Lecture 15

Complex structures

Definition A complex stucture on A Real vector space V is A Linear Endomorphism J of V such that

$$J^2 = -Id|_V \qquad \det J^2 = (-1)^n \Rightarrow n \text{ is even}$$

- Rem① A Real vector space with A cplex stuctue can be given the stuctue of A complex vector space by defining:

$$((a+ib), X) = aX + bJX.$$

- Rem② Conversely if we are given A cplex vector spa V of complex dim n we can define the linear endomorphism J of V by $J(X) = iX$.

Then V considered As Real vector space of dim 2n has J As its complex stucture.

- By setting $z_k = x_k + iy_k$ the complex vector space $\mathbb{C}^n$ can be identified with $\mathbb{R}^{2n} = (x_1, \dots, x_n, y_1, \dots, y_n)$

J_{can} is Represented by $\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$ canonical cplex stucture of $\mathbb{R}^{2n}$

V vector space real of dim $2n$

$V^c = V \otimes_{\mathbb{R}} \mathbb{C}$ complexification of V

Action of J on V^c has two Eigenvalues $+i$ and $-i$

then $\quad V^c = V^{1,0} \oplus V^{0,1}$

where $V^{1,0} = \{ \underbrace{X - iJX}_{= X^{1,0}}, X \in V \}$ and $V^{0,1} = \{ \underbrace{X + iJX}_{= X^{0,1}}, X \in V$

$JX^{1,0} = iX^{1,0}$ $\qquad$ $JX^{0,1} = -iX^{0,1}$

J induces an Action on Dual V^* of V

$$(J\alpha)(X) = -\alpha(JX) \qquad \alpha \in V^* \quad X \in V.$$

$\Rightarrow \qquad V^{*c} = V_{1,0} \oplus V_{0,1}$

$V_{1,0} = \{ \alpha + iJ\alpha \mid \alpha \in V^* \}$ $\qquad$ $V_{0,1} = \{ \alpha - iJ\alpha \mid \alpha \in V^*$

$\alpha^{1,0}$ with $\alpha^{1,0}(X^{0,1}) = 0$ $\qquad$ $\alpha^{0,1}(X^{1,0}) = 0$

annihilator

Almost - cplex Manifold

An Almost cplex ~~manifold (M, J) is a~~ stucture J on smooth Manifold M is A smooth field of automorphisms of T

such that $J^2 = -Id$ (i.e $J_p^2 = -Id|_{T_pM}$)

A Riemannian Metric g is J-compatible if

$$g(X,Y) = g(JX,JY) \quad \forall X,Y \in \mathfrak{X}(M)$$

Then (M, J, g) is called almost - Hermitian Manifold

Claim On (M,J) there are always J-compatible Riemannian Metrics. Take Random Metric $\tilde{g}$ then

$$g(X,Y) = \frac{1}{2}\left(\tilde{g}(X,Y) + \tilde{g}(JX,JY)\right)$$ is J compatible Riemannian Metric

Rem ① If g is J-compatible then X is g-orthogonal to JX

$$g(X,JX) = -g(JX,X) = -g(X,JX)$$

$$\Rightarrow g(X,JX) = 0.$$

Def Given An Almost-Hermitian Structure (J,g) we define the Fundamental 2-Form ω by $\omega(X,Y) := g(JX,Y) \quad \forall X,Y \in \mathfrak{X}(M)$

ω is Non degenerate

① ω is skew-symmetric

$$\omega(X,Y) = g(JX,Y) = -g(X,JY) = -g(JY,X)$$
$$= -\omega(Y,X)$$

② ω is J-invariant

$$\omega(JX,JY) = -g(X,JY) = -g(JY,X) = -\omega(Y,X)$$
$$= \omega(X,Y).$$

The structure (J,g,ω) is Almost-Hermitian structure.

$\Rightarrow$ $V_g = \dfrac{\omega^n}{n!}$ is volume Form.

Back to Almost-cplex structures

An even dimensional oriented real manifold does not ~~have~~ necessarily have an almost-cplex structure.

