Chapter 8

Radical expressions

Vocabulary

- Quadratic equation
- Rational numbers
- Irrational numbers
- Radicand
- Rationalizing (a denominator)
- Imaginary unit
- Real numbers and complex numbers

8.1 Introduction: Quadratic equations and number systems

For the last two chapters, we have worked with polynomials as algebraic objects on which we can perform "symbolic arithmetic." We will now return to the question of solving equations. Up to this point, we have developed an approach to solving *linear* equations: degree one polynomial equations in one or two variables.

In particular, we will attempt to solve quadratic equations, or polynomial equations of degree two. However, we will see in this introductory section that these equations—even simple ones—force us to face some fundamental problems not just about algebra, but about the number systems we have been working with. That will be the topic of this chapter.

Up to now, we have been working almost exclusively with **rational numbers**—numbers that can be expressed as a ratio of two integers. That means the "worst" numbers we have had to work with have been fractions or (repeating) decimals. We will see that the setting of rational numbers is not adequate to solve any but the simplest quadratic equations. In fact, solving quadratic equations will force us to face "numbers" that have properties that are quite different from any we have seen up to now.

Let's start our discussion of quadratic equations with a simple example of a quadratic equation in one variable:

$$x^2 = 9.$$

This equation (in one variable) is quadratic, since the highest power of x is 2. A little bit of trial and error quickly reveals that 3 is a solution to this equation: substituting, $(3)^2 = 9$ is a true equation. After all, 9 is a perfect square; said in a different way, 3 is a square root of 9.

However, remember that to solve the equation means to find *all* solutions. So far we have found one. Keeping in mind, though, that the square of a negative number is positive, we can see that -3 is also a solution to $x^2 = 9$, since $(-3)^2 = 9$ is a true equation. We can already see a major difference between quadratic equations and linear equations: *a typical quadratic equation may have more than one solution*. (Recall that if a linear equation in one variable has more than one solution, then every number is a solution to that equation.) It shouldn't be hard to convince yourself that no other number is a solution.

Our first simple example points to a crucial fact about quadratic equations that will be at the core of all of our strategies to solve quadratic equations in Chapter 9:

Basic fact about quadratic equations: A typical quadratic equation in one variable has no more than two solutions.

We will return to this fact shortly to state it in a more precise way. (It can be proved with more detailed knowledge of polynomials, usually discussed in a precalculus course.)

Let's look at another simple example, which on the surface looks no different from the last one:

$$x^2 = 5.$$

The problem with this equation is that when we try to guess a solution, like we did last time, no obvious solution appears. After all, unlike 9 that appeared in the last equation, 5 is not a perfect square.

We might try a more refined version of guessing. We see, for example, that $(2)^2 = 4$ is less than 5, while $(3)^2 = 9$ is greater than 5. So we might guess a number between 2 and 3—say 2.5. With a little bit more work calculating, we see that $(2.5)^2 = 6.25$, still larger than 5. So we'll guess again, this time between 2 and 2.5—maybe 2.25. We see that $(2.25)^2 = 5.0625$, close, but still a little too large! If we try 2.24, we see that $(2.24)^2 = 5.0176$—closer, but still too large. If we try 2.235, we see that $(2.235)^2 = 4.995225$, which is less than 5 but closer still. We might say that a solution to $x^2 = 5$ is "approximately 2.235." That seems a little unsatisfying, though, as mathematics aims to be a precise discipline.

At the core of this problem is a fundamental fact about whole numbers: *If a whole number is not a perfect square, than its square root is not a rational number.* Since rational numbers either have a terminating or a repeating decimal expansion, this means that if $x^2 = 5$ has a solution, its decimal expansion will never terminate and it will never repeat. This is somewhat depressing from the point of view of the last paragraph, where we tried to "guess" a solution of $x^2 = 5$. Having said that, our guessing attempts do seem to point to the fact that a solution does, in fact, exist, even though it might be hard to pin down exactly what that solution is.

Our discussion of the quadratic equation $x^2 = 5$ has led us to numbers that might be completely unnecessary from the point of view of linear equations. We can rephrase the statement of the previous paragraph: *The solutions to $x^2 = k$, where k represents a positive integer which is not a perfect square, are* ***irrational numbers****.*[1] The next section will be devoted to working with irrational numbers, which will play an essential role in solving quadratic equations.

One final example will reveal another basic problem in solving quadratic equations. Consider the quadratic equation

$$x^2 = -4.$$

We might have an initial glimmer of hope, seeing that 4 is a perfect square. But we are looking for a number which, when multiplied by itself, gives a negative number. This reveals a problem that is much more basic than the problem of irrational numbers in the previous problem: *No number, rational or irrational, when squared, will give a negative number.* This fact might tempt us to simply say that the equation has no solution. In the Section 8.3, we give another way to handle this problem by introducing a new kind of "number" called *complex numbers.*

Let's summarize what this short discussion of the most simple quadratic equations in one variable has revealed.

[1] The discovery of irrational numbers—numbers which cannot be written as the ratio of two integers—is usually credited to the ancient Greek school of thinkers known as the Pythagoreans nearly 2,500 years ago. It is an irony of history that this discovery, derived logically, should have come from the Pythagoreans, according to whose world view all things could be understood as a ratio of whole numbers.

