Math 71000 - Homework #4 Spring 2025
Gradaute Center

Problem 1. Let $H = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in \mathbb{R} \right\}$ be the Heisenberg group (with matrix multiplication). The vectors $\left\{ E_1 = \frac{\partial}{\partial x} + z\frac{\partial}{\partial y}, E_2 = \frac{\partial}{\partial y}, E_3 = \frac{\partial}{\partial z} \right\}$ are right-invariant and form a basis of the Lie algebra of H. We consider on H the right-invariant metric g defined when we declare the vectors E_1, E_2, E_3 to be g-orthonormal.

- Compute the curvature of the metric g.
- Does (H, g) have a constant sectional curvature?
- Compute the Riemannian Ricci tensor of (H, g)? Is (H, g) an Einstein Riemannian manifold?
- Compute the Riemannian scalar curvature of (H, g)? Does (H, g) have a constant Riemannian scalar curvature?

Problem 2. Prove that a 3-dimensional Einstein Riemannian manifold is a manifold of constant sectional curvature.

Problem 3. Let (M, g) be a 3-dimensional manifold. Prove that the curvature tensor is entirely determined by the Riemannian Ricci tensor

Problem 4. Consider the connection ∇ on $\mathbb{R}^2$ defined by the nonvanishing Christoffel symbol

$$\Gamma^1_{11} = f(y),$$

for some function f (here $\{x, y\}$ are the coordinates on $\mathbb{R}^2$). Define the curvature R^∇ and the Ricci tensor Ric^∇ of ∇ as for the Levi-Civita connection. Is Ric^∇ a symmetric tensor? Justify.