemannian Metrics

Let's continue with Examples:

S^2 immersed submanifold of $\mathbb{R}^3$

$$\iota : S^2 \to \mathbb{R}^3$$

$$\iota(\theta, \alpha) = (\sin\theta \cdot \cos\alpha, \sin\theta \sin\alpha, \cos\theta)$$

is an immersion.

$$\iota_* = \begin{bmatrix} \cos\theta \cos\alpha & -\sin\theta \sin\alpha \\ \cos\theta \sin\alpha & \sin\theta \cos\alpha \\ -\sin\theta & 0 \end{bmatrix}$$

$$\frac{\partial}{\partial\theta} = (1,0) \qquad \frac{\partial}{\partial\alpha} = (0,1) \qquad \frac{\partial}{\partial x} = (1,0,0) \text{ etc}$$

$$\iota_*\left(\frac{\partial}{\partial\theta}\right) = \begin{bmatrix} \cos\theta \cos\alpha & -\sin\theta \sin\alpha \\ \cos\theta \sin\alpha & \sin\theta \cos\alpha \\ -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\theta \cos\alpha \\ \cos\theta \sin\alpha \\ -\sin\theta \end{bmatrix}$$

$$\rightarrow \iota_*\left(\frac{\partial}{\partial\theta}\right) = \cos\theta\cos\alpha \frac{\partial}{\partial x} + \cos\theta \sin\alpha \frac{\partial}{\partial y} - \sin\theta \frac{\partial}{\partial z}$$

$$\iota_*\left(\frac{\partial}{\partial\alpha}\right) = -\sin\theta \sin\alpha \frac{\partial}{\partial x} + \sin\theta \cos\alpha \frac{\partial}{\partial y}$$

The induced Metric on S^2 by Euclidean Metric $g = dx^2 + dy^2 + dz^2$ is

$$h = h_{11}\, d\theta^2 + h_{12}\,(d\theta \otimes d\alpha + d\alpha \otimes d\theta) + h_{22}\, d\alpha^2$$

compute

$$h_{11} = g\left(\iota_*\left(\frac{\partial}{\partial\theta}\right), \iota_*\left(\frac{\partial}{\partial\theta}\right)\right) \overset{\downarrow}{=} 1$$

$$h_{12} = g\left(\iota_*\left(\frac{\partial}{\partial\theta}\right), \iota_*\left(\frac{\partial}{\partial\alpha}\right)\right) = 0 = h_{21} = g\left(\iota_*\left(\frac{\partial}{\partial\alpha}\right), \iota_*\left(\frac{\partial}{\partial\theta}\right)\right)$$

$$h_{22} = g\left(\iota_*\left(\frac{\partial}{\partial\alpha}\right), \iota_*\left(\frac{\partial}{\partial\alpha}\right)\right) = \sin^2\theta$$

$$h = d\theta\otimes d\theta + \sin^2\theta \; d\alpha\otimes d\alpha \qquad \left(h = \begin{bmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{bmatrix}\right)$$

OR we can use parametrization $\begin{cases} x = \sin\theta\cos\alpha \\ y = \sin\theta\cos\theta \\ z = \cos\theta. \end{cases}$

Expression dx, dy, dz AND substitute it in
$g = dx^2 + dy^2 + dz^2$.

Exercise In Stereographic coordinates the inversion is given by

$$\iota(u,v) = \left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1}\right)$$

prove that $\ell = \iota^* g = \dfrac{4}{(u^2+v^2+1)^2}\left(du^2 + dv^2\right)$

This can be generalized to any dimension

Back to S^1: $g = d\theta^2$ the induced Metric by The Euclidean.
(The Round Metric)

we can also have Metrics on S^1 of the Form

$h = f(\theta)^2 \, d\theta^2$ another Metric where $f: S^1 \to \mathbb{R}$ is positive eg $f(\theta) = 2 + \cos\theta$

Are the circles (S^1, g) and (S^1, h) "the same"?

