Lecture 13

Computing R^g the curvature

We consider a simple case namely the parametrization is CONFORMAL to the Euclidean Metric:

$$g_{iJ} = f\delta_{iJ} \qquad \text{For some function } f.$$
$$g^{iJ} = f^{-1}\delta_{iJ}^{iJ} \quad (\text{or } \cancel{f^{-1}\delta^{iJ}})$$

Kronecker

$$\Rightarrow \Gamma_{iJ}^{K} = \frac{1}{2}\sum_{l} g^{Kl}\left(\frac{\partial}{\partial x_i} g_{Jl} + \frac{\partial}{\partial x_J} g_{il} - \frac{\partial}{\partial x_l} g_{iJ}\right)$$

$R = \sum R_{iJK}^{\ l}\, dx^i \otimes dx^J \otimes dx^K \otimes \frac{\partial}{\partial x_l}$

$$\Rightarrow \Gamma_{iJ}^{K} = \frac{1}{2}\left(\delta_{KJ}\frac{\partial \log f}{\partial x_i} + \delta_{Ki}\frac{\partial \log f}{\partial x_J} - \delta_{iJ}\delta_{Kl}\frac{\partial \log f}{\partial x_l}\right)$$

then the curvature $(u = \log\sqrt{F})$ $\left[R\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_J}\right)\frac{\partial}{\partial x_K} = \sum R_{iJK}^{\ l}\frac{\partial}{\partial x_l}\right.$

$$R_{iJK}^{\ l} = \sum_q \Gamma_{iK}^{q}\Gamma_{Jq}^{l} - \sum_q \Gamma_{JK}^{q}\Gamma_{iq}^{l} + \frac{\partial}{\partial x_J}\Gamma_{iK}^{l} - \frac{\partial}{\partial x_i}\Gamma_{JK}^{l}$$

skip steps

$$= \delta_{il}^{l}\frac{\partial^2 u}{\partial x_J \partial x_K} - \cancel{\delta_{iJ}}\,\delta_{Jl}^{l}\frac{\partial^2 u}{\partial x_i \partial x_K}$$

$$- \sum_p \delta_{iK}\delta_{lp}^{lp}\frac{\partial^2 u}{\partial x_J \partial x_p} + \sum \delta_{JK}\delta_{lp}^{lp}\frac{\partial^2 u}{\partial x_i \partial x_p}$$

$$+ |Du|^2\left(\delta_{KJ}\delta_{il}^{l} - \delta_{iK}\delta_{lJ}^{l}\right)$$

$$+ \delta_{Jl}^{l}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_K} - \delta_{il}^{l}\frac{\partial u}{\partial x_K}\frac{\partial u}{\partial x_J}$$

$$+ \sum_p \delta_{iK}\delta_{lp}^{lp}\frac{\partial u}{\partial x_p}\frac{\partial u}{\partial x_J} - \sum_p \delta_{JK}\delta_{lp}^{lp}\frac{\partial u}{\partial x_p}\frac{\partial u}{\partial x_i}$$

Taking the trace over J and ℓ gives Ricci curvature:

$$Ric_{iJ} = \sum \delta^k_\ell R_{ikJ}{}^{\ell} = \sum g^{k\ell} R_{ikJ\ell}$$

$$Ric_{iJ} = -\delta_{iJ}\left(\Delta u + (n-2)|Du|^2\right) - (n-2)\left(\frac{\partial u}{\partial x_i \partial x_J} - \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_J}\right)$$

where $\Delta u = \sum_{k\ell} \delta^{k\ell} \frac{\partial^2 u}{\partial x_k \partial x_\ell}$ is Laplacian of u.

