ARITHMETIC:
A Textbook for Math 01
4th edition (2015)

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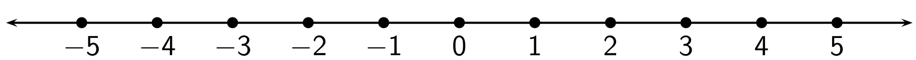
Contents

Chapter 1

Whole numbers

The *natural numbers* are the counting numbers: $1, 2, 3, 4, 5, 6, \ldots$. The dots indicate that the sequence is *infinite* – counting can go on forever, since you can always get the next number by simply adding 1 to the previous number. In order to write numbers efficiently, and for other reasons, we also need the number 0. Later on, we will need the sequence of *negative* numbers $-1, -2, -3, -4, -5, -6, \ldots$. Taken together, all these numbers are called the **integers**.

It helps to visualize the integers laid out on a **number line**, with 0 in the middle, and the natural numbers increasing to the right. There are numbers between any two integers on the number line. In fact, every location on the line represents some number. Some locations represent *fractions* such as one-half (between 0 and 1) or four-thirds (between 1 and 2). Other locations represent numbers which cannot be expressed as fractions, such as π. (π is located between 3 and 4 and expresses the ratio of the circumference to the diameter of any circle.)



For now, we concentrate on the non-negative integers (including 0), which we call **whole numbers**. We need only ten symbols to write any whole number. These symbols are the *digits*

$$0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9.$$

We write larger whole numbers using a **place-value** system. The digit in the right-most place indicates how many *ones* the number contains, the digit in the second-from-right place indicates how many *tens* the number contains, the digit in the third-from-right place indicates how many *hundreds* the number contains, etc.

Example 1.

7 stands for 7 *ones*

72 stands for 7 *tens* + 2 *ones*

349 stands for 3 *hundreds* + 4 *tens* + 9 *ones*

6040 stands for 6 *thousands* + 0 *hundreds* + 4 *tens* + 0 *ones*

Notice that when you move left, the place value increases ten-fold. So if a number has five digits, the fifth-from-right place indicates how many ten-thousands the number contains. (Ten-thousand is ten times a thousand.)

1.0.1 Exercises

1. 35 stands for ____
2. 209 stands for ____
3. 9532 stands for ____
4. 21045 stands for ____

1.1 Adding Whole Numbers

When we add two or more integers, the result is called the **sum**. We assume you know the sums of single-digit numbers. For practice, do the following example.

Example 2. Fill in the missing squares in the *digit-addition table* below. For example, the number in the row labelled 3 and the column labelled 4 is the sum $3 + 4 = 7$.

+	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	5	6	7	8	9
1										
2										
3					7					
4										
5										
6										
7						12				
8										
9										

Table 1.1: The digit addition table

1. Do you notice any patterns or regularities in the digit-addition table? Can you explain them?
2. Why is the second line from the top identical to the top line?
3. What can you say about the left-most column and the second-from-left column?

1.1.1 Commutativity, Associativity, Identity

The first question above leads us to an important property of addition, namely, that for any two numbers x and y,

$$x + y = y + x.$$

In other words, the order in which two numbers are added does not effect the sum. This property of addition is called **commutativity**.

The last two questions lead us to an important property of 0, namely, for any number x,

$$x + 0 = x = 0 + x.$$

In other words, when 0 is added to any number, x, you get the identical number, x, again. Because of this property, 0 is called the **additive identity**.

One final property of addition is expressed in the following equation

$$(x + y) + z = x + (y + z),$$

which says that if three numbers are added, it doesn't matter how you "associate" the additions: you can add the first two numbers first, and then add the third to that, or, you could add the second two numbers first, and then add the first to that. This property of addition is called **associativity**.

Example 3. Find the sum of 2, 3, and 5 by associating in two different ways.

Solution. Associating 2 and 3, we calculate

$$(2 + 3) + 5 = 5 + 5 = 10.$$

On the other hand, associating 3 and 5, we calculate

$$2 + (3 + 5) = 2 + 8 = 10.$$

The sum is the same in both cases. □

1.1.2 Multi-digit addition

To add numbers with more than one digit, we line up the numbers vertically so that the *ones* places (right-most) are directly on top of each other, and all other places are similarly arranged. Then we add the digits in each place to obtain the sum.

Example 4. Find the sum of 25 and 134.

Solution. We line up the numbers vertically so that the 5 in the ones place of 25 is over the 4 in the ones place of 134. If we do this carefully, all the other places line up vertically, too. So there is a "ones" column, a "tens" column to the left of it, and a "hundreds" column to the left of that:

$$\begin{array}{r} 2\,5 \\ 1\,3\,4 \end{array}$$

Then we draw a line and add the digits in each column to get the sum:

$$\begin{array}{rr} & 2\,5 \\ + & 1\,3\,4 \\ \hline & 1\,5\,9 \end{array}$$

□

Sometimes we need to "carry" a digit from one place to the next higher place. For example, when adding $38 + 47$, we first add the ones places, $8 + 7 = 15$. But 15 has two digits: it stands for 1 *ten* + 5 *ones*. We "put down" the 5 in the *ones* place, and "carry" the 1 (standing for a *ten*) to the *tens* column. So we have:

Example 5. Find the sum

$$\begin{array}{r} 3\,8 \\ +\;\;4\,7 \\ \hline \end{array}$$

Solution. Put down the 5 in the *ones* place:

$$\begin{array}{r} 3\,8 \\ +\;\;4\,7 \\ \hline 5 \end{array}$$

and carry the 1 to the top of the *tens* column:

$$\begin{array}{r} \{1\}\;\; \\ 3\,8 \\ +\;\;4\,7 \\ \hline 5 \end{array}$$

and finish the job by adding up the *tens* column, including the carried one:

$$\begin{array}{r} \{1\}\;\; \\ 3\,8 \\ +\;\;4\,7 \\ \hline 8\,5 \end{array}$$

The sum is 85. □

Since carrying alters the sum in the column immediately to the left of the one we are working on, in multi-digit addition, we always start with the right-most column and proceed leftward.

1.1.3 Exercises

Find the sums, carrying where necessary.

1.

$$\begin{array}{r} 2\,6 \\ +\;\;5\,5 \\ \hline \end{array}$$

2.

$$\begin{array}{r} 383 \\ +\quad 47 \\ \hline \end{array}$$

3.

$$\begin{array}{r} 32 \\ 45 \\ +\quad 64 \\ \hline \end{array}$$

4.

$$\begin{array}{r} 129 \\ 377 \\ +\quad 4503 \\ \hline \end{array}$$

5.

$$\begin{array}{r} 909 \\ 777 \\ +\quad 6964 \\ \hline \end{array}$$

6.

$$\begin{array}{r} 320 \\ 984 \\ 902 \\ +\quad 4503 \\ \hline \end{array}$$

7.

$$\begin{array}{r} 9 \\ 999 \\ 902 \\ + \quad 9502 \\ \hline \end{array}$$

8.

$$\begin{array}{r} 56320 \\ 9864 \\ 904 \\ + \quad 6503 \\ \hline \end{array}$$

9.

$$\begin{array}{r} 32000 \\ 9844 \\ 902 \\ + \quad 4503 \\ \hline \end{array}$$

10.

$$\begin{array}{r} 2997 \\ 9844 \\ 205 \\ + \quad 54908 \\ \hline \end{array}$$

1.2 Subtracting Whole Numbers

Another way to say that $5 + 2 = 7$ is to say that $5 = 7 - 2$. In words, "5 is the difference of 7 and 2," or "5 is the result of taking away 2 from 7." The operation of taking away one number from another, or finding their **difference**, is called *subtraction*. For now, we have to be careful that the number we take away is no larger than the number we start with: we cannot have 3 marbles and take 7 of them away! Later on, when we introduce negative numbers, we won't have to worry about this.

We assume you remember the differences of single-digit numbers. Just to make sure, do the following example.

Example 6. Fill in the missing squares in the *digit-subtraction table* below. Here's a start: the number in the row labelled 7 and the column labelled 2 is the difference $7 - 2 = 5$. The digit in the row labelled 3 and the column labelled 3 is the difference $3 - 3 = 0$. (Squares that have an asterisk ($*$) will be filled in later with negative numbers.)

−	0	1	2	3	4	5	6	7	8	9
0		*	*	*	*	*	*	*	*	*
1			*	*	*	*	*	*	*	*
2				*	*	*	*	*	*	*
3				0	*	*	*	*	*	*
4						*	*	*	*	*
5							*	*	*	*
6								*	*	*
7			5						*	*
8										*
9						4				

Table 1.2: The digit subtraction table

1.2.1 Commutativity, Associativity, Identity

When we study negative numbers, we will see that subtraction is **not commutative**. We can see by a simple example that subtraction is also **not associative**.

Example 7. Verify that $(7-4)-2$ is not equal to $7-(4-2)$.

Solution. Associating the 7 and 4, we get

$$(7-4)-2=3-2=1,$$

but associating the 4 and the 2, we get

$$7-(4-2)=7-2=5,$$

a different answer. Until we establish an order of operations, we will avoid examples like this! □

It is true that

$$x-0=x$$

for any number x. However, 0 is not an identity for subtraction, since $0-x$ is *not* equal to 0 (unless $x=0$). To make sense of $0-x$, we will need negative numbers.

1.2.2 Multi-digit subtractions

To perform subtractions of multi-digit numbers, we need to distinguish the number "being diminished" from the number which is "doing the diminishing" (being taken away). The latter number is called the **subtrahend**, and the former, the **minuend**. For now, we take care that the subtrahend is no larger than the minuend.

To set up the subtraction, we line the numbers up vertically, with the minuend over the subtrahend, and the *ones* places lined up on the right.

Example 8. Find the difference of 196 and 43.

Solution. The subtrahend is 43 (the smaller number), so it goes on the bottom. We line up the numbers vertically so that the 6 in the ones place of 196 is over the 3 in the ones place of 43.

```
1 9 6
  4 3
```

Then we draw a line and subtract the digits in each column, starting with the *ones* column and working right to left, to get the difference:

$$
\begin{array}{r}
196 \\
-\quad 43 \\
\hline
153
\end{array}
$$

□

1.2.3 Checking Subtractions

Subtraction is the "opposite" of addition, so any subtraction problem can be restated in terms of addition. Using the previous example, and adding the difference to the subtrahend, we obtain

$$
\begin{array}{r}
153 \\
+\quad 43 \\
\hline
196
\end{array}
$$

which is the original minuend. In general, **if subtraction has been performed correctly, adding the difference to the subtrahend returns the minuend.** This gives us a good way to check subtractions.

