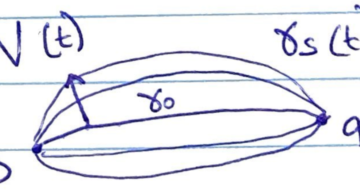
Lecture 11

Def A curve γ from p to q is minimizing curve (or minimal) if $\mathrm{dist}(p,q) = L_g(\gamma)$
i.e. it realizes the inf in $\mathrm{dist}(p,q)$.

Prop A unit speed minimizing curve is a geodesic.

Proof Let $\gamma_s(t)$, $|s|<\varepsilon$ be a smooth family of curves $\gamma = \gamma_0$ is minimizing from $p = \gamma(a)$ to $q = \gamma(b)$
and $\gamma_s(a) = p$ $\gamma_s(b) = q$



Then letting $V = \frac{d}{ds}\gamma_s\Big|_{s=0}$ we compute

$$\frac{d}{ds} L_g(\gamma_s)\Big|_{s=0} = \int_a^b \frac{d}{ds} g(\dot\gamma_s, \dot\gamma_s)^{1/2}\Big|_{s=0} dt$$

First variation of length

$$= \frac{1}{2}\int_a^b \frac{1}{|\dot\gamma_s|}\left(g\left(\frac{D}{ds}\dot\gamma_s, \dot\gamma_s\right) + g\left(\dot\gamma_s, \frac{D}{ds}\dot\gamma_s\right)\right)\Big|_{s=0} dt$$

$$* = \int_a^b \frac{1}{|\dot\gamma|} g\left(\frac{DV}{dt}, \dot\gamma\right) dt =$$

Lemma

$$\frac{DV}{dt} = \frac{D}{dt}\frac{d}{ds}\gamma_s(t)\Big|_{s=0} = \frac{D}{ds}\frac{d}{dt}\gamma_s(t)\Big|_{s=0} = \frac{D}{ds}\dot\gamma_s\Big|_{s=0}$$

compute Both side in local coordinates
use $\Gamma^k_{ij} = \Gamma^k_{ji}$

$$= g(V,\dot\gamma)\Big|_a^b - \int_a^b g\left(V, \underbrace{\frac{D\dot\gamma}{dt}}_{=\nabla_{\dot\gamma}\dot\gamma}\right) dt$$

integration by parts

$=0$ because $V(a)=V(b)=0$

IF The above vanishes FoR ALL smooth families of cuves with Fixed Endpoints at p AND q then $\nabla_{\dot\gamma}\dot\gamma = 0$ i.e γ is a geodesic

Note this shows that smooth cuves with Fixed endpoints that minimize are geodesics (Smoothness can be assumed see Lee p. 156)
"cutting corners" argument

Prop Up to Reparametrization (Radial) geodesics $t \mapsto \exp_p(tv)$ where $|t| < \varepsilon$ is small enough so that cuve stays in A Normal Neighborhood of p are the only minimizing geodesics From p to $\exp_p(tv)$.

Proof LaFontaine p89 theorem 2.92
Idea: Take Another cuve. AND show that its length is At least as large as the length oF Radial geodesics.

Cor Geodesics are <u>locally</u> distance Minimizing.

(AT SOME point it can stop minimizing distance)

Recall A unit speed minimizing curve is a geodesic.
Energy vs Length

Geodesics are locally distance minimizing (using normal neighborhood)

$E_g(\gamma) = \cancel{\frac{1}{2}} \int_a^b \|\dot\gamma(t)\|^2 dt$ is not invariant under reparametrization

$L_g(\gamma) = \cancel{\frac{1}{2}} \int_a^b \|\dot\gamma(t)\| dt$ is invariant

By Cauchy–Schwarz $L_g(\gamma)^2 = \left(\int_a^b \|\dot\gamma(t)\| dt\right)^2 \leq \int_a^b \|\dot\gamma(t)\|^2 dt \int_a^b 1$

$= 2(b-a)\, E_g(\gamma)$

and equality holds iff $\|\dot\gamma\| = \text{const}$ i.e iff γ has constant speed

Prop Let $p, q \in M$ and $\gamma : [a,b] \to M$ a curve joining p to q. Then γ is a minimizer for E_g iff it's minimizer for L_g and has constant speed.