Recall smooth Manifolds are same "the same" if diffeo

Def The Riemannian Manifolds (M^n, g) and (N^n, h) are isometric if there is a diffeomorphism $\phi : (M, g) \to (N, h)$ (ϕ bijection ϕ, ϕ^{-1} differen) such that $\forall p \in M$ $\forall X, Y \in \mathfrak{X}(M)$

$$h_{\phi(p)}\left(f_*(X(p)), f_*(Y(p))\right) = g_p\left(X(p), Y(p)\right)$$

(i.e. $g = \phi^* h$) Such ϕ is called isometry.

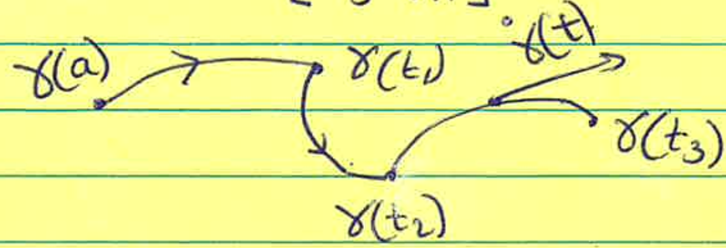
Idea Roughly speaking isometry is A transformation that preserves distances/shapes/size.

→ two Riemannian Manifold are "The same" if they are isometric.

→ Isometries preserves Anything that depends on Metric: distance, volume, curvature, scalar curvature

How Do we Measure distances on Riemannian Manifolds?

Length: Let $\gamma : [a,b] \to (M,g)$ be A piecewise C^1 curve i.e $\gamma|[t_i, t_{i+1}]$ is C^1 $(i = 0, \dots, n)$ $t_0 = a$, $t_n = b$



Length of γ in (M,g) is

$$L_g(\gamma) = \int_a^b g(\dot\gamma(t), \dot\gamma(t))^{1/2}\, dt.$$

Exercise ① Show that L_g is invariant under Reparametrization ie if $\eta : [a,b] \to [c,d]$ is A diffeomorphism then

$$L_g(\gamma) = L_g(\gamma \circ \eta).$$

Exercise ② Suppose that $\phi : (M,g) \to (M,h)$ is

AN isometry AND $\gamma : [a,b] \to M$ is C^1 curve (piecewise) show that $L_g(\gamma) = L_h(\phi \circ \gamma)$.

$g = d\theta$ **Example** $\gamma : [0, 2\pi] \to S^1$ (parametrization of S^1) if we take $S^1 \to \mathbb{R}^2$

$h = (2+\cos\theta)^2 d\theta$ $\quad \gamma(\theta) = (\cos\theta, \sin\theta)$ Note $\dot\gamma(\theta) = \frac{\partial}{\partial\theta} = -\sin\theta\frac{\partial}{\partial x} + \cos\theta\frac{\partial}{\partial y}$

Then

$$L_g(\gamma) = \int_0^{2\pi} g\left(\frac{\partial}{\partial\theta}, \frac{\partial}{\partial\theta}\right)^{1/2} d\theta = 2\pi \qquad g = d\theta^2$$

$$L_h(\gamma) = \int_0^{2\pi} h\left(\frac{\partial}{\partial\theta}, \frac{\partial}{\partial\theta}\right)^{1/2} d\theta = \int_0^{2\pi} 2 + \cos\theta \, d\theta = 4\pi$$

Upshot

$\rightarrow$ (S^1, g) AND (S^1, h) are Not isometric!!

Claim (S^1, h) is isometric to $\left([0, 4\pi]/\sim, ds^2 \right)$

circle of length 4π

Proof

Let $\phi : [0, 2\pi] \longrightarrow [0, 4\pi]$ be increasing smooth

$\theta \qquad\qquad s$

Function $\phi(\theta) = \int_0^\theta \underbrace{(2 + \cos t)}_{f(t)} dt = 2\theta + \sin\theta$

ϕ induces A diffeomorphism between $\phi : [0, 2\pi]/\sim \rightarrow [0, 4\pi]/\sim$

such that $\phi^* ds^2 = \phi'(\theta)^2 d\theta^2 = (2 + \cos\theta)^2 d\theta^2 = h$