Example 9. Check whether the following subtraction is correct:

$$
\begin{array}{r}
94 \\
-\quad 51 \\
\hline
33
\end{array}
$$

Solution. Adding the (supposed) difference to the subtrahend, we get

$$
\begin{array}{r}
33 \\
+\quad 51 \\
\hline
84
\end{array}
$$

which is not equal to the minuend (94). Thus the subtraction is incorrect. We leave it to you to fix it! □

1.2.4 Borrowing

Sometimes, when subtracting, we need to "borrow" a digit from a higher place and add its equivalent to a lower place.

Example 10. Find the difference:

$$
\begin{array}{r}
85 \\
-\quad 46 \\
\hline
\end{array}
$$

Solution. The digit subtraction in the *ones* column is not possible (we can't take 6 from 5). Instead we remove or "borrow" 1 *ten* from the *tens* place of the minuend, and convert it into 10 *ones*, which we add to the *ones* in the ones place of the minuend. The minuend is now represented as {7}{15}, standing for 7 *tens* + 15 *ones*. It looks funny (as if 15 were a digit), but it doesn't change the value of the minuend, which is still $7 \times 10 + 15 = 85$. We represent the borrowing operation like this:

$$\begin{array}{rcc} & \{7\} & \{15\} \\ & 8 & 5 \\ - & 4 & 6 \\ \hline \end{array}$$

Ignoring the original minuend, we have $15 - 6 = 9$ for the *ones* place, and $7 - 3 = 4$ for the *tens* place, as follows:

$$\begin{array}{rcc} & \{7\} & \{15\} \\ & 8 & 5 \\ - & 4 & 6 \\ \hline & 3 & 9 \end{array}$$

The difference is 39. We can check this by verifying that the difference + the subtrahend = the original minuend:

$$\begin{array}{rr} & 39 \\ + & 46 \\ \hline & 85 \end{array}$$

□

Sometimes you have to go more than one place to the left to borrow successfully. This happens when the next higher place has a 0 digit – there is nothing to borrow from.

Example 11. Find the difference:

$$\begin{array}{rr} & 207 \\ - & 69 \\ \hline \end{array}$$

Solution. In the *ones* column we can't take 9 from 7, so we need to borrow from a higher place. We can't borrow from the *tens* place, because it has 0 *tens*. But we can borrow from the *hundreds* place. We borrow 1 *hundred*, and convert it into 9 *tens* and 10 *ones*. The minuend is now represented as {1}{9}{17}, standing for 1 *hundred* + 9 *tens* + 17 *ones*, (as if 17 were a digit). We represent the borrowing as before:

$$\begin{array}{rccc} & \{1\} & \{9\} & \{17\} \\ & 2 & 0 & 7 \\ - & & 6 & 9 \\ \hline \end{array}$$

Ignoring the original minuend, we have $17 - 9 = 8$ for the *ones* place, $9 - 6 = 3$ for the *tens* place, and $1 - 0 = 1$ for the *hundreds* place:

$$\begin{array}{rccc} & \{1\} & \{9\} & \{17\} \\ & 2 & 0 & 7 \\ - & & 6 & 9 \\ \hline & 1 & 3 & 8 \end{array}$$

The difference is 138. To check, we verify that the difference + the subtrahend = the original minuend:

$$\begin{array}{r} 1\,3\,8 \\ +\ \ 6\,9 \\ \hline 2\,0\,7 \end{array}$$

(You may have noticed that the *carrying* you do in the addition check simply reverses the *borrowing* done in the original subtraction!) □

1.2.5 Exercises

Find the differences, borrowing where necessary. Check that the difference + the subtrahend = the minuend.

1.

$$\begin{array}{r} 9\,4 \\ -\ \ 3\,7 \\ \hline \end{array}$$

2.

$$\begin{array}{r} 2\,7\,5 \\ -\ \ 1\,8\,1 \\ \hline \end{array}$$

3.

$$\begin{array}{r} 3\,5\,0 \\ -\ \ 7\,6 \\ \hline \end{array}$$

4.

$$\begin{array}{r} 5\,0\,0 \\ -\ \ 1\,9\,1 \\ \hline \end{array}$$

5.

$$\begin{array}{r} 6\,0\,0 \\ -\ \ 1\,9\,9 \\ \hline \end{array}$$

6.

$$\begin{array}{r} 1\,5\,0\,0 \\ - \quad 1\,1\,9\,1 \\ \hline \end{array}$$

7.

$$\begin{array}{r} 5\,6\,7\,8 \\ - \quad 4\,5\,6\,7 \\ \hline \end{array}$$

8.

$$\begin{array}{r} 5\,0\,0\,0\,0 \\ - \quad 4\,9\,9\,9 \\ \hline \end{array}$$

9.

$$\begin{array}{r} 8\,0\,1 \\ - \quad 7\,9\,0 \\ \hline \end{array}$$

10.

$$\begin{array}{r} 6\,3\,8\,9 \\ - \quad 9\,9\,9 \\ \hline \end{array}$$

11.

$$\begin{array}{r} 5\,0\,0\,0\,0\,0 \\ - \quad 4\,3\,2\,1\,0 \\ \hline \end{array}$$

12.

$$\begin{array}{r} 9\,0\,0\,1\,0\,1\,0 \\ - \quad 1\,1\,1\,1\,1\,1\,1 \\ \hline \end{array}$$

1.3 Multiplying Whole Numbers

Multiplication is really just repeated addition. When we say "4 times 3 equals 12," we can think of it as starting at 0 and adding 3 four times over:

$$0 + 3 + 3 + 3 + 3 = 12.$$

We can leave out the 0, since 0 is the additive identity ($0{+}3 = 3$). Using the symbol $\times$ for multiplication, we write

$$3 + 3 + 3 + 3 = 4 \times 3 = 12.$$

The result of multiplying two or more numbers is called the **product** of the numbers. Instead of the $\times$ symbol, we often use a **central dot** ($\cdot$) to indicate a product. Thus, for example, instead of $2 \times 4 = 8$, we can write

$$2 \cdot 4 = 8.$$

Example 12. We assume you remember the products of small whole numbers, so it should be easy for you to reproduce the partial *multiplication table* below. For example, the number in the row labelled **7** and the column labelled **5** is the product $7 \cdot 5 = 35$.

×	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9	10	11	12
2	0	2	4	6	8	10	12	14	16	18	20	22	24
3	0	3	6	9	12	15	18	21	24	27	30	33	36
4	0	4	8	12	16	20	24	28	32	36	40	44	48
5	0	5	10	15	20	25	30	35	40	45	50	55	60
6	0	6	12	18	24	30	36	42	48	54	60	66	72
7	0	7	14	21	28	35	42	49	56	63	70	77	84
8	0	8	16	24	32	40	48	56	64	72	80	88	96
9	0	9	18	27	36	45	54	63	72	81	90	99	108
10	0	10	20	30	40	50	60	70	80	90	100	110	120
11	0	11	22	33	44	55	66	77	88	99	110	121	132
12	0	12	24	36	48	60	72	84	96	108	120	132	144

Table 1.3: Multiplication table

1. Do you notice any patterns or regularities in the multiplication table? Can you explain them?
2. Why does the second row from the top contain only 0's?
3. Why is the third row from the top identical to the first row?
4. Is multiplication commutative? How can you tell from the table?

1.3.1 Commutativity, Associativity, Identity, the Zero Property

An examination of the multiplication table leads us to an important property of 0, namely, when any number, N, is multiplied by 0, the product is 0:

$$0 \cdot N = N \cdot 0 = 0.$$

It also shows us an important property of 1, namely, when any number, N, is multiplied by 1, the product is the identical number, x, again:

$$1 \cdot N = N \cdot 1 = N.$$

For this reason, 1 is called the **multiplicative identity**.

The following example should help you to see that **multiplication is commutative**.

Example 13. The figure shows two ways of piling up twelve small squares. On the left, we have piled up 3 rows of 4 squares (3×4); on the right, we have piled up 4 rows of 3 squares (4×3). In both cases, of course, the total number of squares is the product $3 \times 4 = 4 \times 3 = 12$.

Figure 1.1: $3 \times 4 = 4 \times 3 = 12$

Examples like the following help you to see that multiplication is associative.

Example 14. We can find the product $3 \times 4 \times 5$ in two different ways. We could first associate 3 and 4, getting

$$(3 \times 4) \times 5 = 12 \times 5 = 60,$$

or we could first associate 4 and 5, getting

$$3 \times (4 \times 5) = 3 \times 20 = 60.$$

The product is the same in both cases.

1.3.2 Multi-digit multiplications

To multiply numbers when one of them has more than one digit, we need to distinguish the number "being multiplied" from the number which is "doing the multiplying." The latter number is called the **multiplier**, and the former, the **multiplicand**. It really makes no difference which number is called the multiplier and which the multiplicand (because multiplication is commutative!). But it saves space if we choose the multiplier to be the number with the fewest digits.

To set up the multiplication, we line the numbers up vertically, with the multiplicand over the multiplier, and the *ones* places lined up on the right.

Example 15. To multiply 232 by 3, we write

$$\begin{array}{r} 2\,3\,2 \\ \times \quad\ 3 \\ \hline \end{array}$$

We multiply place-by-place, putting the products in the appropriate column. 3×2 *ones* is 6 *ones*, so we put 6 in the *ones* place

$$\begin{array}{r} 2\,3\,2 \\ \times \quad\ 3 \\ \hline 6 \end{array}$$

3×3 *tens* is 9 *tens*, so we put 9 in the *tens* place

$$\begin{array}{r} 2\,3\,2 \\ \times \quad\ 3 \\ \hline 9\,6 \end{array}$$

Finally, 3×2 *hundreds* is 6 *hundreds*, so that we put down 6 in the hundreds place, and we are done:

$$\begin{array}{r} 2\,3\,2 \\ \times \quad\ 3 \\ \hline 6\,9\,6 \end{array}$$

Sometimes we need to carry a digit from one place to the next higher place, as in addition. For example, $4 \times 5 = 20$, a number with two digits. So we would "put down" the 0 in the current place, and "carry" the 2 to the column representing the next higher place, as in the next example.

Example 16. Multiply 251 by 4.

