Lecture 16

Definition An Almost-Hermitian Manifold (M, J, g, ω) is

① Hermitian if J is integrable (cplex structure)

② Almost-Kähler if ω is symplectic ie $d\omega = 0$ ω is closed

③ Kähler if J is integrable + ω is symplectic (Almost-Kähler + Hermitian)

Rem There are MORE very interesting Metrics to investigate.

LEMMA An Almost Hermitian (M, J, g, ω) is Kähler iff $D^g J = 0$ (**Rem**: $Hol(D^g)$ of A Kähler is in $U(n)$)

Rem D^g preserves the Kähler structure.

Proof

Suppose $D^g J = 0$ Then

$$(*) \quad d\omega(X, Y, Z) = ((D^g_X J) Y, Z) + ((D^g_Y J) Z, X) + ((D_Z J) X, Y)$$

$$4N(X, Y) = J(D_Y J) X - J(D_X J) Y - (D^g_{JY} J) X + (D^g_{JX} J) Y.$$

$$\Rightarrow d\omega = N = 0$$

suppose $d\omega = N = 0$ then

$$(4N(X,Y), Z) = g(J(D_Y J)X + (D^g_{JX} J)Y, Z) + g(-J(D^g_X J)Y - (D^g_{JY} J)X, Z)$$

$$(*) \quad \Rightarrow = d\omega(JX, Y, Z) + d\omega(X, JY, Z) - 2g((D_Z J)X, JY)$$

$$g((D^g_X J)Y, Z) = \tfrac{1}{2} d\omega(X, Y, Z) - d\omega(X, JY, JZ) + 2g(JX, N($$

Examples of Kähler Metrics

① Any 2-dim Almost-Hermitian manifold is Kähler:

$$d\omega = 0$$
$$N(X, JX) = 0.$$

② $\mathbb{C}^n$ with Flat Metric:

$$\omega = \frac{i}{2} \sum dz_i \wedge d\bar{z}_i = \frac{i}{2} \partial\bar{\partial} |z|^2 = \sum dx_i \wedge dy$$

$$J = J_0$$

③ $\mathbb{C}P^n$ Fubini-Study Metric $\quad S^1 \hookrightarrow S^{2n+1}$ Free action and isometrically

$S^{2n+1}/S^1 = \mathbb{C}P^n$

$\pi : S^{2n+1} \to \mathbb{C}P^n$ Riemannian submersion.

Rem For a Non Kähler Manifold Lemma suggests that D^g is Not Natural connection because it Doesn't preserve J

$\Rightarrow$ Hermitian connection

Example 2 $\quad H = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \right\} \times \mathbb{R} = \{t\}.$

The left invariant vector fields:

$$\left\{ E_1 = \frac{\partial}{\partial x},\ E_2 = \frac{\partial}{\partial y},\ E_3 = x\frac{\partial}{\partial y} + \frac{\partial}{\partial z},\ E_4 = \frac{\partial}{\partial t} \right\}$$

$$[E_1, E_3] = E_2$$

$$g = dx^2 + dy^2 - 2x\, dy \odot dz + (1+x^2)\, dz^2 + dt^2$$

$$\Rightarrow \begin{cases} E_1^* = g(E_1) = dx \\ E_2^* = g(E_2) = dy - x\,dz \\ E_3^* = g(E_3) = dz \end{cases}$$

$$E_4^* = g(E_3) = dt$$

Define $\qquad J E_1 = E_2 \qquad J E_3 = E_4$

is J-integrable?

$$N(E_1, E_3) = [E_2, E_4] - [E_1, E_3] - J[E_1, E_4] - J[E_2, E_3]$$

$$= -E_2 \quad \underline{\text{No}}$$

~~Define~~ $\quad \cancel{J} \qquad g = \sum E_i^* \otimes E_i^* \Rightarrow \omega = -JE_1^* \otimes E_1^*$
$$-JE_2^* \otimes E_2^*$$
$$-JE_3^* \otimes E_3^*$$
$$-JE_4^* \otimes E_4^*$$

