Solutions to Review Sheet 2

Problem ①

$$g^{-1} = \begin{pmatrix} e^{-2z} & 0 & 0 \\ 0 & e^{-2z} & 0 \\ 0 & 0 & e^{-2z} \end{pmatrix}$$

The only non vanishing Christoffel symbols are:

taking $x = x_1$, $y = x_2$, $z = x_3$

$$\Gamma^1_{13} = \Gamma^2_{23} = -\Gamma^3_{11} = -\Gamma^3_{22} = \Gamma^3_{33} = 1$$

then

$$R\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial z}, \frac{\partial}{\partial x}, \frac{\partial}{\partial z}\right) = g\left(R\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial z}\right)\frac{\partial}{\partial z}, \frac{\partial}{\partial x}\right) = 0$$

Problem ②

$\{e_1, \dots, e_{n-1}, X\}$ is orthonormal

$$Sec(e_i \wedge X) = \frac{R(e_i, X, e_i, X)}{g(X,X)g(e_i,e_i) - g(X,e_i)} \overset{\downarrow}{=} g(R(e_i, X, e_i, X)$$

$$\text{But } Ric(X,X) = \sum_{i=1}^{n-1} R(X, e_i, X, e_i) + \overset{0}{\cancel{R(X,X,X,X)}}$$

curvature symmetries $\rightarrow$

$$= \sum_{i=1}^{n} R(e_i, X, e_i, X) = \sum_{i=1}^{n} Sec(e_i \wedge X).$$

Problem ③ $D^g_{E_1} E_1 = ?$ Apply Koszul Formula

$$g(D^g_{E_1} E_1, E_2) = \tfrac{1}{2}\left[g([E_2, E_1], E_1) + g([E_2, E_1], E_1)\right] = g(E_2, E_1) = 0$$

$$g(D_{E_1} E_1, E_3) = \tfrac{1}{2}\left[g([E_3, E_1], E_1) + g([E_3, E_1], E_1)\right] = -g(E_3, E_1) \neq 0$$

$$\Rightarrow D^g_{E_1} E_1 = 0$$

$$g(D^g_{E_2} E_1, E_2) = 1/2\left(g([E_2,E_1], E_2) + g([E_2,E_1],E_2)\right) = -g(E_2,E_2)$$
$$= -1$$

$$g(D^g_{E_2} E_1, E_3) = 1/2\left(g([E_2,E_1], E_3) + g([E_3,E_2], E_1)\right) \cancel{= g(E_3,E_3)}$$
$$\cancel{= -1} \quad + g([E_3,E_1],E_2)$$
$$\Rightarrow \cancel{D_{E_3} E_1}$$
$$= 1/2\left[-g(E_2,E_3) - g(E_3,E_2)\right] = 0$$

$$\Rightarrow D^g_{E_2} E_1 = -E_2.$$

$$g(D^g_{E_1} E_2, E_1) = 1/2\, g([E_1,E_2],E_1) + g([E_1,E_2],E_1))$$
$$= 1/2\, g(E_2,E_1) + g(E_2,E_1) = 0$$

$$g(D^g_{E_1} E_2, E_3) = 1/2\, g([E_1,E_2],E_3) + g([E_3,E_1],E_2) +$$
$$g([E_3,E_2],E_1) = 1/2\left(g(E_2,E_3) - g(E_3,E_2)\right) = 0$$

$$\Rightarrow D^g_{E_1} E_2 = 0$$

$$g(D^g_{E_2} E_2, E_1) = 1/2\left(g([E_1,E_2],E_2) + g([E_1,E_2],E_2)\right)$$
$$= 1/2\left(g(E_2,E_2) + g(E_2,E_2)\right) = 1$$

$$g(D^g_{E_2} E_2, E_3) = 0$$

$$\Rightarrow D^g_{E_2} E_2 = E_1$$

$$\mathrm{Sec}(E_1 \wedge E_2) = R^g(E_1, E_2, E_1, E_2) = g(R^g(E_1, E_2)E_1, E_2)$$

$$= g\left(\cancelto{0}{D^g_{E_2}(D^g_{E_1}E_1)} - D^g_{E_1}(D^g_{E_2}E_1) + D^g_{[E_1,E_2]}E_1, E_2\right)$$

$$= g\left(+ D^g_{E_1}E_2 + D^g_{E_2}E_1, E_2\right)$$

$$= g(-E_2, E_2) = -1 \quad \Rightarrow \quad \boxed{\mathrm{Sec}(E_1 \wedge E_2) = -1}$$

$$\mathrm{Ric}^g(E_1, E_2) = \cancelto{0}{R^g(E_1, E_1, E_2, E_1)} + \cancelto{0}{R^g(E_1, E_2, E_2, E_2)} + R^g(E_1, E_3, E_2, E_3)$$

So $g(D_{E_3}E_2, E_1) = \frac{1}{2}\left[g([E_3, E_2], E_1) + g([E_1, E_3], E_2) + g([E_1, E_2], E_3)\right.$

$$= \frac{1}{2}\left(g(E_3, E_2) + g(E_2, E_3)\right) = 0$$

$$g(D_{E_3}E_2, E_3) = 0$$

$$\Rightarrow D_{E_3}E_2 = 0$$

So $\mathrm{Ric}^g(E_1, E_3) = R^g(E_1, E_3, E_2, E_3)$

$$= g\left(\cancelto{0}{D^g_{E_3}(D^g_{E_1}E_2)} - \cancelto{0}{D^g_{E_1}(D^g_{E_3}E_2)} + D_{[E_1,E_3]}E_2, E_3\right)$$

$$= g(D^g_{E_3}E_2, E_3) = 0$$

$$\Rightarrow \mathrm{Ric}^g(E_1, E_3) = 0.$$

Problem 4

Since M is compact orientable 2-dim Riemannian manifold with positive Gauss curvature K its Euler characteristic is positive AND so M is Homeomorphic to A sphere. Let $c_1, c_2 : [0,T] \to M$ be two NoN self intersecting closed geodesics on M.