(there are topological restrictions: given ∃? non zero section of $\frac{SO(2n)}{U(n)}$ fiberbundle) ← Reduction of group structure of TM.

Example: S^2 and S^6 are only sphere which admit almost-cplex structure (Husemoller '74)

Example $S^2 \hookrightarrow \mathbb{R}^3$ $\quad \vec{x} \perp T_x(S^2)$

$$J_x V = \vec{x} \times V \qquad V \in T_x(S^2)$$

↑ cross product

indeed $J^2 V = \vec{x} \times (\vec{x} \times V) = \underbrace{(\vec{x} \cdot V)}_{=0} \vec{x} - (\vec{x} \cdot \vec{x}) V = -V$

(same thing cross product on $\mathbb{R}^7$ induces J on S^6)

Definition An almost-cplex structure is integrable if $[T^{1,0}, T^{1,0}] \subset T^{1,0}$ as distribution.

Famous Newlander-Nirenberg Theorem $\Longleftrightarrow$ local existence of holomorphic coordinates $\{z_i = x_i + i y_i\}$

$\Longleftrightarrow$ J is a complex structure (locally J is isomorphic to J_{can})

$\Rightarrow \psi_* J = J_0 \psi_* \Rightarrow (\psi \circ \Phi^{-1})_* J_0 = J_0 (\psi \circ \Phi^{-1})_*$

Easy to prove $\Longleftarrow$ Given $\{z_i = x_i + i y_i\} \Rightarrow J\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial y_i}$

local $\Rightarrow$ Basis of $T^{1,0} = \left\{ \frac{\partial}{\partial z_i} = \frac{1}{2}\left(\frac{\partial}{\partial x_i} - i \frac{\partial}{\partial y_i}\right) \right\}$

$\Rightarrow [T^{1,0}, T^{1,0}] \subset T^{1,0}$.

Definition Nijenhuis tensor on (M,J) is $(1,2)$

$$4N(X,Y) = [JX,JY] - [X,Y] - J[X,JY] - J[JX,Y]$$

N is $(1,2)$-tensor.

J is integrable $\Leftrightarrow N \equiv 0$.

Proof **Rem** $N(JX,JY) = -N(X,Y)$ and $N(X,Y) = -N(Y,X)$
and $N(JX,Y) = -JN(X,Y)$

$$4N(X^{1,0},Y^{1,0}) = -2\left([X^{1,0},Y^{1,0}] + iJ[X^{1,0},Y^{1,0}]\right)$$

or $[X^{1,0},Y^{1,0}]^{0,1} = N(X,Y) + iJN(X,Y)$

$$= -2\left([X^{1,0},Y^{1,0}]^{0,1}\right)$$

Def An Almost-Hermitian structure (J,g) is Hermitian if J is integrable

~~Definition~~ **Prop** An Almost-Hermitian structure is Hermitian iff

$$(*)\ D^g_{JX}J = JD^g_XJ \quad \forall X \in \mathfrak{X}(M)$$

Proof D^g – Levi-Civita connection

$$(**)\ 4N(X,Y) = (D^g_{JX}J)Y - J(D^g_XJ)Y - (D^g_{JY}J)X + J(D^g_YJ)X$$

If $(*) \Rightarrow N \equiv 0$

$$g(JX,N(Y,Z)) - g(JY,N(Z,X)) - g(JZ,N(X,Y)) \overset{(**)}{=}$$

$$\frac{1}{2}g((D^g_XJ)Y,Z) + g(J(D_{JX}J)Y,Z)$$

$D^g_XJ = (D^g_XF)^t$

Prop J is integrable iff $\mathcal{L}_{JX}J = J\mathcal{L}_XJ \quad \forall X \in \mathfrak{X}$

Proof $(\mathcal{L}_XJ)(Y) = [X,JY] - J[X,Y] \Rightarrow$

$$(\mathcal{L}_{JX}J - J\mathcal{L}_XJ)(Y) = 4N(X,Y)$$