Solution. The steps are

$$\begin{array}{r} 2\,5\,1 \\ \times \quad\ 4 \\ \hline \end{array}$$

For the *ones* place,

$$\begin{array}{r} 2\,5\,1 \\ \times \quad\ 4 \\ \hline 4 \end{array}$$

For the *tens* place, $4 \times 5 = 20$, so we put down the 0 in the *tens* place and carry the 2 to the *hundreds* column:

$$\begin{array}{r} \{2\}\quad\ \ \\ 2\,5\,1 \\ \times \quad\ 4 \\ \hline 0\,4 \end{array}$$

For the hundreds place, 4×2 *hundreds* is 8 *hundreds*, to which we add the 2 *hundreds* that were carried. This gives us 10 *hundreds*, or 1 *thousand*. We put down 0 in the *hundreds* place, and 1 in the *thousands* place.

$$\begin{array}{r} \{2\} \\ 2\,5\,1 \\ \times \qquad 4 \\ \hline 1\,0\,0\,4 \end{array}$$

The product of 251 and 4 is 1004. □

If the multiplier has more than one digit, the procedure is a little more complicated. We get *partial products* (one for each digit of the multiplier) which are added to yield the total product.

Example 17. Consider the product

$$\begin{array}{r} 2\,4 \\ \times \qquad 3\,2 \\ \hline \end{array}$$

Since the multiplier stands for 3 *tens* + 2 *ones*, we can split the product into two partial products

$$\begin{array}{r} 2\,4 \\ \times \qquad 2 \\ \hline 4\,8 \end{array}$$

and

$$\begin{array}{r} 2\,4 \\ \times \qquad 3 \\ \hline \end{array}$$

Notice that in the second partial product the multiplier is in the *tens* column. This is almost exactly like having a 1-digit multiplier. The second partial product is obtained by simply putting down a 0 in the *ones* place and shifting the digit products one place to the left:

$$\begin{array}{r} 2\,4 \\ \times \qquad 3 \\ \hline 7\,2\,0 \end{array}$$

(Notice that we put down 2 and carried 1 when we performed the digit product $3 \times 4 = 12$.) The total product is the sum of the two partial products: $48 + 720 = 768$. We can write the whole procedure compactly by aligning the two partial products vertically

$$\begin{array}{r} 2\,4 \\ \times \qquad 3\,2 \\ \hline 4\,8 \\ 7\,2\,0 \end{array}$$

and then performing the addition

$$\begin{array}{rr} & 2\,4 \\ \times & 3\,2 \\ \hline & 4\,8 \\ + & 7\,2\,0 \\ \hline & 7\,6\,8 \end{array}$$

Here's another example.

Example 18. Find the product of 29 and 135.

Solution. We choose 29 as the multiplier since it has the fewest digits.

$$\begin{array}{rr} & 1\,3\,5 \\ \times & 2\,9 \\ \hline \end{array}$$

We use the 1-digit multiplier 9 to obtain the first partial product

$$\begin{array}{rr} & \{3\}\{4\} \\ & 1\;\;3\;\;5 \\ \times & 2\;\;9 \\ \hline & 1\,2\;\;1\;\;5 \end{array}$$

Notice that we put down 5 and carried 4 to the *tens* place, and also put down 1 and carried 3 to the *hundreds* place. Next we use the 1-digit multiplier 2 (standing for 2 *tens*) to obtain the second partial product, shifted left by putting a 0 in the *ones* place

$$\begin{array}{rr} & \{1\} \\ & 1\,3\,5 \\ \times & 2\,9 \\ \hline & 1\,2\,1\,5 \\ & 2\,7\,0\,0 \end{array}$$

(What carry did we perform?) Finally, we add the partial products to obtain the (total) product

$$\begin{array}{rr} & 1\,3\,5 \\ \times & 2\,9 \\ \hline & 1\,2\,1\,5 \\ + & 2\,7\,0\,0 \\ \hline & 3\,9\,1\,5 \end{array}$$

Note that the whole procedure is compactly recorded in the last step, which is all that you need to write down. □

1.3.3 Exercises

Find the products.

1.

$$\begin{array}{r} 122 \\ \times \quad 4 \\ \hline \end{array}$$

2.

$$\begin{array}{r} 83 \\ \times \quad 5 \\ \hline \end{array}$$

3.

$$\begin{array}{r} 104 \\ \times \quad 7 \\ \hline \end{array}$$

4.

$$\begin{array}{r} 3008 \\ \times \quad 9 \\ \hline \end{array}$$

5.

$$\begin{array}{r} 212 \\ \times \quad 43 \\ \hline \end{array}$$

6.

$$\begin{array}{r} 83 \\ \times \quad 56 \\ \hline \end{array}$$

7.

$$\begin{array}{r} 136 \\ \times \quad 27 \\ \hline \end{array}$$

8.

$$\begin{array}{r} 308 \\ \times \quad 109 \\ \hline \end{array}$$

9.

$$\begin{array}{r} 2103 \\ \times \quad 44 \\ \hline \end{array}$$

10.

$$\begin{array}{r} 837 \\ \times \quad 54 \\ \hline \end{array}$$

1.4 Powers of Whole Numbers

If we start with 1 and repeatedly *multiply* by 3, 4 times over, we get a number that is called the 4*th power of* 3, written

$$3^4 = 1 \times 3 \times 3 \times 3 \times 3.$$

The factor of 1 is understood and usually omitted. Instead we simply write

$$3^4 = 3 \times 3 \times 3 \times 3.$$

In the expression 3^4, 3 is called the *base*, and 4 the *exponent* (or power).

Example 19. The 5th power of 2, or 2^5, is the product

$$2 \times 2 \times 2 \times 2 \times 2 = 32.$$

The 3rd power of 4, or 4^3, is the product

$$4 \times 4 \times 4 = 64.$$

We make the following definition in the cases where the exponent is 0.

For any nonzero number N,

$$N^0 = 1.$$

(0^0 is undefined.)

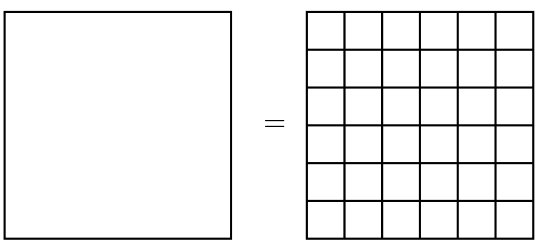
This makes sense if you remember the harmless factor of 1 that is understood in every exponential expression. (You'll see another justification when you study algebra.)

Although 0^0 is undefined, expressions such as 0^2, 0^3, etc., with 0 base and nonzero exponent, make perfect sense (e.g., $0^3 = 0 \times 0 \times 0 = 0$).

Example 20. $17^0 = 1$. $0^5 = 0$. $0^1 = 0$. 0^0 is undefined.

1.4.1 Squares and Cubes

Certain powers are so familiar that they have special names. For example, the 2nd power is called the **square** and the 3rd power is called the **cube**. Thus 5^2 is read "5 squared," and 7^3 is read as "7 cubed." The source of these special names is geometric (see Section 1.9). The **area** of a square, x units on a side, is x^2 *square units*. This means that the square contains x^2 small squares, each one unit on a side. For example, the figure below shows a square 6 units on a side, with area $6^2 = 36$ square units.



Similarly, the **volume** of a cube, y units on a side, is y^3 *cubic units*. This means that the cube contains y^3 little cubes, each one unit on a side. For example, the volume of an ice cube that measures 2 cm (centimeters) on a side is $2^3 = 8$ cubic centimeters.

1.4.2 Exercises

1. Rewrite using an exponent: $8 \times 8 \times 8 \times 8$
2. Rewrite using an exponent: $4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4$
3. Evaluate 2^5
4. Evaluate 9^0
5. Evaluate 0^7
6. Evaluate 5^4
7. Evaluate 10^2

8. Evaluate 10^3

9. Complete the following table of squares:

$0^2 = 0$	$6^2 =$	$12^2 =$	$18^2 =$
$1^2 =$	$7^2 = 49$	$13^2 =$	$19^2 =$
$2^2 =$	$8^2 =$	$14^2 =$	$20^2 =$
$3^2 =$	$9^2 =$	$15^2 =$	$30^2 =$
$4^2 =$	$10^2 =$	$16^2 =$	$40^2 =$
$5^2 =$	$11^2 =$	$17^2 =$	$50^2 =$

10. Complete the following table of cubes:

$0^3 =$	$3^3 =$	$6^3 =$	$9^3 =$
$1^3 =$	$4^3 = 64$	$7^3 =$	$10^3 =$
$2^3 =$	$5^3 =$	$8^3 =$	$100^3 =$

1.4.3 Square Roots

If there is a number, whose square is the number n, we call it the **square root** of n, and symbolize it by

$$\sqrt{n}.$$

For example, $2^2 = 4$, so 2 is the square root of 4, and we write

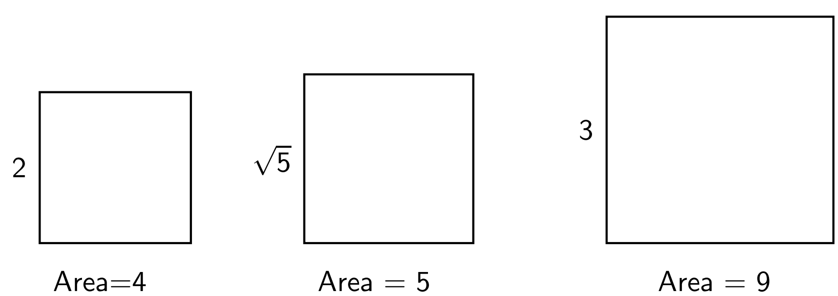
$$\sqrt{4} = 2.$$

Similarly, $3^2 = 9$, so 3 is the square root of 9, and we write

$$\sqrt{9} = 3.$$

Actually, every positive whole number has *two* square roots, one positive and one negative. The positive square root is called the **principal square root**, and, for now, when we say square root, we mean the principal one.

A whole number whose square root is also a whole number is called a **perfect square**. The first few perfect squares are 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121. Clearly, there are lots of whole numbers that are *not* perfect squares. But these numbers must have square roots. Using our geometric intuition, it is easy to believe that there are geometric squares whose areas are not perfect square numbers. For example, if we want to construct a square whose area is 5 square units, we could start with a square whose area is 4 square units, and steadily expand it on all sides until we get the desired square.



The side length of the middle square is a number which, when squared, yields 5, that is, $\sqrt{5}$. We can see that $\sqrt{5}$ is bigger than $\sqrt{4} = 2$, and less than $\sqrt{9} = 3$.

In general, if a whole number lies between two perfect squares, its square root must lie between the two corresponding square roots.

Example 21. Since 21 lies between the perfect squares 16 and 25, $\sqrt{21}$ must lie between $\sqrt{16} = 4$ and $\sqrt{25} = 5$.

Example 22. Between what two consecutive whole numbers does $\sqrt{53}$ lie?

