Lecture 12

(M,g) Riemannian D^g its Levi-Civita connection

The curvature of D^g is the $(1,3)$-tensor

$$R : TM \otimes TM \otimes TM \longrightarrow TM$$

$$\ominus \quad R^g_{X,Y} Z = D^g_X(D^g_Y Z) - D^g_Y(D^g_X Z) - D^g_{[X,Y]} Z$$

Define $(0,4)$-tensor

$$R^g(X,Y,Z,W) = g(R^g_{X,Y} Z, W) \text{ then}$$

(so $R^g(fX,Y,Z,W) = f R^g(X,Y,Z,W)$ etc.

① $R^g(X,Y,Z,W) = -R^g(Y,X,Z,W) \underset{\uparrow\, D^g g = 0}{=} -R^g(X,Y,W,Z)$

② $R^g(X,Y,Z,W) \underset{\uparrow\, D^g \text{ has No torsion}}{=} R^g(Z,W,X,Y)$

③ 1st Bianchi identity

$$R^g(X,Y,Z,W) + R^g(Y,Z,X,W) + R^g(Z,X,Y,W) = 0$$

③ Uses Jacobi identity $\cancel{[[X,Y],Z] + [[Y,}$

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$$

Indeed

$$R^g(X,Y,Z,W) + R^g(Y,Z,X,W) + R^g(Z,X,Y,W)$$

$$= -g([X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]], W) = 0$$

② Follows From ① + ③

$$R^g(X,Y,Z,W) = -R^g(Y,Z,X,W) - R^g(Z,X,Y,W)$$
$$= R^g(Y,Z,W,X) + R^g(Z,X,W,Y)$$
$$\textcircled{3} \to = -R^g(Z,W,Y,X) - R^g(W,Y,Z,X)$$
$$- R^g(X,W,Z,Y) - R^g(W,Z,X,Y)$$

$$= 2R^g(Z,W,X,Y) + R^g(W,Y,X,Z)$$
$$+ R^g(X,W,Y,Z)$$
$$\textcircled{3} \to = 2R^g(Z,W,X,Y) - R^g(Y,X,W,Z)$$

We can think of R as Asymmetric Endomorphism (bilinear Form)
$R^g : \Lambda^2 TM \to \Lambda^2 TM$ called curvature operator.
$g(R^g(X \wedge Y), Z \wedge W)$

Remk② in normal coordinates: $g_{ij}(x_i) = \delta_{ij} + 1/3 R_{ikjl}(p) x_k x_l + o(|x|^3)$
Curvature measures deviation from Euclidean metric.

Remk① R^g can be expressed in terms of Christoffel symbols.

Def The sectional curvature of plane spanned by X, Y

$$\sec(X \wedge Y) = \frac{R^g(X,Y,X,Y)}{g(X\wedge Y, X \wedge Y)} = \frac{R^g(X,Y,X,Y)}{\|X\|^2\|Y\|^2 - g(X,Y)^2}$$

Note: If $\operatorname{span}\{X,Y\} = \operatorname{span}\{X',Y'\}$ then
$$\sec(X \wedge Y) = \sec(X' \wedge Y')$$

Prop The sectional curvature determines the curvature operator.

Proof Suppose R' such that $\frac{g(R^g(X,Y)X,Y)}{\|X \wedge Y\|^2}$ show $R' = R$.

sec of R'

By Hypothesis

$$\cancel{R'(X+Z, X, Y, X+Z) = R(X+Z, Y}$$

$$R^g(X+Z, Y, X+Z, Y) = R'(X+Z, Y, X+Z, Y)$$

so

$$R'(X,Y,X,Y) + 2R'(X,Y,Z,Y) + R'(Z,Y,Z,Y)$$
$$= R(X,Y,X,Y) + 2R(X,Y,Z,Y) + R(Z,Y,Z,Y)$$
$$\Rightarrow \quad R'(X,Y,Z,Y) = R(X,Y,Z,Y) \quad (*)$$

$$\Rightarrow R'(X, Y+W, Z, Y+W) = R(X, Y+W, Z, Y+W)$$

$$\Rightarrow R'(X,Y,Z,Y) + R'(X,Y,Z,W) + R'(X,W,Z,Y) + R'(X,W,Z,W$$

using $(*)$ $= R(X,Y,Z,Y) + R(X,Y,Z,W) + R(X,W,Z,Y) + R(X,W,Z,W$

$$\Rightarrow R'(X,Y,Z,W) + R'(X,W,Z,Y)$$
$$= R(X,Y,Z,W) + R(X,W,Z,Y)$$

$$\Rightarrow R'(X,Y,Z,W) - R(X,Y,Z,W) = -R'(X,W,Z,Y) + R(X,W,Z,Y)$$
$$= \cancel{R'(W,X,Z,Y) - R(W,X,Z,Y)}$$

