Math 71000 - Homework #1 Spring 2025
Gradaute Center

Problem 1. In stereographic coordinates with respect to the north pole, the immersion ι of the unit 2-dimensional sphere S^2 in $\mathbb{R}^3$ is given by

$$\iota(u,v) = \left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1}\right) = (x,y,z).$$

Let $g = dx\otimes dx + dy\otimes dy + dz\otimes dz$ be the Euclidean metric on $\mathbb{R}^3$. Prove that

$$\iota^* g = \frac{4}{(u^2+v^2+1)^2}\,(du\otimes du + dv\otimes dv)$$

Problem 2. On a Riemannian manifold (M,g), we consider a C^1 curve $\gamma : [a,b] \to (M,g)$. We denote by $L_g(\gamma)$ the length of γ with respect to the metric g. Show that L_g is invariant under reparametrization.

Problem 3 Suppose that $\phi : (M,g) \to (M,h)$ is an isometry where g,h are Riemannian metrics. We consider a C^1 curve $\gamma : [a,b] \to (M,g)$. We denote by $L_g(\gamma)$ the length of γ with respect to the metric g. Show that $L_g(\gamma) = L_h(\phi\circ\gamma)$.

Problem 4 Consider the hyperboloid model of the hyperbolic space

$$\mathbb{H}^n = \{(x_0,x_1,\cdots,x_n)\in\mathbb{R}^{n+1} \mid \sum_{i=0}^{n-1} x_i^2 - x_n^2 = -1,\ \ x_n > 0\}.$$

Consider the Riemannian metric g on $\mathbb{H}^n$ given by $dx_0^2 + \cdots + dx_{n-1}^2 - dx_n^2$ restricted to $\mathbb{H}^n$.

Let $s = (0,0,\cdots,0,-1)$ and define

$$f(x) = s - 2\frac{(x-s)}{\langle x-s, x-s\rangle},$$

where $\langle x,y\rangle = \sum_{i=0}^{n-1} x_i y_i - x_n y_n$, here $x = (x_0,\cdots,x_n)$ and $y = (y_0,\cdots,y_n)$.

Show that f is a diffeomorphism from $\mathbb{H}^n$ onto the unit disk$=\{x = (x_0,\cdots,x_{n-1})\in\mathbb{R}^n \mid \sum_{i=0}^{n-1} x_i^2 < 1\}$ and that

$$(f^{-1})^*(g) = \frac{4}{\left(1-\sum_{j=0}^{n-1} x_j^2\right)^2}\sum_{i=0}^{n-1} dx_i^2.$$

Problem 5 Show that the Riemannian circles (S^1,g) and (S^1,h) are isometric if and only they have the same length. Here g and h are Riemannian metrics.

Problem 6 Let (M,g) be a 1-dimensional Riemannian manifold. Show that LOCALLY the metric g can be expressed as $g = ds\otimes ds$, where s is the arc-length parameter.