Bronx Community College of City University of New York
Department of Mathematics and Computer Science

MTH 42 Linear Algebra Review Sheet

1. For the linear system $\left\{\begin{array}{rl} x_1 + \ \ x_2 + \ \ x_3 &= 1 \\ 2x_1 + 2x_2 + \ \ x_3 &= 2 \\ x_1 + \ \ x_2 + 2x_3 &= 3 \end{array}\right.$, answer the following:

(a) Find matrices $\mathbf{A}$, $\mathbf{x}$ and $\mathbf{b}$ that express the linear system above as a matrix equation.

(b) Find $\mathbf{A}^{-1}$ and express it as a product of elementary matrices.

(c) Solve the system by multiplying $\mathbf{A}^{-1}$ and $\mathbf{b}$.

2. Let $\left\{\begin{array}{rl} ax_1 + \ \ bx_2 &= 2 \\ x_1 + 2\,bx_2 &= 1 \end{array}\right.$. Find all values of a, b so that the system has

(a) exactly one solution; (b) no solution; (c) infinitely many solutions.

3. Find $\mathbf{A}$ if $(5\mathbf{A}^T)^{-1} = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}$.

4. Find values of a, b and c for which $\mathbf{A} = \begin{bmatrix} 1 & a+b & c \\ 1 & 1 & a-b \\ 2 & 0 & 1 \end{bmatrix}$ is symmetric.

5. Given $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = n$, find the determinant of each matrix below.

$(a) \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ $\quad (b) \begin{bmatrix} c & d \\ -a & -b \end{bmatrix}$ $\quad (c) \begin{bmatrix} a & b \\ a+c & b+d \end{bmatrix}$ $\quad (d) \frac{1}{4} \begin{bmatrix} 4a & 4b \\ 3a+c & 3b+d \end{bmatrix}$ $\quad (e) \begin{bmatrix} a & b \\ 4a & 4b \end{bmatrix}$

6. Find the determinant of the matrix: (a) $\begin{bmatrix} 2 & 3 & /4 & 5 \\ 0 & 1 & 3 & 5 \\ 2 & 3 & 4 & 6 \\ 4 & 5 & 6 & 7 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 3 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & 0 & 4 & 6 \\ 0 & 0 & 6 & 7 \end{bmatrix}$.

7. Let $\mathbf{u} = (2, 5)$ and $\mathbf{v} = (-4, 3)$ be vectors in $\mathbb{R}^2$.

(a) Find the inner product $\mathbf{u} \cdot \mathbf{v}$. (b) Are $\mathbf{u}$ and $\mathbf{v}$ orthogonal?

8. Let $\mathbb{R}^4$ have the Euclidean inner product. Use the Gram-Schmidt process to transform the basis $\{(0, 2, 1, 0), (1, -1, 1, 0), (1, 2, 1, 0), (1, 0, 0, 1)\}$ for $\mathbb{R}^4$ into an orthonormal basis.

9. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $w_1 = 3x_1 + 5x_2$, $w_2 = 2x_1 - 9x_2$.

(a) Find the matrix for T in the standard basis of $\mathbb{R}^2$.

(b) Calculate $T\left(\begin{bmatrix} -2 \\ -3 \end{bmatrix}\right)$.

(c) Is T one-to-one? If so, find the matrix for the inverse linear transformation T^{-1} in the standard basis.

10. Find the matrix of T in the standard bases matrix for the linear transformations T defined by the following formulae:

(a) $T(x_1, x_2, x_3, x_4) = (7x_1 + 2x_2 - x_3 + x_4, x_2 + x_3, -x_1)$.

(b) $T(x_1, x_2, x_3, x_4) = (x_4, x_1, x_3, x_2, x_1 - x_3, x_2 + x_3)$.

11. Let $\mathcal{M}_{22}$ be the vector space of 2×2 matrices with real number entries and with standard matrix addition and scalar multiplication. Determine which of the following are subspaces of $\mathcal{M}_{22}$.

(a) 2×2 matrices A with $\det A = 1$.

(b) 2×2 diagonal matrices.

(c) 2×2 matrices with integer entries.

(d) 2×2 matrices A such that $\operatorname{tr}(A) = 0$.

(e) 2×2 matrices A with nonzero entries.

12. Determine which of the following are subspaces of $\mathbb{R}^3$.

(a) Vectors of the form (a, a, a).

(b) Vectors of the form $(a, 1, 1)$.