Proof If γ has constant speed and minimizes L_g then any other curve $\alpha : [a,b] \to M$ with $\alpha(a) = p$ $\alpha(b) = q$ has

$L_g(\alpha) \geq L_g(\gamma)$ so $E_g(\alpha) \geq \frac{1}{2(b-a)} L_g(\alpha)^2 \geq \frac{1}{2(b-a)} L_g(\gamma)^2$

$\underset{\|\dot\gamma\| = \text{constant}}{=} E_g(\gamma)$ i.e γ minimizes E_g.

The converse will follow from 1st variation of Energy (still to come)

$\|f\|_{k,p} = \left(\sum_{i=0}^{k} \|f^{(i)}\|^p\right)^{1/p}$

1^{st} variation of Energy:

Hilbert space

Let γ_s be A Family of curves $|s|<\varepsilon$ $\left(\gamma_s \in W^{1,2}([a,b], M)\right)$

Hilbert Manifold of paths in M.

Set $V = \frac{d}{ds}\gamma_s\Big|_{s=0}$

Note that $\frac{DV}{dt} = \frac{d}{ds}\dot{\gamma}_s\Big|_{s=0}$ (Last Lecture)

$$dE_g(\gamma)V = \frac{d}{ds}E_g(\gamma_s)\Big|_{s=0} = \frac{1}{2}\int_a^b \frac{d}{ds} g(\dot{\gamma}_s, \dot{\gamma}_s)\Big|_{s=0} dt$$

$$= \int_a^b g\left(\frac{DV}{dt}, \dot{\gamma}\right) dt \underset{\substack{\uparrow \\ \text{integration by parts}}}{=} \underbrace{g(V, \dot{\gamma})\Big|_a^b}_{=0} - \int_a^b g\left(V, \frac{D\dot{\gamma}}{dt}\right) dt$$

Thus if $\gamma = \gamma_0$ is A critical pt of E_g then $dE_g(\gamma)V = 0$ FOR ALL V.

$\int_a^b f(x)h(x)dx = 0$ (f continuous, h "smooth") $\Rightarrow f \equiv 0$.

It follows (From Fundamental lemma of calculus of variations) that $\frac{D\dot{\gamma}}{dt} = 0$ i.e $\nabla_{\dot{\gamma}}\dot{\gamma} = 0$ (γ is geodesic hence constant speed joining p to q.)

2^{nd} variation of Energy

Remark Given a vector field $W(s,t)$ along $\gamma_s(t)$ we have

$$\frac{D}{ds}\frac{D}{dt}W - \frac{D}{dt}\frac{D}{ds}W = R\left(\frac{d}{ds}\gamma_s, \frac{d}{dt}\gamma_s\right)W$$

where R is the $(1,3)$-tensor given by:

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$$

"curvature tensor". (more later)

$\frac{D}{dt} = (\gamma^*\nabla)_{\frac{\partial}{\partial t}}$

Proof Compute in coordinates using $X = (\gamma_s)_* \frac{\partial}{\partial s}$, $Y = (\gamma_s)_* \frac{\partial}{\partial t}$

so that $[X,Y]=0$ and $\frac{D}{ds} = \nabla_X$ $\frac{D}{dt} = \nabla_Y$. ◻

Suppose $\gamma = \gamma_0$ is a geodesic then $\left(V = \frac{d}{ds}\gamma_s\Big|_{s=0}\right)$

$$d^2E_g(\gamma)(V,V) = \frac{d^2}{ds^2}E_g(\gamma_s)\Big|_{s=0} = \int_a^b \frac{d}{ds} g\left(\frac{DV}{dt}, \dot\gamma_s\right)\Big|_{s=0} dt$$

$\partial^2_{uu} f = u^T D^2 f u$

$$= \int_a^b g\left(\frac{D}{ds}\frac{D}{dt}V, \dot\gamma\right) + g\left(\frac{DV}{dt}, \frac{DV}{dt}\right) dt$$