then scalar curvature

$$S = \sum g^{iJ} R_{iJ} = \sum_{iJ} f^{-1} \delta_{iJ} R_{iJ}$$

$$S = -(n-1) f^{-1}\left(2\Delta u + (n-2)|Du|^2\right)$$

Example ① upper Half plane $(x_1, \dots, x_{n-1}, y)$, $y>0$: $g_{iJ} = \frac{1}{y^2}\delta_{iJ}$ $(y = x_n)$ $\Rightarrow$

$$f = y^{-2} \Rightarrow u = -\log y$$

$$\rightarrow \frac{\partial u}{\partial x_i} = -y^{-1}\delta_{in} \qquad \frac{\partial^2 u}{\partial x_i \partial x_J} = \frac{1}{y^2}\delta_i^n \delta_J^n$$

and $|Du|^2 = y^{-2}$

$$R_{iJk\ell} = \sum g_{p\ell} R^{p}_{iJk}$$

$$\rightarrow R_{iJk\ell} = y^{-4}\left(\delta_{Jk}\delta_{i\ell} - \delta_{ik}\delta_{J\ell}\right)$$

An orthonormal Basis is given $\{X_i = y\frac{\partial}{\partial x_i}\}$ so for any sectional curvature

$$Sec(X_i \wedge X_J) = \frac{R^g(X_i, X_J, X_i, X_J)}{1} = \cancel{-\frac{R^g(X_J, X_i, X_i, X_J)}{1}} = \boxed{-1}$$

Example ② upper Half plane

$$g = \frac{1}{y^2}(dx^2 + dy^2)$$

orthonormal Frame left-invariant E_1, E_2

$$[E_1, E_2] = -E_1 \qquad \left(E_1 = y\frac{\partial}{\partial x} \quad E_2 = y\frac{\partial}{\partial y}\right)$$

$$D^g_{E_1} E_1 = ? \quad D^g_{E_2} E_1 = ? \quad D^g_{E_1} E_2 = ? \quad D^g_{E_2} E_2 = ?$$

Koszul Formula:

$$2g(D^g_X Y, Z) = (X \cdot g(Y, Z) + Y \cdot g(Z, X) - Z \cdot g(X, Y)) - g([X, Z], Y) - g([Y, Z], X) - g([X, Y], Z)$$

$g(X_h, Y_h)$
$= (L^*_g)(X_e, Y_e)$
$= g(X_e, Y_e)$

$$\Rightarrow 2g(D^g_{E_1} E_1, E_1) = 0$$

$$2g(D^g_{E_1} E_1, E_2) = -g([E_1, E_2], E_1) - g([E_1, E_2], E_1)$$

$$2 D^g_{E_1} E_1 = -2g([E_1, E_2], E_1) = 1$$

$$\Rightarrow \boxed{D^g_{E_1} E_1 = E_2}$$

$$2g(D^g_{E_2} E_1, E_1) = 0$$

$$2g(D^g_{E_2} E_1, E_2) = -g([E_1, E_2], E_2) - g([E_2, E_1], E_2) = 0$$

$$\boxed{D^g_{E_2} E_1 = 0}$$

$$g(D^g_{E_1} E_2, E_1) = -g(D^g_{E_1} E_1, E_2) = -1$$

$$g(D^g_{E_1} E_2, E_2) = 0$$

$$\boxed{D^g_{E_1} E_2 = -E_1}$$

$$g(D^g_{E_2} E_2, E_1) = -g(D^g_{E_2} E_1, E_2) = 0$$

$$g(D^g_{E_2} E_2, E_2) = 0$$

$$\boxed{D^g_{E_2} E_2 = 0}$$

$$R^g(E_1, E_2) E_1 = D^g_{E_2}(D^g_{E_1} E_1) - D^g_{E_1}(D^g_{E_2} E_1) + D^g_{[E_1, E_2]} E_1$$

$$= D^g_{E_2}(E_2) - 0 \cdot D^g_{E_1} E_1 = -E_2$$

So $g(R^g(E_1, E_2) E_1, E_2) = \boxed{-1}$

$$\sec(E_1 \wedge E_2) = -1$$

Remark IF we have A Lie Group with bi-invariant Metric then it's very simple to compute the curvature:

Using Fact of Biinvariance

Ad adjoint Repr.