Solution. Since 53 lies between the perfect squares 49 and 64, the square root $\sqrt{53}$ must lie between $\sqrt{49} = 7$ and $\sqrt{64} = 8$. □

1.4.4 Exercises

Find the square roots:

1. $\sqrt{49}$
2. $\sqrt{81}$
3. $\sqrt{169}$
4. $\sqrt{121}$
5. $\sqrt{64}$

Between what two consecutive whole numbers do the following square roots lie?

6. $\sqrt{19}$
7. $\sqrt{75}$
8. $\sqrt{26}$
9. $\sqrt{32}$
10. Complete the square-root table:

$\sqrt{0} =$	$= 6$	$= 12$	$= 18$
$\sqrt{1} =$	$= 7$	$= 13$	$= 19$
$\sqrt{4} =$	$= 8$	$= 14$	$= 20$
$\sqrt{9} =$	$= 9$	$= 15$	$= 30$
$= 4$	$= 10$	$= 16$	$= 40$
$= 5$	$= 11$	$= 17$	$= 50$

1.5 Division of Whole Numbers

How many times does 23 "go into" 100? Put another way, starting with 100, how many times can we subtract 23 without obtaining a negative number? The answer is easily seen to be 4. Moreover, it is easy to see that there is something "left over," namely, 8. Here are the computations:

$$\begin{array}{r} 100 \\ -\ 23 \\ \hline 77 \\ -\ 23 \\ \hline 54 \\ -\ 23 \\ \hline 31 \\ -\ 23 \\ \hline 8 \end{array}$$

This operation (repeated subtraction) is called **division**. The number we start with (100 in the example) is called the **dividend**, and the number we repeatedly subtract (23 in the example) is called the **divisor**. We use the symbol $\div$, and note that the dividend is written first:

$$\text{dividend} \div \text{divisor}.$$

1.5.1 Quotient and Remainder

Unlike the other three operations (addition, subtraction, multiplication), the result of a division of whole numbers consists of not one but *two* whole numbers: the number of subtractions performed (4 in the example), and the number left over (8 in the example). These two numbers are called the **quotient** and the **remainder**, respectively.

Whole number divisions with remainder 0 are called **exact**. For example, $48 \div 6$ has quotient 8 and remainder 0, so the division is exact and we can write

$$48 \div 6 = 8,$$

with the understanding that the remainder is 0. Exact divisions can be restated in terms of multiplication. Subtracting 6 (8 times) from 48 yields exactly 0. On the other hand, starting at 0 and adding 6 (8 times) returns 48. Recalling that multiplication is a shorthand for this kind of repeated addition, we see that the two statements

$$48 \div 6 = 8 \qquad \text{and} \qquad 48 = 6 \cdot 8$$

say exactly the same thing. In general,

$$a \div b = c \qquad \text{and} \qquad a = b \cdot c$$

are equivalent statements.

Example 23. Express the statement $72 = 8 \cdot 9$ as an exact division in two ways.

Solution. We can get to 0 by starting at 72 and repeatedly subtracting 8 (9 times), or by repeatedly subtracting 8 (9 times). So, using the division symbol, we can write

$$72 \div 8 = 9$$

or we can write

$$72 \div 9 = 8.$$

□

Example 24. The division $63 \div 7$ is exact. Express this fact using an appropriate multiplication.

Solution. Since the quotient is 9 and the remainder is 0, we can write

$$63 \div 7 = 9 \qquad \text{or} \qquad 63 = 9 \cdot 7.$$

□

1.5.2 Long Division

Divisions with multi-digit divisors and/or dividends can get complicated, so we remind you of a standard way (long division) of organizing the computations. Here is what it looks like:

$$\begin{array}{r|l}
 & \text{quotient} \\ \hline
\text{divisor} & \text{dividend} \\
 & {*}{*}{*} \\ \hline
 & \quad\quad {*}{*}{*} \\
 & \quad\quad {*}{*}{*} \\ \hline
 & \quad\quad\quad {*}{*}{*} \\
 & \quad\quad\quad {*}{*}{*} \\ \hline
 & \quad\quad\quad \text{remainder}
\end{array}$$

The horizontal lines indicate subtractions of intermediate numbers; there is one subtraction for each digit in the quotient. For example, the fact that the division $100 \div 23$ has quotient 4 and remainder 8 is expressed in the long division form as follows:

$$\begin{array}{r|r}
 & 4 \\ \hline
23 & 100 \\
 & -\ 92 \\ \hline
 & 8
\end{array}$$

The 4 repeated subtractions of 23 are summarized as the single subtraction of $4 \cdot 23 = 92$.

In long division, we try to estimate the number of repeated subtractions that will be needed, multiply this estimate by the divisor, and hope for a number that is close to, but not greater than, the dividend. It will be easy to see when our estimate is too large, and to adjust it downward. If it is too small, the result of the subtraction will be too big – there was actually "room" for further subtraction. Going back to our example, $100 \div 23$, it is more or less clear that we will need more than 2 subtractions, since $23 \times 2 = 46$, which leaves a big remainder of 54 (bigger than the divisor, 23). $23 \times 3 = 69$, which also leaves a remainder that is too big. Since $23 \times 5 = 115$, which is bigger than the dividend, 100, we know that the best estimate for the quotient is 4. Now $23 \times 4 = 92$. We subtract 92 from 100, leaving the remainder 8, which is less than the divisor, as it should be. To check our calculations, we verify that $23 \times 4 + 8 = 100$. (In this check, the multiplication is done *before* the addition – this is the standard *order of operations*, which we will say more about later.)

Let's try another simple division problem.

Example 25. Find the quotient and remainder of the division $162 \div 42$.

Solution. Putting the dividend and divisor into the long division form, we have

$$
\begin{array}{r|r}
\cline{2-2}
42 & 162
\end{array}
$$

Let's estimate the number of subtractions that we'll need to perform. 40 goes into 160 four times, so, perhaps four is a good guess. But $4 \times 42 = 168$, which is too big (bigger than the dividend). So we lower our estimate by 1. We get $3 \times 42 = 126$, which is less than the dividend, so this must be the right choice.

$$
\begin{array}{r|r}
\multicolumn{2}{r}{3} \\
\cline{2-2}
42 & 162 \\
 & -126
\end{array}
$$

Subtracting, we obtain the remainder

$$
\begin{array}{r|r}
\multicolumn{2}{r}{3} \\
\cline{2-2}
42 & 162 \\
 & -126 \\
\cline{2-2}
 & 36
\end{array}
$$

which is less than the divisor, as it should be. Thus, the quotient is 3 and the remainder is 36. As a check, we verify that $3 \times 42 + 36 = 162$. (Multiplication before addition.) □

When the dividend is large, estimating the quotient is not so easy. The next example shows how to break the problem down by considering related, but smaller dividends.

Example 26. Find the quotient and remainder of the division $3060 \div 15$.

Solution. If we read the dividend from the left, one digit at a time, we get, successively, the numbers 3, 30, 306, and 3060. Of course, according to the place-value system, these numbers stand for 3 *thousands*, 30 *hundreds*, etc., but we do not need to think this way. To find the left-most digit of the quotient, observe that the divisor, 15, does not "go into" 3, but it does go into 30 (2 times, exactly). To indicate this we put, for the first digit of the quotient, the digit 2, directly over the right-most digit (0) in 30:

```
         2
     ---------
15  |  3 0 6 0
```

Now we compute the product $15 \times 2 = 30$ and subtract it from the initially selected part of the dividend, i.e., from 30:

```
         2
     ---------
15  |  3 0 6 0
     - 3 0
     ---------
         0
```

In this case, we get a remainder of 0. But we are not done yet. We treat this intermediate remainder as if it were a new dividend. Does 15 "go into" 0? No – we need a larger dividend. To get one, we "bring down" the digit 6:

```
         2
     ---------
15  |   3060
     -  30
     ---------
         06
```

Now $06 = 6$ (why?), and 15 does not go into 6. More precisely, it goes into it 0 times. We indicate this by putting a 0 directly above the digit 6 in the dividend:

$$\begin{array}{r|l} & \;\;20 \\ \hline 15 & 3060 \\ & -30 \\ \hline & \;\;06 \end{array}$$

Then we bring down the next (and the last) digit (0) of the dividend:

$$\begin{array}{r|l} & \;\;20 \\ \hline 15 & 3060 \\ & -30 \\ \hline & \;\;060 \end{array}$$

This gives us a new dividend of $060 = 60$, and 15 goes into 60 four times. So we put down 4 above the final digit in the original dividend.

$$\begin{array}{r|l} & \;\;204 \\ \hline 15 & 3060 \\ & -30 \\ \hline & \;\;060 \end{array}$$

Multiplying 15 by 4, we subtract this product from 60. This final difference is the remainder.

$$
\begin{array}{r|l}
 & \ \ 204 \\ \hline
15 & \ 3060 \\
 & -\ 30 \\ \hline
 & \ \ 060 \\
 & -\ \ \ 60 \\ \hline
 & \ \ \ \ 0
\end{array}
$$

We have reached the end of our dividend – there are no more digits to bring down. Hence, our division stops. The quotient is 204, and the remainder is 0. We check by verifying that $15 \times 204 + 0 = 3060$. □

1.5.3 Exercises

Express each exact division as an equivalent multiplication.

1. $88 \div 22 = 4$
2. $42 \div 6 = 7$
3. $51 \div 17 = 3$
4. $168 \div 14 = 12$
5. $96 \div 12 = 8$

Find the quotient and remainder of each division.

6. $37 \div 11$
7. $712 \div 101$
8. $3007 \div 110$
9. $3333 \div 111$
10. $3456 \div 241$
11. $457 \div 41$
12. $578 \div 19$
13. $317 \div 21$
14. $907 \div 201$
15. $712 \div 21$

1.6 Division and 0

Can 0 be the divisor in a division problem? If you think so, then answer this: how many times does 0 go into 7? Thinking back to our definition of division as repeated subtraction, the question is "how many times can 0 be subtracted from 7 before we arrive at a negative number?" The answer is "as many times as you like!," since subtracting 0 has no effect on 7, or any other number. For this reason, we say that division by 0 is *undefined*.

On the other hand, it is easy to answer the question "how many times does 7 go into 0?" The answer is 0 times! You can't take away any 7's (or any other positive numbers) from 0 without obtaining a negative number. So $0 \div 7 = 0$.

1.7 Order of operations

We often do calculations that involve more than one operation. For example

$$1 + 2 \times 3$$

involves both addition and multiplication. Which do we do first? If we do the multiplication first, the result is $1 + 6 = 7$, and if we do the addition first, the result is $3 \times 3 = 9$. Obviously, if we want the expression

$$1 + 2 \times 3$$

to have a definite and unambiguous meaning, we need a convention or agreement about the order of operations. It could have been otherwise, but the convention in this case is:

multiplication before addition.