Important features of quadratic equations in one variable

- A typical quadratic equation in one variable will have two solutions.
- A quadratic equation may have irrational solutions (even if its coefficients are rational numbers).
- A quadratic equation may have solutions which are complex numbers.

8.2 Radical expressions

In this section, we will establish some conventions about how we will treat the types of irrational numbers that arise in solving quadratic equations. In particular, we will develop a way of writing square roots symbolically.

We already saw in the previous section that solutions to the equation $x^2 = a$ are irrational numbers whenever a is a whole number that is not a perfect square. In that discussion, we saw that solutions to this simple type of quadratic equation is closely related to the notion of a square root, which we treated in Chapter 1 as an operation. One way to treat irrational square roots would be to agree, in advance, that we will estimate them to a given decimal accuracy. For example, if we agree to estimate square roots to 8 decimal places, a calculator might tell us that the square root of 5 is given by $\sqrt{5} \approx 2.23606798$. If, on the other hand, we agree to estimate to 12 decimal places, we would write $\sqrt{5} \approx 2.236067977500$.

We are going to handle irrational square roots in a different way. Instead of estimating (which depends in practice on using a calculator), we will adopt a symbolic approach.

The square root as a symbol

For any non-negative number k, the symbol $\sqrt{k}$ ("the square root of k") represents the non-negative solution to the equation $x^2 = k$.

Said in another way, the symbol $\sqrt{k}$ represents the non-negative number which satisfies

$$(\sqrt{k})^2 = k.$$

It is worthwhile pointing out what is new about this definition. In our previous understanding of square roots, $\sqrt{k}$ actually consisted of two separate symbols: k (representing a number) and the *radical sign* $\sqrt{}$, which represented an operation performed on the number k. In our new definition, by contrast, $\sqrt{k}$ represents *one symbol* with two parts: the radical sign and the *radicand*, as the quantity k inside the radical sign is known.

Keep in mind from the introductory section that the equation $x^2 = k$ typically has two solutions, one positive and one negative. In this case, $\sqrt{k}$ represents the *positive* solution.

From the point of view of solving quadratic equations, at least for simple ones, this definition in a certain sense "cheats." We have *defined* the symbol $\sqrt{k}$ to be the nonnegative solution of $x^2 = k$; the other solution will then be its opposite, written as $-\sqrt{k}$. The following examples illustrate this point.

In fact, we can write this as our first strategy to solve quadratic equations.

> **First strategy to solve quadratic equations of the form $x^2 = k$**
>
> An equation having the form $x^2 = k$ has two solutions, written symbolically as $\sqrt{k}$ and $-\sqrt{k}$.

For now, we will be applying this strategy when k represents a nonnegative number. In Section 8.3, we will consider what happens when k is negative.

Notice that in the special case $x^2 = 0$, the "two" solutions $\sqrt{0}$ and $-\sqrt{0}$ are the same—they are both 0. But rather than thinking of this as a special case of a quadratic equation with only one solution, it is more convenient to think of this as a quadratic equation with two solutions that just happen to be the same.

Example 8.2.1. *Solve the following quadratic equations:*

(a) $x^2 = 7$*;*

(b) $x^2 = 129$*;*

(c) $x^2 = 15$.

Answer. *(a) The positive solution to* $x^2 = 7$ *is* $\sqrt{7}$ *(by definition!). The negative solution is written* $-\sqrt{7}$*. So the solutions are* $\sqrt{7}$ *and* $-\sqrt{7}$*.*

(b) The solutions are $\sqrt{129}$ *and* $-\sqrt{129}$*.*

(c) The solutions are $\sqrt{15}$ *and* $-\sqrt{15}$*.*

The point of these simple examples is not really to show how to solve a quadratic equation—although we have technically done so. The point is to illustrate the fact that we have used a symbol to represent a solution to an equation. It has the advantage of needing no estimation. For example, $\sqrt{129}$ is a symbol representing the *exact value* of the positive solution to $x^2 = 129$. It has the disadvantage, however, of hiding the fact that this symbol $\sqrt{129}$ is a symbol for an actual (irrational) number, whose value is approximately $11.357816691600547221\ldots$.

There is another, more serious, disadvantage to this symbolic approach. If we get too excited about this new notation, we would be tempted to say that the solutions to the quadratic equation $x^2 = 9$ are $\sqrt{9}$ and $-\sqrt{9}$, and done! This seems like a wrong way to answer a question, when it is much more easily understood to say that the solutions are 3 and -3. Unlike the examples above, 9 is a perfect square, and so there is no need to have to have a special notation made specifically to address irrational numbers.

For these (and other related) reasons, we are going to agree to a series of rules about how we write square roots symbolically. These rules have evolved over the course of history and are generally accepted.