Remark $\rightarrow$ $$= \int_a^b g\left(\frac{DV}{dt}, \frac{DV}{dt}\right) + g\left(\frac{D}{dt}\frac{D}{ds}V, \dot\gamma\right) + g\left(R(V,\dot\gamma)V, \dot\gamma\right) dt$$

integration by parts $\rightarrow$ $$= g\left(\frac{DV}{ds}, \dot\gamma\right)\Big|_a^b - \int_a^b g\left(\frac{DV}{ds}, \cancel{\frac{D\dot\gamma}{dt}}\right)$$

$0 =$ γ geodesic

$$+ \int_a^b g\left(\frac{DV}{dt}, \frac{DV}{dt}\right) + g\left(R(V,\dot\gamma)V, \dot\gamma\right) dt$$

$$\overset{0}{\overset{\parallel}{}}\qquad\qquad\qquad\qquad V(a)=V(b)=0$$

integration by parts + symmetry of curvature R.

$$= g\left(\frac{DV}{ds}, \dot\gamma\right)\Big|_a^b + g\left(\frac{DV}{dt}, V\right)\Big|_a^b -$$

often noted by V''

$$\int_a^b g\Big(\underbrace{\frac{D^2V}{dt^2} + R(V,\dot\gamma)\dot\gamma}_{\text{Jacobi operator}}, V\Big)\, dt$$

Def A vector field $V : [a,b] \longrightarrow TM$ along a geodesic is a Jacobi Field if it solves the Jacobi Equation

$$V'' + R(V,\dot\gamma)\dot\gamma = 0.$$

(describes difference between geodesic AND an infinitesimally close geodesic)

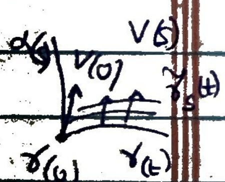
Prop The variational field $V(t) = \frac{d}{ds}\gamma_s\big|_{s=0}$ is a Jacobi Field along the geodesic $\gamma = \gamma_0$ iff curves $t \longrightarrow \gamma_s(t)$ are geodesics for $|s| < \varepsilon$.

Proof Suppose $\gamma_s(t)$ are geodesics.

$$V''(t) = \frac{D}{dt}\frac{D}{dt}\frac{d}{ds}\gamma_s(t) = \frac{D}{dt}\frac{D}{ds}\frac{d}{dt}\gamma_s(t) = \frac{D}{ds}\underbrace{\frac{D}{dt}\dot\gamma_s(t)}_{\substack{=0 \text{ because} \\ \gamma_s(t) \text{ is geod.}}}$$

$$- R(V,\dot\gamma)\dot\gamma$$

So V is a Jacobi Field.



Conversely if V is a Jacobi Field then let

$\alpha(s) = \exp_{\gamma(0)}(sV(0))$ and let $X(s)$ be vector field

along $\alpha(s)$ with $X(0) = \dot\gamma(0)$ $X'(0) = V'(0)$

Set $\tilde\gamma_s(t) = \exp_{\alpha(s)}(tX(s))$ $\left(\tilde\gamma_s(0) = \alpha(s)\right)$

$\alpha(0) = p = \gamma(0)$

Since $t \to \tilde{\gamma}_s(t)$ are geodesics $\Rightarrow$ $\tilde{V}(t) = \frac{d}{ds}\tilde{\gamma}_s(t)\Big|_{s=0}$

satisfies $\tilde{V} + R(\tilde{V}, \tilde{\gamma}')\tilde{\gamma}' = 0$

Moreover $\tilde{V}(0) = \frac{d}{ds}\tilde{\gamma}_s(0)\Big|_{s=0} = \alpha'(0) = V(0)$.

and $\tilde{V}'(0) = \frac{D}{dt}\frac{d}{ds}\tilde{\gamma}_s(t)\Big|_{\substack{s=0\\t=0}} = \frac{D}{ds}\frac{d}{dt}\tilde{\gamma}_s(t)\Big|_{\substack{s=0\\t=0}}$

$$= \frac{D}{ds}X(s)\Big|_{s=0} = X'(0) = V'(0).$$

so by uniqueness of sol to ODE with same initial conditions

$$V(t) = \tilde{V}(t) = \frac{d}{ds}\tilde{\gamma}_s(t)\Big|_{s=0}$$

Hence V is the variational field of family of geodesics $\tilde{\gamma}_s(t)$.