With this convention, when I write $1 + 2 \times 3$, you know that I mean 7 (not 9). The precedence of multiplication can be made explicit using the **grouping symbols** () (parentheses):

$$1 + (2 \times 3) = 1 + 6 = 7.$$

If one of us insists that the addition be done first, we can do that by re-setting the parentheses:

$$(1 + 2) \times 3 = 3 \times 3 = 9.$$

Thus grouping symbols can be used to force any desired order of operations. Common grouping symbols, besides parentheses, are brackets, [ ], and braces, $\{\}$. The square root symbol $\sqrt{\ }$ is also a grouping symbol. For example

$$\sqrt{4 + 5} = \sqrt{9} = 3.$$

The $\sqrt{\ }$ symbol acts like a pair of parentheses, telling us to evaluate what is inside (in this case, the sum $4 + 5$) first, *before* taking the square root.

The **order of operations** is:

1. operations within grouping symbols first;
2. exponents and roots next;
3. multiplications and divisions (in order of appearance) next;

4. additions and subtractions (in order of appearance) last.

"In order of appearance" means in order from left to right. Thus in the expression

$$2 + 5 - 3,$$

the addition comes first, so it is evaluated first,

$$2 + 5 - 3 = 7 - 3 = 4,$$

while in the expression

$$8 - 6 + 11,$$

the subtraction is done first because it comes first,

$$8 - 6 + 11 = 2 + 11 = 13.$$

Example 27. Evaluate $6 \cdot 5 - 4 \div 2 + 2$.

Solution. All four operations appear here. There are no grouping symbols, exponents or roots. Following the order of operations, as well as the order of appearance, we do the computations in the following order: multiplication, division, subtraction, addition. The computations are as follows:

$$\begin{gathered} 6 \cdot 5 - 4 \div 2 + 2 = \\ 30 - 4 \div 2 + 2 = \\ 30 - 2 + 2 = \\ 28 + 2 = \\ 30 \end{gathered}$$

□

In the next two examples, we use the same numbers and the same operations, but we insert grouping symbols to change the order of operations. As expected, this changes the final result.

Example 28. Evaluate $6 \cdot (5 - 4) \div 2 + 2$.

Solution. The grouping symbols force us to do the subtraction first. After that, the usual order of operations is followed.

$$\begin{gathered} 6 \cdot (5 - 4) \div 2 + 2 = \\ 6 \cdot 1 \div 2 + 2 = \\ 6 \div 2 + 2 = \\ 3 + 2 = \\ 5 \end{gathered}$$

□

Example 29. Evaluate $6 \cdot 5 - 4 \div [2 + 2]$.

Solution. Now the grouping symbols (brackets) force us to do the addition first.

$$6 \cdot 5 - 4 \div [2 + 2] =$$
$$6 \cdot 5 - 4 \div 4 =$$
$$30 - 4 \div 4 =$$
$$30 - 1 =$$
$$29$$

□

In the next example, there are grouping symbols within grouping symbols. The strategy here is to work *from the inside outward*.

Example 30. Evaluate $6^2 - [3 + (3 - 1)]^2$.

Solution. We have parentheses within brackets. The parentheses enclose up the innermost group, which is where we start. (That is what "from the inside outward" means.) Thus the first operation to be done is $3 - 1 = 2$, yielding

$$6^2 - [3 + 2]^2.$$

Next, the bracketed group, $[3 + 2]$ is evaluated, yielding

$$6^2 - 5^2.$$

At this point, there are no more grouping symbols, and the order of operations tells us to evaluate the expressions with exponents next, yielding

$$36 - 25.$$

All that is left is the remaining subtraction,

$$36 - 25 = 11.$$

□

Example 31. Evaluate $\sqrt{(11 - 5)^2 + (24 \div 2 - 4)^2}$.

Solution. We evaluate the two inner groups $(11 - 5)$ and $(24 \div 2 - 4)$ first. In the second group, division precedes subtraction, yielding $12 - 4 = 8$. Thus the expression is reduced to

$$\sqrt{6^2 + 8^2}.$$

Next we evaluate the two expressions with exponents, obtaining

$$\sqrt{36 + 64}.$$

Then, remembering that $\sqrt{}$ is a grouping symbol, we evaluate the group $36 + 64 = 100$, and finally, evaluate the square root

$$\sqrt{100} = 10.$$

□

1.7.1 Exercises

Evaluate the expressions using the correct order of operations.

1. $6 + 16 \div 4$
2. $16 \cdot 4 - 48$
3. $15 - 9 - 4$
4. $2 \cdot 6 + 2(\sqrt{36} - 1)$
5. $4 \times 3 \times 2 \div 8 - 3$
6. $(2 \cdot 5)^2$
7. $\sqrt{21 - 30 \div 6}$
8. $[18 \div (9 \div 3)]^2$
9. $2 + 2 \times 8 - (4 + 4 \times 3)$

1.8 Average

The average of 2 numbers is their sum, divided by 2. The average of 3 numbers is their sum, divided by 3. In general, **the average of n numbers is their sum, divided by n.**

Example 32. Find the average of each of the following multi-sets of numbers. (A **multi-set** is a set in which the same number can appear more than once.) (a) $\{10, 12\}$; (b) $\{5, 6, 13\}$; (c) $\{8, 12, 9, 7, 14\}$.

Solution. In each case, we take the sum of all the numbers in the multi-set, and then (following the order of operations), divide by number of numbers:

(a) $(10 + 12) \div 2 = 11$; (b) $(5 + 6 + 13) \div 3 = 8$; (c) $(8 + 12 + 9 + 7 + 14) \div 5 = 10$.

□

The average of a multi-set of numbers is a description of the whole multi-set, in terms of only one number. If all the numbers in the multi-set were the same, the average would be that number. For example, the average of the multi-set $\{5, 5, 5\}$ is

$$(5 + 5 + 5) \div 3 = 5,$$

and the average of the multi-set $\{2, 2, 2, 2, 2, 2\}$ is

$$(2 + 2 + 2 + 2 + 2 + 2) \div 6 = 2.$$

This gives us another way to define the average of a multi-set of n numbers: it is that number which, added to itself n times, gives the same total sum as the sum of all the original numbers in the multi-set.

1.8.1 Exercises

Find the average of each of the following multi-sets of numbers.

1. $\{1, 2, 3, 4, 5, 6, 7\}$

2. $\{13, 13, 19, 15\}$

3. $\{206, 196, 204\}$

4. $\{85, 81, 92\}$

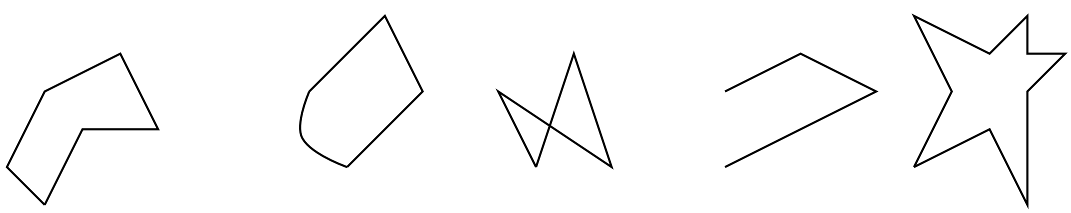
5. A baseball team had 7 games cancelled due to rain in the 2010 season. The number of cancelled games in the 2002-2009 seasons were 5, 6, 2, 10, 9, 4, 6, 5. What was the average number of cancelled games for the 2002-2010 seasons?

6. Suppose the average of the multi-set $\{20, 22, N, 28\}$ is 25, where N stands for an unknown number. Find the value of N.

1.9 Perimeter, Area and the Pythagorean Theorem

Squares and rectangles are examples of **polygons** – closed shapes that can be drawn on a flat surface, using segments of straight lines which do not cross each other. "Closed" means that the line segments form a boundary, with no gaps, which encloses a unique "inside" region, and separates it from the "outside" region.

Example 33. Which of the following figures are polygons?



(a) (b) (c) (d) (e)

Solution. (b) is not a polygon since not all of its sides are straight lines. (c) is not a polygon because it does not have a unique "inside" region. (d) is not a polygon because it is not closed. □

There are two useful numerical quantities associated with a polygon: the **perimeter**, which is the length of its boundary, and the **area**, which is (roughly speaking) the "amount of space" it encloses. Perimeters are measured using standard units of length such as feet (ft), inches (in) , meters (m), centimeters (cm). Areas are measured using **square units**, such as square feet (ft^2), square inches (in^2), square meters (m^2), square centimeters (cm^2).

To find the perimeter of a polygon, we simply find the sum of the lengths of its sides.

Example 34. Find the perimeter of each polygon. Assume the lengths are measured in feet.

Solution. Adding the lengths of the sides, we find that the perimeter of polygon (a) is

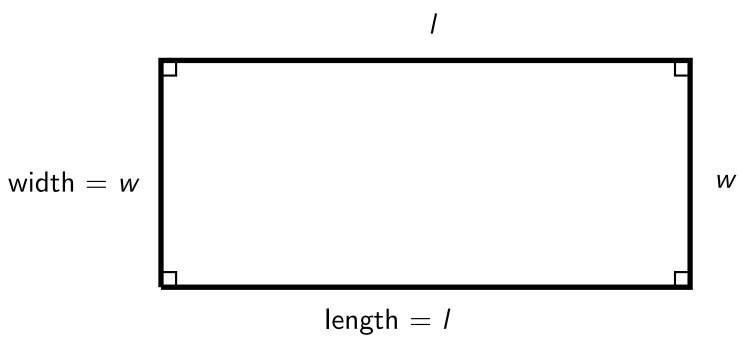
$$3 + 4 + 1 + 5 + 5 = 18 \text{ ft.}$$

Similarly, the perimeter of the polygon (b) is

$$7 + 5 + 6 + 1 + 1 + 1 = 21 \text{ ft.}$$

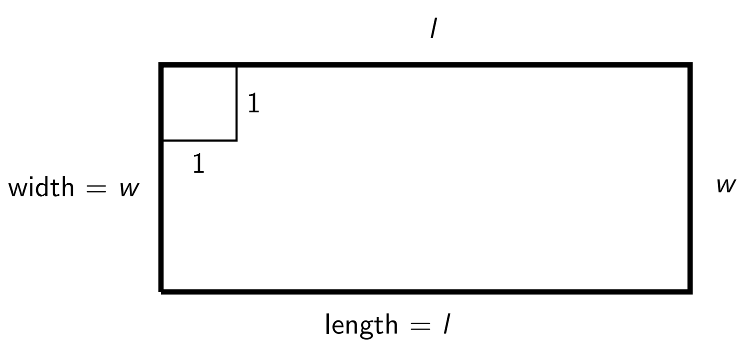
□

Finding the area of a polygon can be complicated, but it is quite simple if the polygon can be divided up into rectangles. A **rectangle** is a four-sided polygon with two pairs of opposite parallel sides, and "square" corners. The square corners – also known as **right angles** – imply that the paired opposite sides have the same length. The length of the shorter pair of sides is often called the *width* and denoted w, and the length of the longer pair is called the *length* and denoted l.