Simplified square root notation

The symbol $\sqrt{k}$ is called **simplified** if the following conditions hold:

1. The radicand k has no perfect square factors;
2. The radicand k contains no fractions.

In addition, any expression containing radicals must satisfy a third condition:

3. No radical expression shall appear in a denominator.

What happens if we encounter a radical expression which is not simplified? We will take the effort to *simplify* it, rewriting it in an equivalent form which is simplified according to the conditions above. In order to do this, we will rely on two basic properties of square roots.

Some properties of square roots

If a and b represent non-negative numbers, then:

(S1) $\sqrt{a \cdot b} = \sqrt{a} \cdot \sqrt{b}$; (8.1)

(S2) $\sqrt{\dfrac{a}{b}} = \dfrac{\sqrt{a}}{\sqrt{b}}$.

These properties are really just versions of the properties of exponents we discussed in Chapter 6. For example, the first property really says that the number represented by $\sqrt{a} \cdot \sqrt{b}$ should be the (non-negative) solution to $x^2 = ab$. Is this true? Since $(\sqrt{a})^2 = a$ and $(\sqrt{b})^2 = b$, $(\sqrt{a} \cdot \sqrt{b})^2 = (\sqrt{a})^2 \cdot (\sqrt{b})^2 = ab$, by property (E4) of exponents. So $\sqrt{a} \cdot \sqrt{b}$ really is a solution to $x^2 = ab$, and the equality of Property (S1) is valid.

We now show through a series of examples how these properties can help us to simplify square roots.

Example 8.2.2. *Simplify the following square roots:*

(a) $\sqrt{12}$*;*

(b) $\sqrt{72}$*;*

(c) $\sqrt{75}$.

Answer. *Before applying the properties of square roots, let's look at the three square roots we are being asked to simplify. Notice that all three represent irrational numbers, since neither* 12, 72 *nor* 75 *are perfect squares. However, all three have perfect square* factors. *For this reason, none of the three square roots are simplified; they all violate Rule 1 of our definition of a simplified square root.*

To simplify them, we will write the radicand as a product of a perfect square (preferably as large as possible, if there are more than one) with another number, and then apply Property (S1).

(a) 12 *has a perfect square factor of* 4*. So*

$$\sqrt{12} = \sqrt{4 \cdot 3} = \sqrt{4} \cdot \sqrt{3} = 2\sqrt{3}.$$

The answer is $2\sqrt{3}$.

(b) 72 *has several perfect square factors, but the largest is* 36.

$$\sqrt{72} = \sqrt{36 \cdot 2} = \sqrt{36} \cdot \sqrt{2} = 6\sqrt{2}.$$

The answer is $6\sqrt{2}$. *(What would have happened if we had factored out the perfect square factor of* 9*?)*

(c) 75 *has a perfect square factor of* 25.

$$\sqrt{75} = \sqrt{25 \cdot 3} = \sqrt{25} \cdot \sqrt{3} = 5\sqrt{3}.$$

The answer is $5\sqrt{3}$.

The next example illustrates how to cope with a radical expression with a fraction in the radicand.

Example 8.2.3. *Simplify the following radical expressions:*

(a) $\sqrt{\frac{3}{4}}$;

(b) $\sqrt{\frac{50}{9}}$;

Answer. *These radical expressions violate Rule 2 for simplified square roots. However, we can apply Property (S2) of square roots directly:*

(a) Applying Property (S2):

$$\sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{\sqrt{4}} = \frac{\sqrt{3}}{2}.$$

Notice we did not eliminate the fraction from the expression. However, the only remaining radicand (which is 3*) does not involve fractions.*

The answer is $\frac{\sqrt{3}}{2}$.*(Now try this: Use a calculator to find an approximate numerical value for the expression* $\sqrt{3}/2$ *(by finding an estimate for* $\sqrt{3}$ *and dividing by* 2*). Square the result. What number do you obtain? Compare your answer to the original expression.)*

(b) Applying Property (S2):

$$\sqrt{\frac{50}{9}} = \frac{\sqrt{50}}{\sqrt{9}} = \frac{\sqrt{50}}{3}.$$

This time, although we have an expression that satisfies Rule 2 for simplified square roots, the remaining radicand of 50 *still has a perfect square factor of* 25. *Hence*

$$\frac{\sqrt{50}}{3} = \frac{\sqrt{25 \cdot 2}}{3} = \frac{\sqrt{25} \cdot \sqrt{2}}{3} = \frac{5\sqrt{2}}{3}.$$

The answer is $\frac{5\sqrt{2}}{3}$.

In the previous example, we were lucky to encounter fractions in the radicand whose denominators were perfect squares. The next example illustrates how to simplify when this is not the case. It involves a technique known as *rationalizing the denominator*. Here, the word "rationalize" implies making an irrational number rational by multiplying by an appropriate number.

Example 8.2.4. *Simplify the following square roots:*

(a) $\sqrt{\frac{1}{2}}$;

(b) $\sqrt{\frac{3}{8}}$.

Answer. *The first thing we notice about both examples is that we have a fraction in the radicand. As in the last example, we begin by using Property (S2) of square roots.*

(a) Applying Property (S2),

$$\sqrt{\frac{1}{2}} = \frac{\sqrt{1}}{\sqrt{2}} = \frac{1}{\sqrt{2}}.$$

Unfortunately, this time, the radicand in the denominator is not a perfect square. According to Rule 3, this expression is not yet completely simplified, since there is a square root symbol remaining in the denominator.

