Math 71000 - Review sheet Spring 2025
Gradaute Center

Problem 1. Consider on $\mathbb{R}^3$ the metric

$$g = e^{2z}\,(dx^2 + dy^2 + dz^2).$$

Compute $R^g(\frac{\partial}{\partial x}, \frac{\partial}{\partial z}, \frac{\partial}{\partial x}, \frac{\partial}{\partial z})$, where R^g is the Riemannian $(0,4)$-curvature tensor.

Problem 2. Let (M,g) be a Riemannian manifold of dimension n. Consider an orthonormal basis $\{e_1, \cdots, e_{n-1}, X\}$ of T_pM. Prove that

$$Ric^g(X,X) = \sum_{i=1}^{n-1} sec(e_i \wedge X),$$

where Ric^g is the Riemannian Ricci tensor and $sec(e_i \wedge X)$ is the sectional curvature of the plane spanned by e_i and X.

Problem 3. Consider the Lie algebra of dimension 3 given by the structure equations

$$[E_1, E_2] = E_2,\ [E_1, E_3] = E_3.$$

Consider the left-invariant Riemannian metric g defined when we declare E_1, E_2, E_3 to be g-orthonormal. Compute $sec(E_1 \wedge E_2)$ and $Ric^g(E_1, E_2)$.

Problem 4. Let (M,g) be a compact 2-dimensional Riemannian manifold with positive Gaussian curvature. Show that any two non-self intersecting closed geodesics must intersect each other (Hint: the Euler characteristic of the cylinder is zero).

Problem 5. Let (M,J) be an almost-complex manifold. If ∇ is a connection on M whose torsion tensor vanishes, define the connection $\bar{\nabla}$ by

$$\bar{\nabla}_X Y = \nabla_X Y - \frac{1}{4}\left((\nabla_{JY} J)\, X + J\,(\nabla_Y J)\, X + 2J\,(\nabla_X J)\, Y\right).$$

Compute the torsion of $\bar{\nabla}$ in terms of the Nijenhuis tensor of J.

Problem 6. Consider the Lie algebra of dimension 4 given by the structure equations

$$[E_1, E_2] = E_3, \ [E_1, E_4] = E_2.$$

Consider the left-invariant almost-complex structure defined by

$$JE_1 = E_2, \quad JE_3 = E_4.$$

is J integrable?

Problem 7. Let (M, J, g) be an almost-Hermitian manifold. Prove that if J is integrable then

$$D^g_{JX} J = J\, D^g_X J,$$

for any vector field X. Here D^g is the Levi-Civita connection of the Riemannian metric g.

Problem 8. Let (M, J, g) be an almost-Hermitian manifold. Prove that if g is an almost-Kähler metric then

$$D^g_{JX} J = -J\, D^g_X J,$$

for any vector field X. Here D^g is the Levi-Civita connection of the Riemannian metric g. Deduce from the previous problem that if g is Kähler then $D^g J = 0$.

Problem 9. Consider the Lie algebra of dimension 4 given by the structure equations

$$[E_1, E_2] = E_3, \ [E_1, E_4] = E_2.$$

Consider the left-invariant almost-complex structure defined by

$$JE_1 = E_2, \quad JE_3 = E_4.$$

A vector field X is said to be holomorphic if $\mathcal{L}_X J = 0$, where $\mathcal{L}$ is the Lie derivative. Is any of the vectors E_i holomorphic?