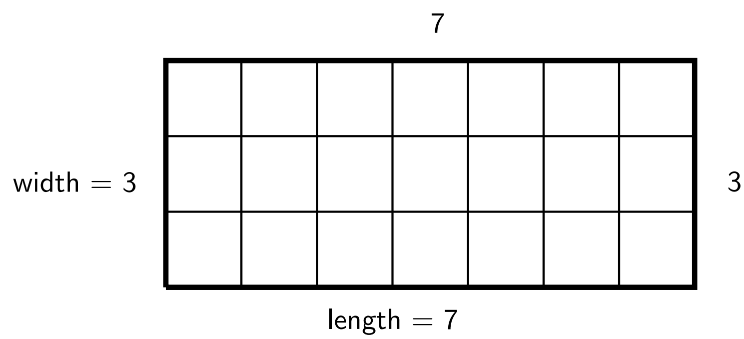


The small squares in the corners are there to indicate that the polygon is a rectangle.

Example 35. Estimate the area of the rectangle below, if the square in the upper left corner is 1 unit on a side.



Solution. If the corner square is 1 unit on a side, the area of the rectangle is the number of squares of that size that fit inside the rectangle. As the next figure shows, 21 such square units fit inside, in 3 rows of 7. The width (w) is evidently 3 units, and the length (l) 7 units. The area is the product $3 \times 7 = 21$ square units.



□

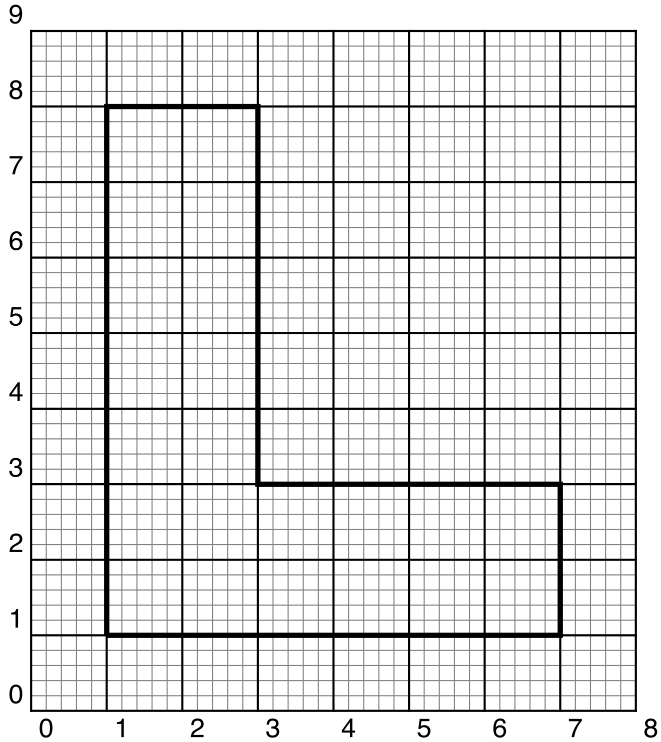
The example illustrates a general fact: **the area of a rectangle is the product of the length and the width.** If A denotes the area, the formula is

$$A = l \cdot w.$$

It is also clear that **the perimeter of a rectangle is the sum of twice the length and twice the width,** since to travel all the way around the boundary of the rectangle, both the length and the width must be traversed twice. If P denotes the perimeter, the formula is

$$P = 2l + 2w.$$

Example 36. In the figure below we have drawn an L-shaped polygon on a grid. Find the area and perimeter of the polygon. Take the unit of area to be one of the large squares of the grid, and assume these squares are 1 cm on a side.



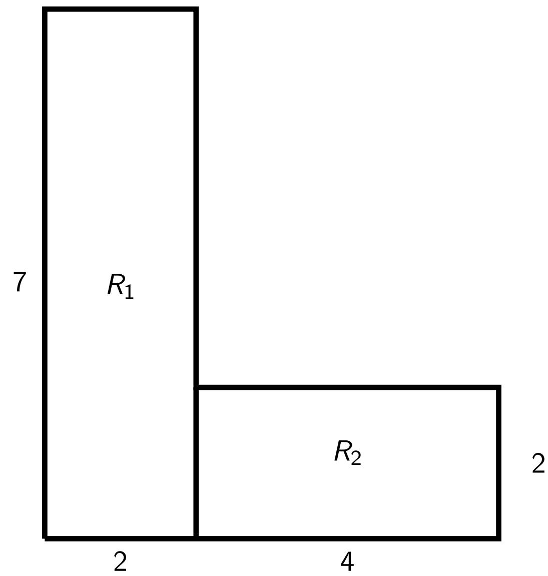
Solution. Using the grid, it is easy to find the lengths of all six sides of the polygon. For example, the left-hand vertical side is 7 cm long (stretching from 1 to 8 on the scale at left), and the top side is 2 cm long. Continuing clockwise around the *L*-shape, we see that the side lengths are

$$7, 2, 5, 4, 2, 6 \text{ cm},$$

respectively. From this, it follows easily that the perimeter P is given by the sum

$$7 + 2 + 5 + 4 + 2 + 6 = 26 \text{ cm}.$$

To find the area, we draw a line which divides the L-shape into two rectangles (labeled R_1 and R_2) in the figure below. (We could have drawn a different line to obtain a different division into two rectangles – do you see it?)



It is evident that R_1 has length 7 cm and width 2 cm. So the area of R_1 is

$$A_1 = 2 \cdot 7 = 14 \text{ cm}^2.$$

Similarly, since the length of R_2 is 4 cm and the width is 2 cm, the area of R_2 is

$$A_2 = 2 \cdot 4 = 8 \text{ cm}^2.$$

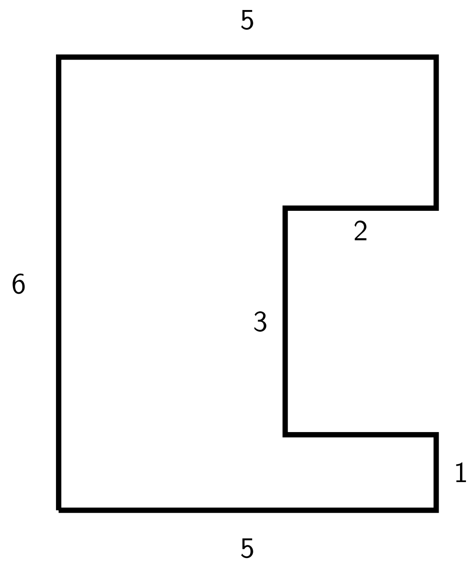
It is visually obvious that the total area A of the L-shaped polygon is the *sum* $A_1 + A_2$ of the areas of R_1 and R_2. Thus

$$A = 14 \text{ cm}^2 + 8 \text{ cm}^2 = 22 \text{ cm}^2.$$

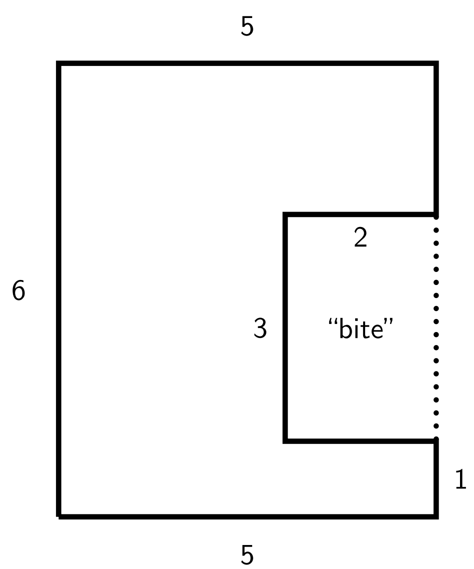
Using the grid, you can easily verify this result by counting the number of large squares inside the polygon. □

The grid is not necessary, as long as we are given enough side-lengths, and assume square corners.

Example 37. Find the area and perimeter of the right-angled (square-cornered) polygon below. Assume the side lengths are measured in feet.



Solution. There are some missing side lengths (for example, the vertical side on the upper right), but the area can be determined without them, using subtraction. We can visualize the polygon as a large rectangle of length 6 ft. and width 5 ft, out of which a rectangular "bite," of length 3 ft and width 2 ft, has been taken.



The area of the large rectangle (bite included) is $6 \cdot 5 = 30$ ft^2, and the area of the bite alone is $3 \cdot 2 = 6$ ft^2. Thus the area of the original polygon is

$$30 - 6 = 24 \text{ ft}^2.$$

To find the perimeter, we need the missing side lengths. For the vertical side on the upper right, we reason as follows. The left edge is 6 ft, and so the total length of all the right vertical edges must also be 6 ft. The two known vertical edges on the right have lengths 1 and 3, and the unkown edge must make up the difference. So its length must be

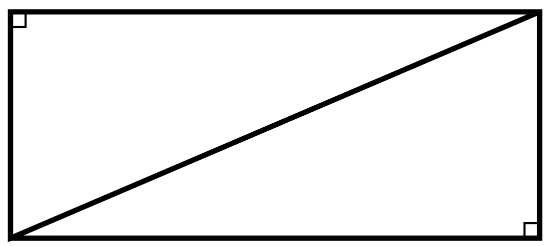
$$6 - (1 + 3) = 2 \text{ ft}.$$

The other missing side (the short horizontal one) is 2 ft. Adding up all the side lengths, starting (arbitrarily) with the bottom edge and proceeding counter-clockwise, yields the perimeter:

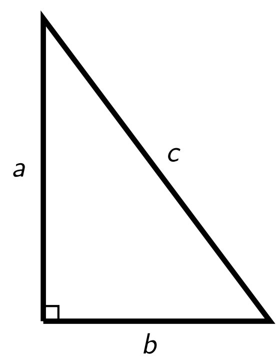
$$5 + 1 + 2 + 3 + 2 + 2 + 5 + 6 = 26 \text{ ft}.$$

□

If we cut a rectangle in two by drawing a **diagonal** from one corner to the opposite corner, we get two **right triangles**, each having exactly the same size and shape, and therefore the same area.