Our strategy will be to multiply the numerator and denominator of the fraction by the same quantity (in other words, multiply the expression by 1*, which does not change the expression). We will choose the quantity in such a way that, after using Property (S1), the radicand in the denominator becomes a perfect square.*

In particular, we will ask: what perfect square has the given radicand as a divisor? In this example, the smallest perfect square that has 2 *as a divisor is* 4*. In order to obtain a radicand of* 4 *in the demoninator, we will multiply the numerator and denominator by* $\sqrt{2}$*:*

$$\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{\sqrt{4}} = \frac{\sqrt{2}}{2}.$$

Notice that we chose what radical expression to multiply the denominator (and the numerator) specifically in order to obtain a perfect square as a radicand in the denominator, after applying Property (S1). Notice that this process does not eliminate the radical expression completely. It only changes the expression is written in such a way that the radical expression appears in the numerator and not in the denominator, in compliance with our Rule 3 of simplified radical expressions.

The answer is $\frac{\sqrt{2}}{2}$.

(b) Again, we begin by applying Property (S2):

$$\sqrt{\frac{3}{8}} = \frac{\sqrt{3}}{\sqrt{8}}.$$

As in Example (a), we are left with a radical expression in the denominator. Although we notice that $\sqrt{8}$ can be simplified, since 8 has a perfect square factor of 4, we will first address the more serious problem of the radical expression in the denominator.

Our goal is to obtain a radicand in the denominator which is a perfect square. We look for a perfect square which has 8 as a factor; the smallest such perfect square is 16. In order to obtain the radicand of 16 in the denominator, we multiply the denominator (and numerator) by $\sqrt{2}$:

$$\frac{\sqrt{3}}{\sqrt{8}} = \frac{\sqrt{3}}{\sqrt{8}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{6}}{\sqrt{16}} = \frac{\sqrt{6}}{4}.$$

The answer is $\dfrac{\sqrt{6}}{4}$.

(In the challenge exercises at the end of Section 8.4, we will come back to the question of rationalizing the denominator for more complicated cases.)

We close this section by incorporating our practice of simplifying radical expressions into solving quadratic equations, following the approach of Example 8.2.1.

Example 8.2.5. *Solve the following quadratic equations.*

(a) $x^2 = 16$.

(b) $x^2 = 98$.

Answer. *In each case, we will follow the approach of Example 8.2.1, finding the positive solution and simplifying if necessary.*

(a) By definition, the positive solution of $x^2 = 16$ is $\sqrt{16}$. Since 16 is a perfect square, we simplify $\sqrt{16}$ as 4. Since 4 is a solution, -4 is a solution as well.

The solutions are 4 and -4.

(b) The positive solution to $x^2 = 98$ is $\sqrt{98}$. While 98 is not a perfect square, it does have a perfect square factor of 49. Simplifying,

$$\sqrt{98} = \sqrt{49 \cdot 2} = \sqrt{49} \cdot \sqrt{2} = 7\sqrt{2}.$$

Since $7\sqrt{2}$ is a solution, so is its opposite $-7\sqrt{2}$.

The solutions are $7\sqrt{2}$ and $-7\sqrt{2}$.

8.2.1 Exercises

Simplify the radical expressions below.

1. $\sqrt{32}$
2. $\sqrt{500}$
3. $\sqrt{98}$
4. $\sqrt{192}$
5. $\sqrt{50}$
6. $\sqrt{\dfrac{3}{16}}$
7. $\sqrt{\dfrac{8}{25}}$
8. $\sqrt{\dfrac{5}{6}}$
9. $\sqrt{\dfrac{1}{12}}$
10. $\sqrt{\dfrac{9}{32}}$
11. (*) Radical notation is also used to handle more general algebraic equations. The n^{th} *root* of a (positive[2]) number k, written $\sqrt[n]{k}$ is defined to be the (positive) solution to the equation $x^n = k$. For example, $\sqrt[3]{64} = 4$, since $(4)^3 = 64$, and $\sqrt[5]{32} = 2$, since $(2)^5 = 32$.
    (a) Evaluate $\sqrt[4]{81}$ and $\sqrt[3]{125}$.
    (b) Using the rules for simplifying square roots as a guide, write the corresponding rules for "simplified n^{th} roots."
    (c) Simplify: $\sqrt[3]{24}$.
    (d) Simplify: $\sqrt[3]{\dfrac{1}{2}}$.
    (e) Find a positive solution to $x^3 = 120$.

Solve the following quadratic equations.

12. $x^2 = 100$
13. $x^2 = 12$
14. $x^2 = 150$
15. $x^2 = \dfrac{3}{4}$

[2] In this example, we will not discuss roots of negative numbers.

8.3 Introduction to complex numbers

We already saw in the chapter introduction that certain quadratic equations might have no "real" number as a solution. For example, for the quadratic equation $x^2 = -1$, no "real" number, when multiplied by itself, can result in any negative number, and in particular cannot be -1.

