Lecture 14

Theorem Gauss Bonnet

$\tilde{g} = \lambda g$
$\tilde{s}_g = \frac{1}{\lambda} s$
$v_g = \lambda v$

Let M be A compact oriented 2-dim Manifold then

Case n Even

$$\int_M K v_g = 2\pi \underbrace{\chi(M)}_{\text{Euler Characteristic}}$$ ← (integral of polynomial of degree $\frac{n}{2}$ in curvature)

← topological invariant (under Homeomor)

$\chi = V - E + F$

($=2$ For sphere)

$=1$ For $\mathbb{R}^n$

$\chi_g = 2-2g$ compact genus g

For Any Riemannian Metric g

relating the curvature to the topology.

Rem ① In Higher dimension $\int K v_g$ varies

→ Einstein-Hilbert Functional (Euler Lagrange ARE Einstein Metric ($n>2 \Rightarrow$ Ricci Flat))

Not invariant under scaling (Normalized version)

Rem ② $\int K v_g$ in constant when g "Kähler" (Chern-Weil Theory) → characteristic classes

Hopf Rinow theorem Let (M, g) be A connected Riemann manifold. Then Following are Equivalent

(i) M is geodesically complete ($\forall p$ the map $\exp_p$ is defined in T_pM because we can extend Any Geodesic on I to $\mathbb{R}$)

(ii) (M, d) is A complete Metric space

Moreover if M is geodesically complete then $\forall p, q$ $\exists$ geodesic c connecting p to q with $L_g(c) = d(p, q)$

Proof Let's prove last statement.

Normal Neighborhood $V \ni p$: $\exp_p : \underset{\overset{\Psi}{0}}{\overset{\text{open set}}{U}} \subset T_pM \xrightarrow[\text{diffeo}]{} V$

(Recall $\exp_p(v) = \gamma_v(1)$ where $\gamma_v(t)$ is unique geodesic with $\gamma_v(0) = p$ $\dot{\gamma}_v(0) = v$)

For $\varepsilon > 0$ define Normal ball with p center and radius ε as open set $B_\varepsilon(p) := \exp_p(B_\varepsilon(0))$ $(\overline{B_\varepsilon(0)} \subset U)$

$\Rightarrow$ Define Normal sphere $S_\varepsilon(p) := \exp_p(\partial B_\varepsilon(0))$ (compact submanifold)

The continuous function $\underset{\overset{\cap}{S_\varepsilon(p)}}{x} \longrightarrow d(x, q)$ will have a

Minimum point $x_0 \in S_\varepsilon(p)$ Moreover $x_0 = \exp_p(\varepsilon v)$ where $\|v\| = 1$

Consider geodesic $c_v(t) = \exp_p(tv)$. We will show that $c_v(\rho) = q$ where $\rho = d(p, q)$. Consider the set:

$A = \{t \in [0, \rho] \mid d(c_v(t), q) = \rho - t\}$

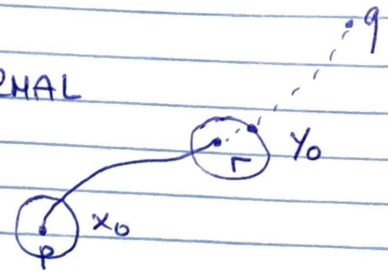
(continuity Method) Since map $t \to d(c_v(t), q)$ is continuous A is closed set.

Moreover $A \neq \emptyset$ as clearly $0 \in A$ $(d(p, q) = \rho)$

We will show that no point $t_0 \in [0, \rho)$ can be Maximum of A (i.e A is open).

Let $t_0 \in A \cap [0, \rho)$, $r = c_v(t_0)$

and $\delta \in (0, \rho - t_0)$ such that $S_\delta(r)$ is A NORMAL sphere



Let y_0 be Minimum pt of continuous Function $y \longrightarrow d(y, q)$ on compact set $S_\delta(r)$

We will show that $y_0 = c_v(t_0 + \delta)$. Indeed we have

$$\rho - t_0 = d(r, q) = \delta + \min_{y \in S_\delta(r)} d(y, q) = \delta + d(y_0, q)$$

AND SO $d(y_0, q) = \rho - t_0 - \delta$ (*)

Triangle inequality implies that

$$d(p, y_0) \geq d(p, q) - d(y_0, q) = \rho - (\rho - t_0 - \delta) = t_0 + \delta.$$

But The curve which connects p to r through c_v AND r to y_0 through A geodesic has A length $t_0 + \delta$

$\Rightarrow$ this is A Minimizing curve hence A (reparametrized) geodesic

(Recall: A unit speed Minimizing curve is geodesic)

$\Rightarrow y_0 = c_v(t + \delta)$

$\overset{(*)}{\Rightarrow} d(c_v(t_0 + \delta), q) = \rho - (t_0 + \delta) \Rightarrow t_0 + \delta \in A$, A is open

See LaFontaine For (i) $\Leftarrow\Rightarrow$ (ii)

Corollary IF M is compact then M is geodesically complete

Proof Any compact Metric space is complete.