Right triangles are interesting in their own right, and we often consider them in isolation, without reference to the rectangle they came from. The side of a right triangle that is opposite the right angle (the longest side) is known as the **hypotenuse**. The two shorter sides are called the **legs**. In the figure below, the legs are labeled a and b, and the hypotenuse is labeled c.



A famous formula, called the **Pythagorean theorem**, states that, in any right triangle with legs of length a and b, and hypotenuse of length c, the following relation holds:

$$a^2 + b^2 = c^2.$$

With this, if we know the lengths of any two of the three sides, we can find the length of the third side. For example, we can easily obtain a formula for the length of the hypotenuse in terms of the lengths of the legs.

$$c = \sqrt{a^2 + b^2}.$$

Since the area of a right triangle is exactly half the area of the rectangle it came from, it follows that **the area of a right triangle is the product of the lengths of the legs, divided by** 2**.** If A denotes the area, and the lengths of the legs are a and b as in the figure, the formula is

$$A = (a \cdot b) \div 2.$$

Example 38. Find the area and perimeter of a triangle whose legs have length 3 feet and 4 feet.

Solution. Putting $a = 4$ and $b = 3$, we find the area using the formula

$$A = (a \cdot b) \div 2 = (4 \cdot 3) \div 2 = 12 \div 2 = 6 \text{ ft}^2.$$

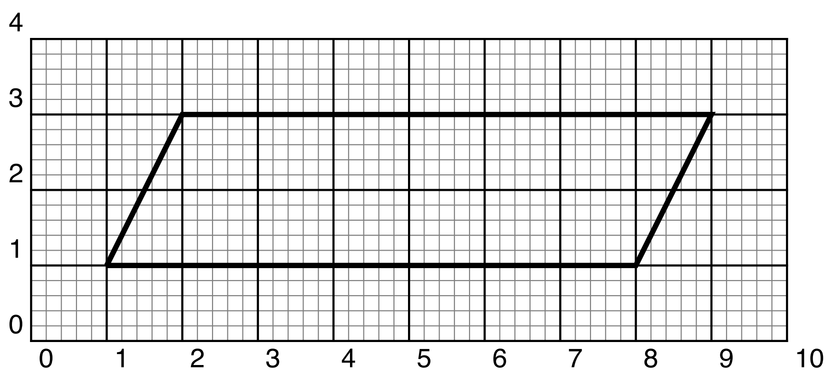
To find the perimeter, we first find the length of the hypotenuse using the formula

$$c = \sqrt{a^2 + b^2} = \sqrt{4^2 + 3^2} = \sqrt{16 + 9} = \sqrt{25} = 5 \text{ ft}.$$

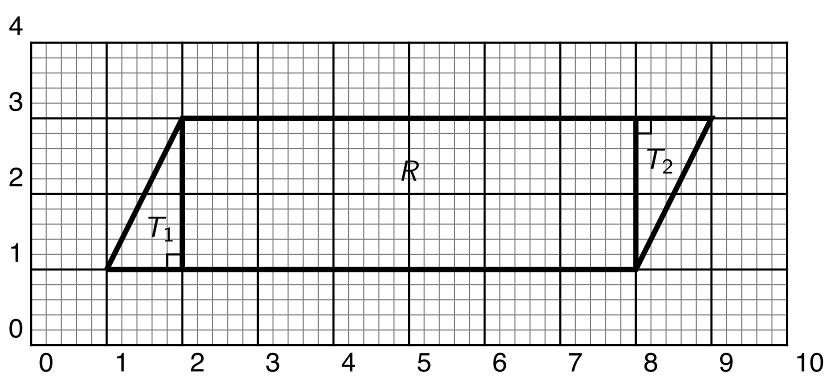
Thus the perimeter is $a + b + c = 4$ ft $+3$ ft $+5$ ft $= 12$ ft. □

Now we can find the area and perimeter of any polygon that can be divided up into rectangles and right triangles.

Example 39. Find the area and perimeter of the polygon below. The large squares of the grid measure 1 centimeter (cm) on a side.



Solution. Drawing two vertical lines, we can divide up the polygon into two right triangles, T_1 and T_2, and a rectangle, R.



The rectangle has length 6 cm and width 2 cm, so it has area $6 \cdot 2 = 12$ cm^2. Both triangles have legs of length 1 and 2, so the area of each is $(2 \cdot 1) \div 2 = 1$ cm^2. Adding the three areas gives the total area of the polygon: $1 + 12 + 1 = 14$ cm^2. To find the perimeter, we use the Pythagorean theorem to find the lengths of the two slanted sides:

$$c = \sqrt{a^2 + b^2} = \sqrt{1^2 + 2^2} = \sqrt{5}.$$

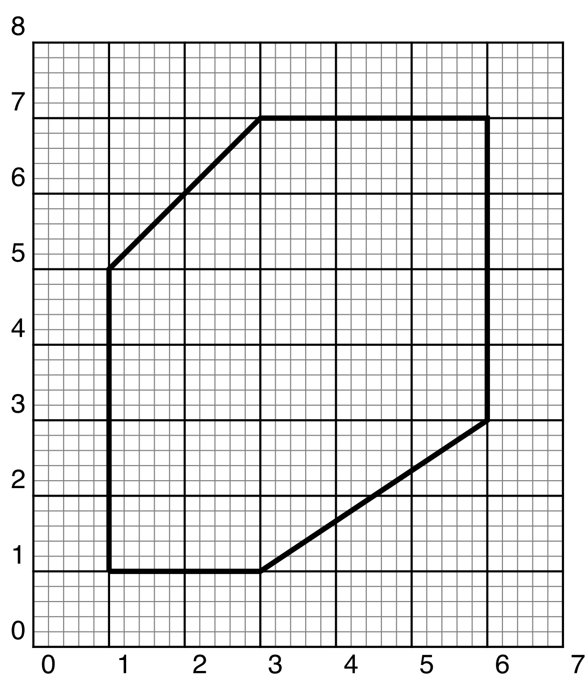
Thus the perimeter is

$$\sqrt{5} + 7 + \sqrt{5} + 7 = 14 + 2\sqrt{5} \text{ cm}.$$

Since $\sqrt{5}$ is not an integer, we do not simplify the expression further. But we can make the following estimate: Since $\sqrt{5}$ is between 2 and 3, $2\sqrt{5}$ is between 4 and 6, and it follows that the perimeter is between 18 and 20 centimeters. □

1.9.1 Exercises

1. Find the area of a right triangle whose legs are 5 in and 12 in.
2. Find the perimeter of the right triangle in the previous example.
3. Find the area of a rectangle whose length is 8 ft and whose width is 7 feet.
4. Find the perimeter of the rectangle in the previous example.
5. Find the length of the diagonal of a square which is 2 cm on a side. Between what two whole numbers does the answer lie?
6. Find the area and perimeter of the polygon below.

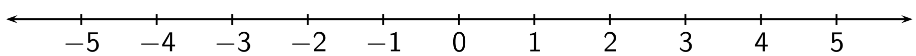


Chapter 2

Signed Numbers

We now start the process of extending the arithmetic of whole numbers to larger sets of numbers. In this chapter, we include the **negative** numbers. We mostly concentrate on the **integers**, which consist of whole numbers, 0, and the negative whole numbers. Later on, we'll put fractions and other non-integers into the mix.

The negative numbers form a kind of "mirror image" of the positive numbers, lying to the left of 0 on the number line. To distinguish them from the positive numbers, we give them a negative **sign**, $-$. Thus, -3 denotes the number "negative 3," which lies 3 units to the left of 0.



The positive numbers have the sign $+$, but we usually don't write it, except for emphasis. Thus $+5$ means "positive 5," but we usually just write 5.

2.1 Absolute value

The **absolute value** of a number is its distance from 0 on the number line. For example, -4 lies at a distance of 4 units from 0, so its absolute value is 4 (positive). Intuitively, distance is a nonnegative quantity: the absolute value of a number is never negative, even if the number itself is negative. The symbol for absolute value is a pair of vertical lines, $|\ |$. Thus we write

$$|-4| = 4.$$

Of course, the absolute value of 0 is 0

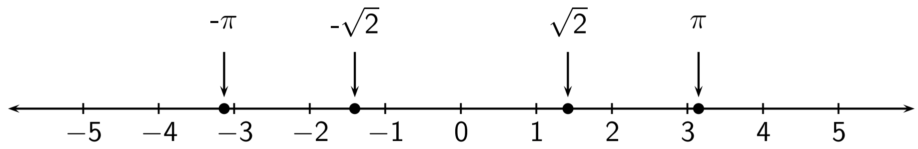
$$|0| = 0,$$

since there is no distance at all between 0 and itself! Every other number, including numbers like $\sqrt{2}$ and π which are not integers, has a partner (its "opposite") which lies at the same distance from, but on the other side of 0, and therefore has the same absolute value. For example, both 5 (positive) and -5 have absolute value equal to 5

$$|-5| = 5 \qquad \text{and} \qquad |5| = 5.$$

Similarly

$$|-\sqrt{2}| = \sqrt{2} \qquad \text{and} \qquad |\sqrt{2}| = \sqrt{2}$$
$$|-\pi| = \pi \qquad \text{and} \qquad |\pi| = \pi.$$



2.1.1 Exercises

Find the absolute value of the following numbers:

1. 12
2. -5
3. $-\pi$
4. $\sqrt{5}$
5. $-\sqrt{5}$
6. 0
7. -260

Find two numbers that have the given absolute value:

8. 99
9. 5
10. π
11. $\sqrt{7}$

2.2 Inequalities and signed numbers

The **inequality symbols** $<$ (less than) and $>$ (greater than) are used to state size comparisons between numbers. It is intuitively clear that

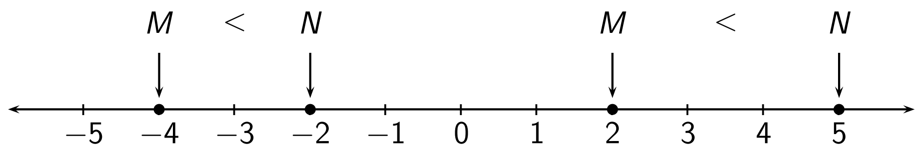
$$2 < 5, \qquad \text{and} \qquad 15 > 10$$

because these are comparisons between *nonnegative* numbers. Can we make sense of the inequality symbols when negative numbers are involved? It seems reasonable to say that any negative number is less than 0, and therefore less than any positive number. How do we compare two negative numbers? For example, is -4 greater or less than -2?