$g((R_{h^{-1}})^*(L_h)^* \cdot, (R_{h^{-1}})^*(L_h)^* \cdot)$

$= g(Ad_h X, Ad_h Y)$

Take $h = e^{tZ}$

$\leftarrow$ $g([X, Y], Z) = g(Y, [Z, X])$ For Any left inv X, Y, Z

Exercise Using Koszul Formula

$$D^g_X Y = \frac{1}{2}[X, Y]$$

Then $R(X, Y) Z = \frac{1}{4}[[X, Y], Z]$

then $\sec(X \wedge Y) = \frac{1}{4} g([X, Y], [X, Y])$ X, Y orthonormal

$\Rightarrow$ Sectional curvature of Lie Group with bi-inv Metric is Non Negative

Surfaces in $\mathbb{R}^3$:

surface is parametrized by $x = x(u,v)$
$y = y(u,v)$
$z = z(u,v)$

$$g = E(u,v)\,du^2 + 2F(u,v)\,du \circ dv + G(u,v)\,dv^2.$$

Then the curvature

$$R_{ijk}{}^{e} = K = -\frac{1}{4(EG-F^2)^2}\Big(2EGE_{vv} - 2F^2E_{vv} + 2EGG_{uu}$$

$$-2F^2G_{uu} - 4EGF_{uv} + 4F^2F_{uv} - EE_vG_v - EG_u^2 - FE_uG_v$$

$$-GE_uG_u + FG_uE_v + 2GE_uF_v - 4FF_uF_v + 2FE_vF_v + 2FF_uG_u$$

$$-GE_v^2 + 2EF_uG_v\Big)$$

(Gauss Theorema "Egregium"
Latin for Remarkable)
K is invariant under local isometry

Rem ① If $K \equiv 0$ then there is new coordinate system in which g = Euclidean.
Conversely if $K \neq 0$ at any point then metric can't be made Euclidean.

Rem ② Isothermal coordinate

$F = 0 \quad E(u,v) = G(u,v)$

$$\Rightarrow K = -\frac{1}{2E^3}\left(EE_{vv} + EE_{uu} - E_u^2 - E_v^2\right)$$

Explicitly:

Euclidean Metric $\Rightarrow K \equiv 0$

S^2 we have $E=1$, $F=0$, $G=\sin^2 u \Rightarrow K \equiv +1$

Hyperbolic space $F=0$, $E=G=\frac{1}{y^2} \Rightarrow K \equiv -1$

Example S^3 as A Lie Group:

$S^3 \hookrightarrow \mathbb{R}^4 \simeq$ quaternions $\mathbb{H} = \{x_0 + x_1 i + x_2 J + x_3 K\}$

$i^2 = J^2 = K^2 = iJK = -I$

Multiplication in $\mathbb{H}$ can be Rewritten:

$$x \cdot y = (x_0, \vec{x}) \cdot (y_0, \vec{y}) = (x_0 y_0 - \vec{x} \cdot \vec{y},\ x_0 \vec{y} + y_0 \vec{x} + \vec{x} \times \vec{y})$$

$$\Rightarrow \underset{\overline{x}}{\overset{\|}{\overline{(x_0, \vec{x})}}} \cdot (x_0, \vec{x}) = \underset{= |x|^2}{x_0^2 + \vec{x} \cdot \vec{x}} \leftarrow \text{NORM OF } x \text{ IN } \mathbb{R}^4$$

Then $|x \cdot y|^2 = \overline{x \cdot y} \cdot x \cdot y = \overline{y} \cdot \overline{x} \cdot x \cdot y = |x|^2 |y|^2$

IF $x, y \in S^3$ then $x \cdot y \in S^3$.