However, rather than simply ending satisfied with the equation having no solution, we will instead adopt the approach of the previous section: we will suppose there is a solution, and denote this solution with a symbol.

The imaginary unit i

The symbol i will be used to denote one solution to the equation $x^2 = -1$.

Stated differently, i is a symbolic "number" having the property that $i^2 = -1$.

Because of the similarity of this definition with the symbolic definition of the square root $\sqrt{a}$ as a solution to $x^2 = a$, we sometimes write

$$i = \sqrt{-1}.$$

Why is i called an "imaginary unit?" The word "unit" (in the sense of "one") is due to the fact that i^2 $(= -1)$, by definition, has *magnitude* one. The word "imaginary" (due to the mathematician–philosopher René Descartes) is meant to emphasize that this symbol has different properties than the "real" numbers we are used to working with, and in particular its square is negative. In fact, the widely-used phrase "*real number*" arose (also due to Descartes) to distinguish these numbers from "imaginary numbers." We will adopt this usage[3]. From now on, we will use the term **real number** to be any number that does not involve the imaginary unit i. By contrast, we call a **complex number** a "number" that may involve the imaginary unit i. (This terminology is meant to be less judgmental than the phrase, "imaginary numbers.")

We emphasize again that complex numbers (involving i) have some very different properties from real numbers (not involving i). We already have seen that it is possible for the square of a complex number to be negative. Another difference, which is not so obvious, is that there is no way to order complex numbers with the usual comparison relations of "less than" or "greater than."

[3]The precise definition of a real number is quite technical. In fact, a rigorous definition of a real number that included all the properties commonly accepted as "real" was only stated toward the end of the 19th Century, some 250 years after Descartes first used the term. Most students (all except some math majors) will never encounter the "real" definition of a real number.

In particular, it makes no sense to call a complex number positive ("greater than zero") or negative ("less than zero").

Once we have made the definition of the imaginary unit, we will treat it (symbolically) exactly as we have treated radical expressions up to now. In this section, we will see how the imaginary unit arises in solving quadratic equations. In the next section, we will see how it is manipulated in the most basic cases.

Notice first that once we have defined i to be a solution to the equation $x^2 = -1$, then we should also admit the symbol $-i$ as another solution:

$$(-i)^2 = (-1 \cdot i)^2 = (-1)^2 \cdot i^2 = 1 \cdot (-1) = -1,$$

assuming that the symbol i should behave in accordance with the properties of exponents. In other words, once we allow for complex numbers, then the equation $x^2 = -1$ has *two* (complex) solutions, i and $-i$. (Remember, complex numbers involving the imaginary are neither positive nor negative. The best we can say is that $i(= 1 \cdot i)$ has a positive *coefficient* of 1, while $-i(= -1 \cdot i)$ has a negative coefficient.)

The following examples show how the imaginary unit arises in a variety of settings, once we introduce the rules of radical expressions that we have seen so far.

Example 8.3.1. *Simplify the following radical expressions.*

(a) $\sqrt{-4}$

(b) $\sqrt{-7}$

(c) $\sqrt{-50}$

Answer. *We adopt exactly the same approach to simplifying square roots with negative radicands as we did in the previous section. The only extra ingredient will be that we will use the symbol i to represent $\sqrt{-1}$.*

(a) Separating the factor of -1 in anticipation of complex number notation,

$$\sqrt{-4} = \sqrt{(-1) \cdot 4} = \sqrt{(-1)} \cdot \sqrt{4} = i \cdot 2 = 2i.$$

The answer is $2i$. (Notice that we will write the integer part as a "coefficient," in the same way that we customarily write $2x$ instead of $x \cdot 2$.)

(b)

$$\sqrt{-7} = \sqrt{(-1) \cdot 7} = \sqrt{-1} \cdot \sqrt{7} = i\sqrt{7}.$$

The answer is $i\sqrt{7}$. (In this case, it is customary to write the radical expression second, *even though $\sqrt{7}$ represents a real number "coefficient." This avoids writing $\sqrt{7}i$, where it might be misunderstood to indicate that the i is part of the radicand.)*

(c) *In this case, in addition to the presence of a complex number, we see that the radicand contains a (real!) perfect square factor.*

$$\begin{aligned}\sqrt{-50} &= \sqrt{(-1)\cdot 50}\\ &= \sqrt{-1}\cdot\sqrt{50}\\ &= i\sqrt{25\cdot 2}\\ &= i\cdot\sqrt{25}\cdot\sqrt{2}\\ &= i\cdot 5\cdot\sqrt{2}\\ &= 5i\sqrt{2}.\end{aligned}$$

The answer is $5i\sqrt{2}$. *(This notation, with the integer factor first, then the imaginary unit, then the irrational radical symbol, is customary. However,* $(5\sqrt{2})i$ *might be more in keeping with using a real number coefficient for the imaginary unit.)*

Complex numbers, really by definition, appear as solutions to quadratic equations. In the following examples, we proceed exactly as in Example 8.2.1, keeping in mind our convention of writing complex numbers using the imaginary unit i.

Example 8.3.2. *Solve the following quadratic equations.*

(a) $x^2 = -15$

(b) $x^2 + 18 = 0$.

