Math 71000 - Review sheet Spring 2025
Gradaute Center

Problem 1. We consider the surface of revolution the paraboloid in $\mathbb{R}^3$ given by the parametrrization

$$\iota(u,v) = \big(u\cos(v), u\sin(v), u^2\big) = (x,y,z).$$

Let $g = dx\otimes dx + dy\otimes dy + dz\otimes dz$ be the Euclidean metric on $\mathbb{R}^3$. Compute $\iota^* g$.

Problem 2. Let $B = \{x^2+y^2<1\}$ be the unit disk. Compute the volume of B with respect to the spherical metric

$$g = \frac{4}{(1+x^2+y^2)^2}\left(dx\otimes dx + dy\otimes dy\right).$$

Problem 3 Consider the upper half plane $\{(x,y)\in\mathbb{R}^2 |\, y>0\}$ with the group operation :

$$(a,b)\dot{(}c,d) = (a+bc, bd).$$

The identity element is $(0,1)$.

Compute the right-invariant metric if we declare $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$ to be orthonormal at $(0,1)$ and show that it's isometric to the left-invariant one.

Problem 4 Consider the metric

$$g = x^2\, dx\otimes dx + \frac{1}{\sqrt{2}} xy\,(dx\otimes dy + dy\otimes dx) + y^2\, dy\otimes dy$$

Compute $(\frac{\partial}{\partial x})^\flat$ and $(dy)^\sharp$ where $\flat$ and $\sharp$ are the musical isomorphism induced by g.
Compute $g(dx,dy)$ and $g(dx\wedge dy, dx\wedge dy)$

Problem 5 We introduce in $\mathbb{R}^3$, with the usual Euclidean metric the connection ∇ defined in Cartesian coordinates $\{x_1,x_2,x_3\}$ by

$$\Gamma^i_{jk} = \omega\,\epsilon_{ijk},$$

where $\omega:\mathbb{R}^3\to\mathbb{R}$ is a smooth function and $\epsilon_{ijk} = \begin{cases} +1 \text{ if } (i,j,k) \text{ is an even permutation of } (1,2,3) \\ -1 \text{ if } (i,j,k) \text{ is an odd permutation of } (1,2,3) \\ 0 \text{ otherwise}\end{cases}$

- Show that ∇ is compatible with the Euclidean metric.
- The geodesics of ∇ are straight lines.
- The torsion of ∇ is not zero in all points where $\omega\neq 0$ (therefore ∇ is not the Levi-Civita connection unless $\omega\equiv 0$.)

Problem 6 Let $f : (M, g) \to (N, h)$ be an isometry between Riemannian manifolds. Show that if $\gamma : I \to M$ is a geodesic with respect to the Levi-Civita connection then $f \circ \gamma : I \to N$ is also a geodesic (Hint: prove first that $f_* \left(D^g_X Y\right) = D^h_{f_* X} f_* Y$, for X, Y vector fields on M.)

Problem 7 Let g be the round metric on S^2 induced by the Euclidean metric

$$g = d\theta \otimes d\theta + \sin^2(\theta) d\alpha \otimes d\alpha$$

- Compute the Christoffel symbols for the Levi-Civita connection.

- Show that the equator is the image of a geodesic.

- Show that any rotation about any axis through the origin in $\mathbb{R}^3$ induces an isometry of S^2

Problem 8 Consider the hyperbolic metric on the upper half plane $\mathbb{H} = \{(x, y) \mid y > 0\}$

$$g = \frac{1}{y^2} \left(dx \otimes dx + dy \otimes dy\right).$$

- Compute the Christoffel symbols for the Levi-Civita connection.

- Show that the curves $\alpha, \beta : \mathbb{R} \to \mathbb{H}$

$$\alpha(t) = (0, e^t)$$
$$\beta(t) = \left(\tanh t, \frac{1}{\cosh t}\right)$$

  are geodesics. What are the sets $\alpha(\mathbb{R}), \beta(\mathbb{R})$?