$J = J^* = -J^T$

$(2)\times$

$$\Rightarrow \omega = E_1^* \wedge E_2^* + E_3^* \wedge E_4^*$$

$$\omega = dx \wedge (dy - x\,dz) + dz \wedge dt$$

$$\Rightarrow d\omega = 0$$

symplectic But NON Kähler

Define $JE_1 = E_3 \quad JE_2 = E_4$

$$N(E_1, E_2) = [E_3, E_4] - [E_1, E_2] - J[E_3, E_2] - J[E_1, E_4]$$

$\Rightarrow N = 0$ integrable.

is g Kähler?

$$\omega = E_1^* \wedge E_3^* + E_2^* \wedge E_4^* = dx \wedge dz + (dy - x\,dz) \wedge dt$$

$d\omega = dx \wedge dz \wedge dt \neq 0$ No.

$H/\Gamma \times S^1$ Kodaira–Thurston 1st compact complex, symplectic But NON Kähler

Γ Lattice discrete subgroup (integral entries)

Twisted Exterior Differential:

$d : \Lambda^p \to \Lambda^{p+1}$ Forms on M

For $\alpha \in \Lambda^p$ $\quad (J\alpha)(X_1, \dots, X_p) = (-1)^p \alpha(JX_1, \dots, JX_p)$

Define twisted Exterior Differential:

$$d^c = J d J^{-1} \quad \text{where } J^{-1} = (-1)^p J \text{ acting on } p\text{-Forms}$$

$\Rightarrow d^c f = J df \Rightarrow (d^c f)(X) = -df(JX)$.

$\Rightarrow d^{c^2} = 0$.

Lemma Let (M, J) be Almost-complex

J integrable $\iff dd^c + d^c d = 0$

Rem on Functions $(dd^c + d^c d) f = 0 \Rightarrow dd^c f$ is J-inv

$$dd^c f(JX, JY) = dd^c f(X, Y)$$

Proof

$$(dd^c f)(X,Y) = (D_X d^c f)(Y) - (D_Y d^c f)(X)$$

$$= X \cdot (d^c f(Y)) - Y \cdot (d^c f(X)) - d^c f([X,Y])$$

$$= -X \cdot (JY \cdot f) + Y \cdot (JX \cdot f) + J[X,Y] \cdot f$$

AND $(d^c d f)_{X,Y} = -(dd^c f)_{JX,JY} = -JX \cdot (Y \cdot f) + JY \cdot (X \cdot f) - J[JX,JY] f$

$\Rightarrow$

$$dd^c f + d^c d f = 4\, d^c f(N(X,Y)) \checkmark$$

Rem Let $\alpha \in T^*M^{1,0}$ $X, Y \in TM \otimes \mathbb{C}$

$$(d\alpha)^{0,2}(X,Y) = d\alpha(X^{0,1}, Y^{0,1}) = (D_{X^{0,1}} \alpha)(Y)^{0,1} - (D_{Y^{0,1}} \alpha) X^{0,1}$$

$$= X^{0,1}(\alpha(Y^{0,1})) - Y^{0,1}(\alpha(X^{0,1})) - \alpha([X^{0,1}, Y^{0,1}])$$

$$= -\alpha^{1,0}(N(X,Y))$$

$$\Rightarrow (d\alpha)^{0,2} = 0 \Leftrightarrow N = 0$$

$$d : \Lambda^{1,0} \to \Lambda^{2,0} \oplus \Lambda^{1,1}$$

Prop J is integrable $\Leftrightarrow d(\Lambda^{p,q}) \subset \Lambda^{p,q+1} \oplus \Lambda^{p+1,q}$

Define Dolbeault operators : For $\psi \in \Lambda^{p,q}$

$$\partial \psi = (d\psi)^{p+1,q} \qquad (\bar\partial \psi) = (d\psi)^{p,q+1}$$

J is integrable $\Leftrightarrow d = \partial + \bar\partial \Leftrightarrow d^c = i(\bar\partial - \partial)$

$\Rightarrow \partial = \frac{1}{2}(d + i d^c) \qquad \bar{\partial} = \frac{1}{2}(d - i d^c)$

$\Rightarrow \partial^2 = \bar{\partial}^2 = \partial\bar{\partial} + \bar{\partial}\partial = 0$

AND $\quad dd^c = 2i\,\partial\bar{\partial}$.