Answer. *We will follow the same approach as Examples 8.2.1 and 8.2.5.*

(a) *One solution to* $x^2 = -15$ *is, by definition,* $\sqrt{-15}$. *Simplifying to indicate the imaginary unit,*

$$\sqrt{-15} = \sqrt{(-1)\cdot 15} = \sqrt{-1}\cdot\sqrt{15} = i\sqrt{15}.$$

The other solution will be $-i\sqrt{15}$.

The solutions are $i\sqrt{15}$ *and* $-i\sqrt{15}$.

(b) *The main thing to notice about this equation is that it does not have the special form* $x^2 = a$ *that we have been relying on so far. Fortunately, that is easy to fix in this case:*

$$\begin{array}{cccccc} x^2 & + & 18 & = & 0 \\ & & -18 & \vdots & -18 \\ \hline x^2 & & & = & -18. \end{array}$$

Now that the x^2*-term is by itself on one side of the equation, we can apply our strategy.*

One solution to $x^2 = -18$ is $\sqrt{-18}$. We see that in addition to being a complex number, the radicand has a perfect square factor of 9. Simplifying,

$$\sqrt{-18} = \sqrt{(-1) \cdot 9 \cdot 2} = \sqrt{-1} \cdot \sqrt{9} \cdot \sqrt{2} = i \cdot 3 \cdot \sqrt{2} = 3i\sqrt{2}.$$

(Notice we performed several simplifications at once.)

One solution is $3i\sqrt{2}$. Hence the other solution is $-3i\sqrt{2}$.

The solutions are $3i\sqrt{2}$ and $-3i\sqrt{2}$.

8.3.1 Exercises

Simplify the following radical expressions using the imaginary unit i.

1. $\sqrt{-16}$
2. $\sqrt{-45}$
3. $\sqrt{-\frac{5}{8}}$
4. (*) Assuming the rules of exponents apply to complex numbers, compute the first 10 powers of i: i^1, i^2, i^3, $\ldots$, i^{10}. (Hint: $i^3 = i^2 \cdot i^1$.)

Solve the following quadratic equations.

5. $x^2 = -36$
6. $x^2 + 24 = 0$

8.4 Arithmetic of radical expressions

We have seen so far that even simple quadratic equations in one variable may have "complicated" solutions—they may be irrational numbers, for example, or even complex numbers that have unusual properties compared to the real numbers we grew up with. So far, we have emphasized a *symbolic* approach to these numbers. In other words, we have used a symbol (a radical expression or an expression involving i) to represent a solution to an equation of a particular form (primarily of the form $x^2 = a$). This has the advantage of sidestepping the exact value of these solutions, but it carries the price of adhering to a set of customary rules about how such symbols will be written.

In this section, we will discuss how to perform arithmetic with these symbols—how to add them, subtract them, multiply them and divide them. In many ways, this will be exactly like how we approached the arithmetic of polynomials, and the reader will notice many similarities to how we approach the arithmetic of radical expressions. That shouldn't be a big surprise: polynomial arithmetic is really a kind of symbolic arithmetic, where the symbols are the variables.

The main difference between the arithmetic of radical expressions and the arithmetic of polynomials is that polynomials involve *indefinite* symbols: the variables are meant to represent an unknown or changing quantity. The symbols we have been using for irrational and complex expressions, like $\sqrt{2}$ and i, are *definite* symbols. They have a specific value or meaning (although it may be hard to write down exactly what they are), and this value is fixed and unchanging. This difference will show up repeatedly in the examples below.

Adding and subtracting radical expressions, like adding or subtracting polynomials, is based on the principle of *combining like terms*. Two expressions involving square roots are considered like terms if their radicands are the same. Like terms are added by adding their coefficients (and leaving the radical symbol the same). For example, the expression $5\sqrt{3} + 3\sqrt{3}$ consists of two like terms, since their symbolic part is the same $\sqrt{3}$. We can write

$$5\sqrt{3} + 3\sqrt{3} = 8\sqrt{3}.$$

(Compare this to $5x + 3x$.) However, $4\sqrt{2} + 6\sqrt{5}$ involves two terms which are not like terms, since the radicands are different, and so cannot be added or further simplified. (Compare this to $4x + 6y$.)

The only thing that needs to be mentioned is that it is important to simplify radical expressions *before* adding or subtracting, as the following examples illustrate.

Example 8.4.1. *Perform the indicated operations:*

(a) $\sqrt{24} - 3\sqrt{150} + 2\sqrt{3} + 15$.

(b) $5\sqrt{18} - 2\sqrt{2}$.

(c) $(3 + 4i) - (2 - 3i)$.