Assuming they do Not intersect they bound A Region S on M that's Homeomorphic to A cylinder. Hence $\chi(S) = 0$. By Gauss Bonnet For surfaces with Boundary we have $\int_S K = 0$ which is impossible since $K > 0$.

Problem 5

$$T^{\bar\nabla}_{X,Y} = \bar\nabla_X Y - \bar\nabla_Y X - [X,Y]$$

$$= T^{\nabla}_{X,Y} - \frac{1}{4}\left(\left(\nabla_{JY}J\right)X + J(\nabla_Y J)X + 2J(\nabla_X J)Y\right)$$

$$+ \frac{1}{4}\left(\left(\nabla_{JX}J\right)Y + J(\nabla_X J)Y + 2J(\nabla_Y J)X\right)$$

$$= T^{\nabla}_{X,Y} - \frac{1}{4}\left((\nabla_{JY}J)X - (\nabla_{JX}J)Y - J(\nabla_Y J)X + J(\nabla_X J)Y\right)$$

But $4N(X,Y) \underset{\uparrow}{=} J(\nabla_Y J) - J(\nabla_X J)Y - (\nabla_{JY}J)X + (\nabla_{JX}J)Y$

∇ Torsion Free
ex. Levi-Civita connection

$$\Rightarrow T^{\bar\nabla}_{X,Y} \underset{\uparrow}{=} N(X,Y)$$

$T^{\nabla}_{X,Y} = 0$

Problem 6

$$N(E_1, E_3) = [JE_1, JE_3] - [E_1, E_3] - J[JE_1, E_3] - J[E_1, JE_3]$$

$$\Rightarrow N(E_1, E_3) = [E_2, E_4] - [E_1, E_3] - J[E_2, E_3] - J[E_1, E_4]$$

$$\Rightarrow N(E_1, E_3) = -JE_2 = E_1 \neq 0$$

$\Rightarrow J$ is Not integrable.

Problem 7 Let D^g be Levi-civita connection then

$$4N(X, Y) = (D^g_{JX} J)Y - J(D^g_X J) - (D^g_{JY} J)X + J(D_Y J)X$$

it is Easy to check that

$$g(JX, N(Y, Z)) - g(JY, N(Z, X)) - g(JZ, N(X, Y)) =$$

$$\frac{1}{2}\left[g((D^g_X J)Y, Z) + g(J(D^g_{JX} J)Y, Z)\right.$$

So if $N \equiv 0 \Rightarrow (D^g_X J) = -J(D^g_{JX} J)$

$$\Rightarrow J \circ (D^g_X J) = D^g_{JX} J$$

Problem 8 we proved in class that:

$$(4N(X, Y), Z) = d\omega(JX, Y, Z) + d\omega(X, JY, Z) - 2g((D^g_Z J)X, JY)$$

so if ω is symplectic:

$$g(N(X,Y),Z) = -\tfrac{1}{2}\, g((D_Z J)X, JY)$$

and so $\Rightarrow g((D_Z J)X, JY) \underset{(*)}{=} -2g(N(X,Y),Z)$

on the other hand

$$g((D_{JZ}J)X, JY) = -2g(N(X,Y), JZ)$$

$$\xrightarrow{g \text{ is } J\text{-invariant}} = 2g(JN(X,Y), Z)$$

$$\xrightarrow{N(JX,Y) = -JN(X,Y)} = -2g(N(JX,Y), Z)$$

$$\overset{(*)}{=} g((D^g_Z J)JX, JY)$$

$$J(D^g_Z J) + (D^g_Z J)J = 0 \rightarrow = -g(J(D^g_Z J)X, JY)$$

Hence $g((D^g_{JZ}J)X, JY) = -g(J(D^g_Z J)X, JY)$

so $(D^g_{JZ}J) = -J(D^g_Z J)$

g is Kähler $\Leftrightarrow$ ~~g is~~ ~~Ke~~ so J is Hermitian and g almost Kähler so

$$(D^g_{JZ}J) = -J(D^g_Z J) \text{ and } (D^g_{JZ}J) = +J(D^g_Z J)$$

$$\Rightarrow \boxed{D^g J = 0}$$

Problem 9

$$(\mathcal{L}_X J)Y = [X, JY] - J[X, Y]$$

$$(\mathcal{L}_{E_1} J)E_2 = [E_1, JE_2] - J[E_1, E_2] = -[E_1, E_1] - JE_3 = -JE_3 = -E_4$$

so $\mathcal{L}_{E_1} J \neq 0 \Rightarrow E_1$ is Not Holomorphic

$$(\mathcal{L}_{E_2} J)E_1 = [E_2, JE_1] - J[E_2, E_1] = [E_2, E_2] \cancel{-JE} + JE_3 = E_4$$

So $\mathcal{L}_{E_2} J \neq 0 \Rightarrow E_2$ is Not Holomorphic

$$(\mathcal{L}_{E_3} J)E_i = [E_3, JE_i] - J[E_3, E_i] = 0 \ \forall i \text{ because } [E_3, \cdot] = 0$$

$\Rightarrow E_3$ is Holomorphic

$$(\mathcal{L}_{E_4} J)E_1 = [E_4, JE_1] - J[E_4, E_1] = [E_4, E_2] + JE_2 = -E_1$$

$\Rightarrow E_4$ is Not Holomorphic

The only Holomorphic vector is E_3