Restriction of Multiplication in $\mathbb{H}$ to S^3 makes S^3 into A Lie group

Claim

The Map

$$\varphi((x_0, x_1, x_2, x_3)) = \begin{bmatrix} x_0 + ix_3 & x_1 + ix_2 \\ -x_1 + ix_2 & x_0 - ix_3 \end{bmatrix}$$

is A group (HOMOMORPHISM) AND (diffeomorphism) From group S^3 to group $SU(2)$ of unitary Matrices with $\det = 1$

Rem S^3 is a Lie group as group of unit Quaternions or as $SU(2)$. (its Lie Algebra $=\{i, J, K\}$ purely imaginary Quaternions or span of

$$\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$$

$\Rightarrow$ Lie Algebra of S^3: E_1, E_2, E_3

$$[E_1, E_2] = 2E_3 \quad [E_2, E_3] = 2E_1$$

$$[E_3, E_1] = 2E_2$$

induced Metric by $\mathbb{R}^4$ (it's Bi-invariant) $\{i, J, K\}$ are orthonormal.

We can compute:

$$\boxed{D^g_{E_i} E_i = 0}$$

$$\boxed{D^g_{E_1} E_2 = E_3} \quad \boxed{D^g_{E_3} E_1 = E_2}$$

$$\boxed{D^g_{E_2} E_3 = E_1}$$

$$D^g_{E_1} E_3 = -E_2 \quad D^g_{E_3} E_2 = -E_1$$

$$D^g_{E_2} E_1 = -E_3$$

The curvature:

$$R^g(E_1, E_2) E_3 = 0$$

$$R^g(E_1, E_2) E_2 = E_1 \quad \text{etc.}$$

The sectional curvatures:

$$sec(E_1 \wedge E_2) = 1 \quad sec(E_2, E_3) = 1 \quad sec(E_1, E_3) = 1$$

$\Rightarrow$ S^3 has constant sectional curvature $+1$.

Heisenberg group $H = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in \mathbb{R} \right\}$
Heisenberggroup

$E_1 = \frac{\partial}{\partial x}$, $E_2 = \frac{\partial}{\partial y}$, $E_3 = x\frac{\partial}{\partial y} + \frac{\partial}{\partial z}$

$[E_1, E_2] = 0 \quad [E_1, E_3] = E_2 \quad [E_2, E_3] = 0$

g Left-inv Metric (E_1, E_2, E_3 are Left-inv g-orthonor)

Koszul Formula $\rightarrow$ $D^g_{E_1} E_1 = 0 \quad D^g_{E_2} E_1 = -1/2\, E_3 \quad D^g_{E_3} E_1 = -1/2\, E_2$

$D^g_{E_1} E_2 = 1/2\, E_3 \quad D^g_{E_2} E_2 = 0 \quad D^g_{E_3} E_2 = 1/2\, E_1$

$D^g_{E_1} E_3 = -1/2\, E_2 \quad D^g_{E_2} E_3 = 1/2\, E_1 \quad D^g_{E_3} E_3 = 0$

curvatures: $R^g(E_1, E_2)E_2 = +\frac{3}{4} E_1 \quad R^g(E_1, E_3)E_3 = 1/4\, E_1$

$R^g(E_2, E_3)E_3 = 1/4\, E_2 \quad R^g(E_2, E_1)E_1 = +\frac{3}{4} E_2$

$R^g(E_3, E_1)E_1 = 1/4\, E_3 \quad R^g(E_3, E_2)E_2 = +1/4\, E_3$

All other curvatures are 0.

$\Rightarrow \sec(E_1 \wedge E_2) = -3/4 \quad \sec(E_1 \wedge E_3) = 1/4 \quad \sec(E_2, E_3) = 1/4$

so H has Both positive/Negative sectional curvatures.

$\mathrm{Ric} = \begin{bmatrix} -1/2 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}$ $\quad \mathrm{Ric}(E_1, E_1) = \sum R^g(E_1, E_i, E_1, E_i)$

$= R^g(E_1, E_2, E_1, E_2) + R^g(E_1, E_3, E_1, E_3)$

$= -\frac{3}{4} + 1/4 = \boxed{-1/2} \Rightarrow \mathrm{Scal} = -1/2$