Answer. *(a) None of the radicands appearing in the expression*

$$\sqrt{24} - 3\sqrt{150} + 2\sqrt{3} + 15$$

are the same, and so there do not appear to be like terms. However, the first two (24 *and* 150*) have perfect square factors. Simplifying,*

$$\begin{aligned}
\sqrt{24} - 3\sqrt{150} + 2\sqrt{3} + 15 \\
\sqrt{4 \cdot 6} - 3\sqrt{25 \cdot 6} + 2\sqrt{3} + 15 \\
\sqrt{4} \cdot \sqrt{6} - 3\left(\sqrt{25} \cdot \sqrt{6}\right) + 2\sqrt{3} + 15 \\
2\sqrt{6} - 3(5\sqrt{6}) + 2\sqrt{3} + 15 \\
2\sqrt{6} - 15\sqrt{6} + 2\sqrt{3} + 15
\end{aligned}$$

Notice that at the last step, we multiplied the coefficient in the term containing $5\sqrt{6}$ *by* -3.

In any case, now there are like terms, namely, the $\sqrt{6}$*-terms. None of the other terms are like terms. Adding the coefficients for the* $\sqrt{6}$*-terms, we obtain*

$$-13\sqrt{6}+2\sqrt{3}+15.$$

The answer is $-13\sqrt{6}+2\sqrt{3}+15$.

(b) As in the previous example, the two radicands appearing in $5\sqrt{18}-2\sqrt{2}$ *are not the same, and so do not appear to be like terms. However,* 18 *contains a perfect square factor, and so can be simplified:*

$$\begin{aligned}
5\sqrt{18}-2\sqrt{2}\\
5\sqrt{9\cdot 2}-2\sqrt{2}\\
5\left(\sqrt{9}\cdot\sqrt{2}\right)-2\sqrt{2}\\
5(3\sqrt{2})-2\sqrt{2}\\
15\sqrt{2}-2\sqrt{2}.
\end{aligned}$$

After simplifying, the two remaining terms are like terms, and so they can be combined to obtain $13\sqrt{2}$.

The answer is $13\sqrt{2}$.

(c) Adding complex numbers, the real number parts are like terms and the imaginary parts (the terms containing $i=\sqrt{-1}$*) are like terms. Proceeding exactly like subtracting polynomials, we will change the problem to one of "adding the opposite" and combine like terms:*

$$\begin{aligned}
(3+4i)-(2-3i)\\
(3+4i)+(-2+3i)\\
(3+(-2))+(4i+3i)\\
1+7i.
\end{aligned}$$

The answer is $1+7i$.

The complex numbers in part (c) of the last example are typical of how complex numbers are written. In fact, any complex number can be written in the form

$$a+bi,$$

where a and b are real numbers.

Multiplying expressions involving radicals will typically involve the distributive law, exactly like multiplying polynomials. However, instead of relying on the rules of exponents (which we needed to multiply powers of a variable), we will use Property (S1) of roots.

Example 8.4.2. *Multiply:* $(8\sqrt{3}+2\sqrt{5})(\sqrt{2}-4\sqrt{5})$.

Answer. *Distributing, we obtain*

$$(8\sqrt{3}+2\sqrt{5})(\sqrt{2}-4\sqrt{5})$$
$$(8\sqrt{3})(\sqrt{2})+(8\sqrt{3})(-4\sqrt{5})+(2\sqrt{5})(\sqrt{2})+(2\sqrt{5})(-4\sqrt{5}).$$

According to Property (S1) of square roots, which states that $\sqrt{a}\cdot\sqrt{b}=\sqrt{ab}$, *we will multiply the coefficients* and *the radicands of each term:*

$$8\sqrt{6}-32\sqrt{15}+2\sqrt{10}-8\sqrt{25}.$$

All the radicands are different. However, one of the terms can be simplified, since 25 *is a perfect square:*

$$8\sqrt{6}-32\sqrt{15}+2\sqrt{10}-8(5)$$
$$8\sqrt{6}-32\sqrt{15}+2\sqrt{10}-40.$$

After simplifying, there are no like terms.

The answer is $8\sqrt{6}-32\sqrt{15}+2\sqrt{10}-40$.

In the preceding example, the factor of 5 in the last term (which first appeared at the second-last step) arose after multiplying $(\sqrt{5})(\sqrt{5})$. We chose to apply Property (S1) to obtain $\sqrt{25}$, then simplified. Notice, though, that by definition

$$(\sqrt{5})(\sqrt{5})=(\sqrt{5})^2=5.$$

Example 8.4.3. *Multiply:* $(5-3\sqrt{2})(4+\sqrt{2})$.

Answer. *We begin by distributing:*

$$\begin{array}{rl}
(5-3\sqrt{2})(4+\sqrt{2}) & \\
(5)(4)+(5)(\sqrt{2})+(-3\sqrt{2})(4)+(-3\sqrt{2})(\sqrt{2}) & \\
20+5\sqrt{2}-12\sqrt{2}-3\sqrt{4} & \\
20+5\sqrt{2}-12\sqrt{2}-3(2) & \textit{Simplifying } \sqrt{4} \\
20+5\sqrt{2}-12\sqrt{2}-6 & \\
14-7\sqrt{2} & \textit{Combining like terms}
\end{array}$$

The answer is $14-7\sqrt{2}$.

The next example involves multiplying two complex numbers. We will use the fact that $i^2=-1$.

Example 8.4.4. *Multiply:* $(3-2i)(-7+5i)$.