$$\Rightarrow -R'(X,Y,W,Z) + R(X,Y,W,Z) = R'(X,W,Y,Z) - R(X,W,Y,Z)$$

$(**) \Rightarrow R'(X,Y,W,Z) - R(X,Y,W,Z) = R'(W,X,Y,Z) - R(W,X,Y,Z)$

1st Bianchi Identity $\Rightarrow (R'-R)(X,Y,Z)$ is invariant By cyclic permutation

$$+R'(Y,W,\cancel{X},Z) + R'(W,\cancel{X},Y,Z) - R(Y,\cancel{W},X,Z) - R(W,\cancel{X},Y,Z)$$
$$= R'(\cancel{X},W,Y,Z) - R(X,\cancel{W},Y,Z) \Rightarrow 3(R'-R) = 0$$

$$R'(X,Y,W,Z) + R'(Y,W,X,Z) + R'(W,X,Y,Z) = 0$$

(**) $$2R'(X,Y,W,Z) + R'(X,Y,W,Z) = 0$$

$$\Rightarrow 3R'(Y,W,X,Z) = 3R(Y,W,X,Z) \quad \forall X,Y,W,Z$$

$$R' = R$$

Rem ① In $\dim = 2$ $R^g : \Lambda^2 TM \longrightarrow \Lambda^2 TM$ is given by A scalar Function called Gaussian curvature

Proposition Schur' 1886 (Proof in Kuhnel Differential Geometry: curves - surfaces - Manifolds)

Let (M,g) be A connected Riemannian manifold.

IF $\dim M \geq 3$ and if $\sec_p(X_p \wedge Y_p)$ Doesn't depend on plane in T_pM but only on the point $p \in M$ then sectional curvature is constant i.e Does Not depend on p.

* use 2^{nd} Bianchi identity $\sigma_{X,Y,W}(D^g_W R^g)(Z,X,Y,W) = 0$

Rem IF $\tilde{g} = \lambda g$ $\lambda > 0$ is constant $\Rightarrow$ Christoffel symbols and Levi-Civita are unchanged ~~However~~ se as well as curvature but sectional curvature changes

$$\sec_{\tilde{g}} = \frac{1}{\lambda} \sec_g.$$

IF M Riemannian of constant sectional curvature then up Rescaling we may assume $se = 0, 1, -1$.

Examples ① sectional curvature of $S^n(1) = 1$

② sectional curvature of $\mathbb{R}^n = 0$

sectional curvature Hyperbolic space is -1

NORMAL coor

volume element

$\sqrt{\det(g_{ij})} =$

$1 - 1/6\, Ric_{ke}(p) x^k x^e + O(|x|^3)$

Def The Riemannian Ricci curvature is a $(0,2)$ tensor it is trace of Endomorphism of T_pM given by

$$V \longrightarrow R_p(X,V)Y$$

IF $\{e_i\}$ is orthonormal Basis of T_pM then

$$Ric_p(X,Y) = \sum_i R^g(X, e_i, Y, e_i)$$

$\Rightarrow$ Ric is symmetric (symmetry (2) of R^g)

Def The Riemannian scalar curvature is trace of Ricci curvature

$Vol(B_r(p)) = r^n\left(1 - \frac{S r^2}{6(n+2)} + O(r^3)\right)$

$$S^g(p) = \sum_i Ric(e_j, e_j) = \sum_{i,j} R^g(e_j, e_i, e_j, e_i)$$

Prop IF M has constant sectional curvature $= k$ then

~~$R^g(X,Y,Z,W) =$~~

$$g(R^g_{X,Y} Z, W) = -K[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]$$

Proof Define $(0,4)$-tensor

$$A(X,Y,Z,W) = -K[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]$$

A satisfies symmetry property of curvature

But $A(X,Y,X,Y) = -K\,[g(X,Y)g(X,Y) - |X|^2|Y|^2]$

$$= R(X,Y,X,Y)$$

$\Rightarrow A = R.$

Exercise For a Manifold with constant Sectional curvature K we have $Ric = (n-1)Kg \Rightarrow$ Einstein Metric

and $S^g = n(n-1)K.$