$s = \phi(\theta)$

So we have AN isometry $\phi : (S^1, h) \rightarrow \left([0, 4\pi]/\sim, ds^2 \right)$

Moreover $\left([0, 4\pi]/\sim, ds^2 \right) \underset{\text{isometric}}{\cong} \left([0, 2\pi]/\sim, 4d\theta^2 \right)$

$s = 2\theta$
$ds^2 = 4\, d\theta^2$

Exercise Show that circles (S^1, g) AND (S^1, h) are isometric iff they have same length = volume

Idea $\forall\, g = f(\theta)^2 d\theta^2$ on S^1 $\exists\, \phi : S^1 \to S^1$ st $\phi^* g = r^2 d\theta^2$

IF two metrics have same r compose ϕ's to get isometry

Hyperbolic geometry ~~In Euclid's Elements the 5th axiom / postulate states that~~ ~~if L_1, L_2, $\alpha+\beta<180°$ then L_1 and L_2 will intersect~~ on that side

Example The Hyperboloid Model of Hyberbolic space $\mathbb{H}^n$ is $(x_0, x_1, \dots, x_n) \in \mathbb{R}^{n+1}$ such that

$$\sum_{i=0}^{n-1} x_i^2 - x_n^2 = -1 \qquad \text{with } x_n > 0$$

(to ensure that $\mathbb{H}^n$ is connected)

Consider the bilinear form

$$(*) \quad \langle X, Y\rangle = \sum_{i=0}^{n-1} x_i y_i - x_n y_n \qquad X = (x_i) \quad Y = (y_i)$$

Claim The Restriction of $\langle \cdot, \cdot \rangle$ to $T_p\mathbb{H}^n$ is positive definite.

Proof $\langle p, p\rangle = -1 \quad p \in \mathbb{H}^n$

The tangent space at p, $T_p\mathbb{H}^n = \{X \in \mathbb{R}^{n+1} \mid \langle X, p\rangle = 0\}$

Suppose that there exists X such that $\langle X, X\rangle < 0$.

$$\rightarrow \sum_{i=0}^{n-1} x_i^2 < x_n^2 \quad \text{and} \quad \sum p_i^2 + 1 = p_n^2 \quad p \in \mathbb{H}^n$$

Therefore

$$\text{and} \quad \sum x_i p_i = x_n p_n \quad \text{tangent}$$

$$\sum x_i^2 \left(1 + \sum p_i^2\right) < x_n^2 p_n^2 = \left(\sum x_i p_i\right)^2 \leq \left(\sum x_i^2\right)\left(\sum p_i^2\right)$$

Cauchy ~~Swarch~~ Schwarz

This is impossible !! and "$=0$" iff $X = 0$

so the Restriction of $\langle \cdot, \cdot \rangle$ to $\mathbb{H}^n$ produces a Riemannian Metric g on $\mathbb{H}^n$ i.e

$$g = dx_0^2 + \dots + dx_{n-1}^2 - dx_n^2 \big|_{\mathbb{H}^n}$$

Exercise ① Poincaré models for (H^n, g)

Let $S = (0, 0, \dots, -1)$ and define

$$f(x) = S - \frac{2(x-s)}{\langle x-s, x-s \rangle}$$

where $X = (x_0, \dots, x_n)$ and $\langle \cdot, \cdot \rangle$ is (*)

① Show that f is diffeomorphism from H^n onto unit disk $\{x \in \mathbb{R}^n, |x| < 1\}$ (here $|x|$ is Euclidean Norm.

and that

$$h = (f^{-1})^* g = 4 \sum_{i=0}^{n-1} \frac{dx_i^2}{(1 - |x|^2)^2}$$

Exercise ② Half space Model for (H^n, g)

$$\phi(x) = S + \frac{2(x-s)}{|x-s|^2}$$

show that ϕ is diffeomorphism for unit disk onto Half space $x_n > 0$

show $\phi^* h = \sum_{i=1}^{n} \frac{dx_i^2}{x_n^2}$