Answer. *Since we are using the symbol i for the radical expression* $\sqrt{-1}$, *our multiplication of complex numbers will look very much like multiplication of two binomials involving one variable—until the last steps.*

$$\begin{aligned}
&(3-2i)(-7+5i) && \\
&(3)(-7)+(3)(5i)+(-2i)(-7)+(-2i)(5i) && \textit{Distributing} \\
&-21+15i+14i-10i^2 && \textit{Multiplying term-by-term} \\
&-21+29i-10i^2 && \textit{Combining like terms} \\
&-21+29i-10(-1) && \textit{Since } i^2=-1 \\
&-21+29i+10 && \\
&-11+29i. && \textit{Combining like terms}
\end{aligned}$$

The answer is $-11+29i$.

The reader should take a moment to compare Examples 8.4.3 and 8.4.4. Both involve multiplying radical expressions with two terms, but one uses radical notation while the other uses imaginary i notation instead of radical notation $\sqrt{-1}$.

We will only consider the simplest examples of division of radical expressions. Some more complicated examples will appear as challenge exercises at the end of the section.

Example 8.4.5. *Simplify:* $\dfrac{\sqrt{3}\cdot\sqrt{66}}{\sqrt{2}}$.

Answer. *In this context, the word "simplify" means to perform all operations, and then simplify according to the rules of radical notation.*

Since the only operations involved are multiplication and division, we can rely on Properties (S1) and (S2) of square roots. In particular, combining the two properties, we can perform all the operations inside *the radicand:*

$$\frac{\sqrt{3}\cdot\sqrt{66}}{\sqrt{2}}=\sqrt{\frac{3\cdot 66}{2}}.$$

For convenience, in performing the operations within the radicand, we will take advantage of the fact that the 2 *in the denominator divides the larger factor* 66 *in the numerator:*

$$\sqrt{\frac{3\cdot 66}{2}}=\sqrt{3\cdot 33}=\sqrt{99}.$$

All that remains is to simplify the result, noticing that 99 *has a perfect square factor of* 9*:*

$$\sqrt{99}=\sqrt{9\cdot 11}=\sqrt{9}\cdot\sqrt{11}=3\sqrt{11}.$$

The answer is $3\sqrt{11}$.

For the record, there are several other approaches to the previous example, due to some flexibility with the order of operations.

8.4.1 Exercises

Perform the indicated operations. Simplify all radical expressions.

1. $2\sqrt{50} - 4\sqrt{8} + 6\sqrt{12}$
2. $-\sqrt{24} + 4\sqrt{3} - \sqrt{27}$
3. $3\sqrt{20} + 2\sqrt{45}$
4. $(3 - 2i) - (5 + 7i)$
5. $\sqrt{5}(2\sqrt{10} - 1)$
6. $\dfrac{\sqrt{5} \cdot \sqrt{30}}{\sqrt{3}}$.
7. $\sqrt{6}(4\sqrt{3} - 5\sqrt{2})$
8. $(1 + 5i)(3 - 2i)$
9. $(1 + \sqrt{2})(1 - \sqrt{2})$
10. $(2 + 3i)(2 - 3i)$
11. $(\sqrt{3} - \sqrt{2})^2$
12. $(1 + i)^2$
13. (*) Use corresponding properties of n^{th} roots (see Exercise 11 of the Section 8.2.1) to simplify
$$5\sqrt[3]{16} - 2\sqrt[3]{54} + 6\sqrt[3]{24}.$$
14. (*) This exercise gives and indication of how to simplify expressions with a binomial involving a radical in the denominator.
    (a) Perform the following multiplication using properties of radicals:
    $$\frac{3 - \sqrt{2}}{5 + 2\sqrt{3}} \cdot \frac{5 - 2\sqrt{3}}{5 - 2\sqrt{3}}.$$
    (Notice that we have really multiplied the expression $\dfrac{3 - \sqrt{2}}{5 + 2\sqrt{3}}$ by 1, so have not changed the value of the expression.)
    (b) Use the idea of the previous exercise to simplify the radical expression
    $$\frac{\sqrt{2}}{4 + \sqrt{5}}.$$
    The technique hinted at in this exercise is known as *rationalizing the denominator*.

15. (*) The technique in the previous exercise can be used to divide complex numbers.

    (a) Perform the following multiplication:

    $$\frac{4+i}{2+3i} \cdot \frac{2-3i}{2-3i}.$$

    (Notice that just like in the last exercise, we are multiplying the expression $\frac{4+i}{2+3i}$ by 1.)

    (b) Use the idea in the previous exercise to write the quotient

    $$\frac{3-2i}{1+6i}$$

    in standard complex form $a+bi$.

16. (*) What is wrong with the following "proof" that $-1 = 1$?

    $$1 = \sqrt{1} = \sqrt{(-1)\cdot(-1)} = \sqrt{-1}\cdot\sqrt{-1} = i \cdot i = i^2 = -1.$$

8.5 Chapter summary

- A typical quadratic equation in one variable will have two solutions.
- The solutions to a quadratic equation in one variable may be rational, irrational, or complex (even when the coefficients of the equation are integers).
- Irrational and complex numbers are generally treated symbolically, according to historically-evolved rules for what is considered "simplified."