
Problem 1. Let $H = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in \mathbb{R} \right\}$ be the Heisenberg group (with matrix multiplication).

Show that the vectors $\left\{ E_1 = \frac{\partial}{\partial x}, E_2 = \frac{\partial}{\partial y}, E_3 = x\frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right\}$ are left-invariant and form a basis of the Lie algebra of H. Compute the left-invariant metric g if we declare the vectors E_1, E_2, E_3 to be orthonormal.

Problem 2. Let (M^n, g) be a Riemannian manifold and D^g the Levi-Civita connection of g. Prove that for any vector field X one has:

$$|\mathcal{L}_X g|^2 = 2|D^g X|^2 + 2tr(D^g X \circ D^g X),$$

where $|\mathcal{L}_X g|^2 = \sum_{i,j=1}^{n} \left((\mathcal{L}_X g)(e_i, e_j) \right)^2$ is the norm square of the Lie derivative of g with respect to X and $|D^g X|^2 = \sum_{i=1}^{n} g(D^g_{e_i} X, D^g_{e_i} X)$ and $tr(D^g X \circ D^g X) = \sum_{i=1}^{n} g(D^g_{D^g_{e_i} X} X, e_i)$. Here $\{e_1, \cdots, e_n\}$ is a local g-orthonormal frame of TM.

Problem 3. Let $\phi : M \to \tilde{M}$ be a diffeomorphism and ∇ be a connection on TM. Define $\tilde{\nabla}$ by

$$\tilde{\nabla}_{\tilde{X}} \tilde{Y} = \phi_* \left(\nabla_{\phi_*^{-1}(\tilde{X})} \left(\phi_*^{-1}(\tilde{Y}) \right) \right),$$

for $\tilde{X}, \tilde{Y}$ vector fields on $\tilde{M}$. Prove that $\tilde{\nabla}$ is a connection on $T\tilde{M}$.

Problem 4. Prove that the Lie derivative is not a connection. Show that there are vector fields V and W on $\mathbb{R}^2$ such that $V = W = \frac{\partial}{\partial x}$ along the x-axis but with $\mathcal{L}_V \frac{\partial}{\partial y} \neq \mathcal{L}_W \frac{\partial}{\partial y}$ along the x-axis. (Remark: this shows that the Lie derivative does not give a well-defined way to take directional derivatives of vector fields along curves).

Problem 5. Consider the linear connection on half plane $y > 0$ on $\mathbb{R}^2$ defined by the components $\Gamma_{ij}^k = 0$ except $\Gamma_{12}^1 = 1$ with respect to the frame $\{e_1 = \frac{\partial}{\partial x}, e_2 = \frac{\partial}{\partial y}\}$. Consider the frame $\{\tilde{e}_1 = \frac{\partial}{\partial x}, \tilde{e}_2 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\}$. Compute the components of the connection and the torsion with respect to this frame.

Problem 6. Let $\{x_1 = x, x_2 = y\}$ be the usual coordinates on $\mathbb{R}^2$. Define a linear connection on $\mathbb{R}^2$ by $\Gamma_{ij}^k = 0$ except $\Gamma_{12}^1 = \Gamma_{21}^1 = 1$.

- write and solve the differential equations of geodesics.
- are the geodesics defined for $t \in (-\infty, +\infty)$?
- find the particular geodesic γ with $\gamma(0) = (2, 1)$ and $\gamma\prime(0) = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}$.
- do the geodesics emanating from the origin go through all the points of the plane.

Problem 7. Let $\{x_1 = x, x_2 = y\}$ be the usual coordinates on $\mathbb{R}^2$. Define a linear connection on $\mathbb{R}^2$ by $\Gamma_{ij}^k = 0$ except $\Gamma_{12}^1 = 1$. Consider the curve $\gamma(t) = (-2e^{-t} + 4, t + 1)$. Compute the vector field obtained by parallel transport along γ of its tangent vector at $\gamma(0)$. Is γ a geodesic curve?