(c) Vectors of the form (a, b, c), where $b = a + c$.

13. Bases and linear independence:

(a) Is $\{(-3, 0, 4), (0, -1, 2), (1, 1, 3)\}$ a linearly independent subset of $\mathbb{R}^3$?

(b) Let $B = \{(3, 9), (-4, a)\}$, find all values of a so that B becomes a basis for $\mathbb{R}^2$.

(c) Let $\mathcal{B} = \{(2, -4), (3, 8)\}$ be a basis for $\mathbb{R}^2$. Find the coordinate vector of $(1, 1)$ relative to the basis $\mathcal{B}$.

14. Determine whether the function (or mapping) is a linear transformation. Here, $\mathcal{M}_{mn}$ denotes the space of m by n matrices.

(a) $T : \mathcal{M}_{nn} \to \mathbb{R}$, where $T(A) = \operatorname{tr}(A)$.

(b) $T : \mathcal{M}_{nn} \to \mathcal{M}_{nn}$, where $T(A) = -A$.

(c) $T : \mathcal{M}_{mn} \to \mathcal{M}_{nm}$, where $T(A) = A^T$.

(d) $T : Gl(n, \mathbb{R}) \to \mathcal{M}_{nn}$, where $T(A) = A^{-1}$, and $Gl(n, \mathbb{R})$ is the set of $n \times n$ invertible matrices.

15. Let $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix}$.

(a) Find all eigenvalues and corresponding eigenvectors of $\mathbf{A}$.

(b) Find an invertible matrix $\mathbf{P}$ such that $\mathbf{P}^{-1}\mathbf{AP}$ is a diagonal matrix.

(c) Compute $\mathbf{A}^{30}$.

16. Determine $T_1 \circ T_2 \circ T_3$ and evaluate $(T_1 \circ T_2 \circ T_3)(\sqrt{3}, -3)$ if $T_1 : \mathbb{R}^2 \to \mathbb{R}^2$ is the counterclockwise rotation through an angle $\pi/4$, $T_2 : \mathbb{R}^2 \to \mathbb{R}^2$ is the orthogonal projection on the y-axis and $T_3 : \mathbb{R}^2 \to \mathbb{R}^2$ is the reflection about the x-axis.

17. Let the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined as $T(x_1, x_2, x_3) = (7x_1 + 2x_2 - x_3, 2x_1 - 2x_2 + x_3, x_2 - x_3)$, and suppose $\mathcal{B} = \{(1, 1, 1), (0, 1, 1), (0, 0, 3)\}$ is a basis for $\mathbb{R}^3$. Find the matrix for T with respect to the basis B. Is T one-to-one? If so, find T^{-1}.

18. Let $A = \begin{bmatrix} 2 & 2 & -1 & 0 & 1 \\ -1 & -1 & 2 & -3 & 0 \\ 1 & 1 & -2 & 0 & -1 \end{bmatrix}$.

(a) Find the bases for the rowspace, column space, and nullspace of A.

(b) What are the rank(A) and nullity(A)?

(c) If T is the multiplication by the matrix A, what are bases for the range and the kernel of T?

19. Fill in the blank.

(a) A square matrix is not invertible iff at least one of its eigenvalues is ______________________________.

(b) A square matrix A of order n is diagonalizable iff A has ___ linearly independent eigenvectors.

(c) A square matrix A of order n has n distinct eigenvalues, then A is ______________________________.

(d) A square matrix A is diagonalizable iff the geometric multiplicity is equal to the __________________ for every eigenvalue.

(e) Any bases for a vector space have the same ____________________ of vectors.

(f) If A is a matrix with n columns, then rank(A) + ____________ $= n$.

(g) The basis for the column space of A is determined by the columns containing ________________________.

(h) If $\dim(V) = n$, and suppose a subset B of V has exactly n vectors, then B is a basis if either B is linearly independent or ______________________________.

(e) W is a subspace of V if ____________ $\in W$ for all vectors $u, v \in V$ and ____________ $\in W$ for all vectors $u \in W$ and scalars k.

20. Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by $T(x_1, x_2) = (x_1 + 2x_2, -x_1, 0)$. Find the matrix $[T]_{B',B}$ with respect to the bases $B = \{(1, 3), (-2, 4)\}$ and $B' = \{(1, 1, 1), (2, 2, 0), (3, 0, 0)\}$. Show $T(2, 3) = (8, -2, 3)$ by using the matrix multiplication.