We laid out the number line so that numbers increase as we move to the right, and decrease as we move to the left. For consistency, we extend that rule to the negative side. The rule of thumb is "left is less" on the number line.

"Left is Less" on the number line:

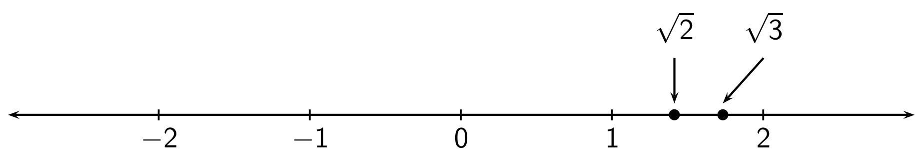
$M < N$ if and only if M is *left of* N



With this rule, we see that −4 is less than −2

$$-4 < -2.$$

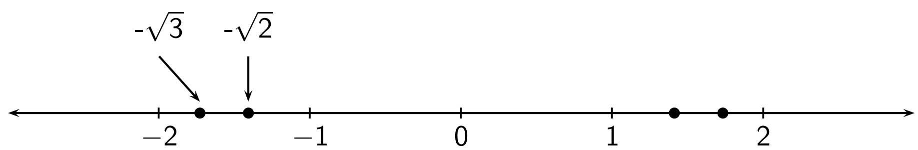
Recall that $\sqrt{2}$ is less than $\sqrt{3}$, and that both are greater than 1 but less than 2. (If you've forgotten why, review the discussion in Section 1.4.3.)



In symbols,

$$1 < \sqrt{2} < \sqrt{3} < 2.$$

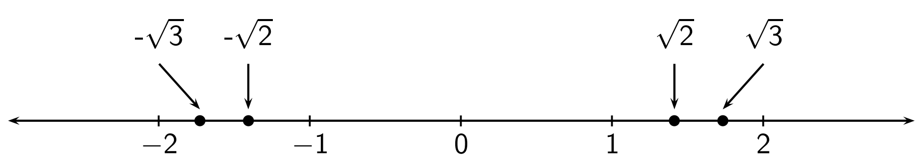
Do these relationships remain true if all the signs are changed? On the number line, 1 is left of 2, but, on the negative side, −2 is left of −1. Since "left is less," it follows that $-2 < -1$. For the same reason, $-\sqrt{3} < -\sqrt{2}$.



In symbols,

$$-2 < -\sqrt{3} < -\sqrt{2} < -1.$$

The mirror analogy helps here: when viewed in a mirror, your right hand 'becomes' a left hand – the thumb switches from one side to the other. Putting everything on one number line,



we see a chain of inequalities, mirror-symmetric (0 acts as the mirror), except for the presence of negative signs on the left:

$$-2 < -\sqrt{3} < -\sqrt{2} < -1 \quad \text{and} \quad 1 < \sqrt{2} < \sqrt{3} < 2.$$

If you prefer a rule of thumb to a visual aid, just remember this:

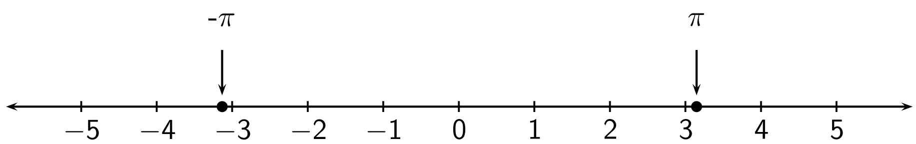
> The relations "less than" and "greater than" are reversed when signs are changed.

Example 40. Is $-\pi$ greater than or less than -3?

Solution. On the 'positive side,' $3 < \pi$. This relation is reversed when the signs are changed.

$$-\pi < -3.$$

Visually, $-\pi$ lies (slightly) to the left of -3 on the number line.



$$-\pi < -3 \quad \text{but} \quad 3 < \pi.$$

□

2.2.1 Exercises

Insert the appropriate inequality symbol ($<$ or $>$).

1. -7 _______ 0

2. 0 _______ -7

3. -5 _______ -7

4. 5 _______ -7

5. $\sqrt{3}$ _______ 2

6. $-\sqrt{3}$ _______ -2

7. $\sqrt{3}$ _______ $\sqrt{2}$

8. $-\sqrt{3}$ _______ $-\sqrt{2}$

9. $-\pi$ _______ -4

10. π _______ 4

Before getting down to the nuts and bolts of arithmetic with signed numbers, we address a question that may have occurred to you: Why bother with negative numbers? For one thing, they are extremely practical. In everyday life, there are many scales of measurement that establish a 0-level, and then go "below" it: temperature can go below 0° on a cold day; an elevator can go from the ground floor into the basement; there are places on land which lie below sea level. We often need to compute sums and differences on these scales. For another thing, banking and finance make constant use of signed numbers: we make deposits and *withdrawals,* which are like negative deposits; in business, we record profits and *losses* (negative profits). Finally, there is a theoretical or 'aesthetic' reason: the operation of subtraction is not yet well-defined. We know what $7-5$ means, but we have avoided $5-7$, and similar subtractions of a larger number from a smaller number. With negative numbers, we will rid ourselves of that restriction.

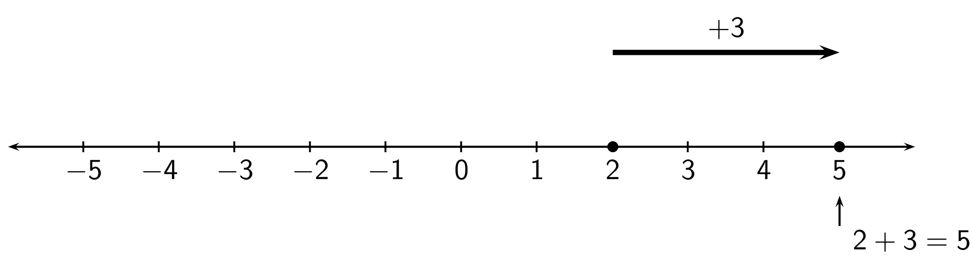
We now extend the ordinary operations of addition, subtraction, multiplication and division to all signed numbers, so that they remain consistent with the familiar operations with nonnegative numbers.

2.3 Adding signed numbers

We define the extended addition operation in terms of motion along the number line, as follows:

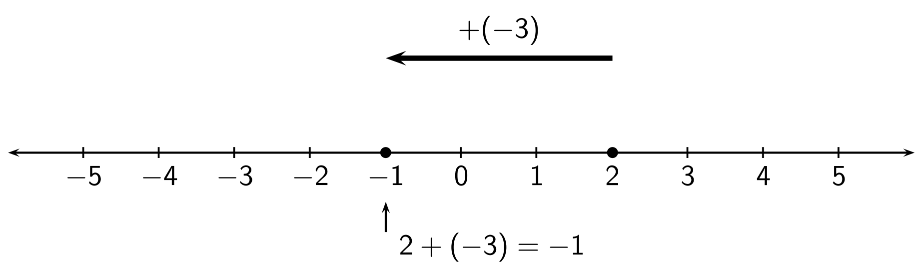
To add a *positive* number,	move to the *right*;
To add a *negative* number,	move to the *left*.

Thus, to add 3 to 2, we imagine starting at 2 on the number line and moving 3 "steps" to the right, arriving at 5.



Example 41. Perform the signed number addition $2+(-3)$.

Solution. We start at 2 as before, but now we move 3 steps *left*, taking the sign of -3 into account.

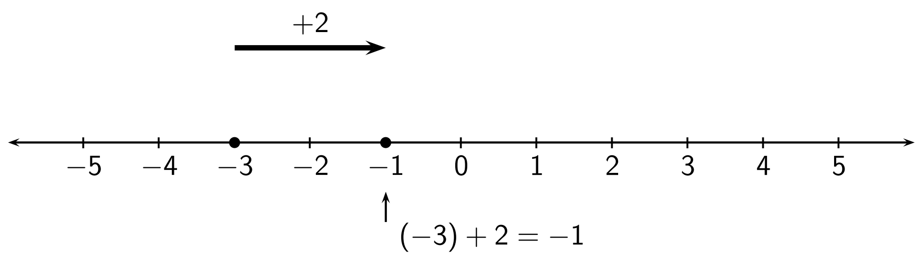


As the picture shows, we end up at -1. We conclude that $2 + (-3) = -1$. □

When two signed numbers are added, the order of addition does not affect the sum.

Example 42. Perform the signed number addition $(-3) + 2$.

Solution. This is just the previous example, with the addition in the reverse order. Now we *start* at -3, and move 2 steps *right*, taking into account the (positive) sign of 2. We arrive at the same result, -1:



□

More generally, for *any* two signed numbers,

$$a + b = b + a.$$

Addition, extended to signed numbers, remains **commutative**.

When numbers with opposite signs are added, a kind of tug-of-war takes place. The negative number "pulls" left, while the positive one "pulls" right. In ordinary tug-of-war, it is usually the heavier side that wins. What do we mean by "heavier" in the context of numbers? A natural substitute is "larger absolute value," that is, larger distance from 0. Notice that in the examples above, the *sign of the sum* is the same as the sign of the number with the larger absolute value. (In the war of signs, the "heavier" side wins.) The *difference* obtained by subtracting the smaller absolute value from the larger gives the *absolute value of the sum*.

If numbers with the same sign are added, it is as if both pullers in the tug-of-war are pulling on the same side of the rope. Of course, there is no contest: if both pullers are on the left side, the rope moves left; if both pullers are on the right, similarly, the rope moves right. In the context of numbers: two numbers with the *same* sign sum to a number which has that same (common) sign. The new absolute value (of the sum) is the sum of the individual absolute values. Here are the general rules.

When two signed numbers are added

- if the numbers have **opposite** signs,
    1. the *sign of the sum* is the sign of the number with the larger absolute value;
    2. the *absolute value of the sum* is the difference between the two individual absolute values (larger − smaller).
- if the two numbers have the **same** sign,
    1. the *sign of the sum* is the common sign of the summands;
    2. the *absolute value of the sum* is the sum of the individual absolute values.

Example 43. Add $15 + (-18)$.

Solution. The numbers have opposite signs, so the sign of the sum is the same as the sign of the number with the larger absolute value (-18), i.e., $-$. The absolute value of the sum is the difference $|-18| - |15| = 18 - 15 = 3$. Thus the sum is -3. This reasoning is summarized as

$$15 + (-18) = -(18 - 15) = -3.$$

□

Example 44. Add $-7 + (-9)$.

Solution. The numbers have the same sign, $-$, so that is also the sign of the sum. The absolute value of the sum is the sum of the individual absolute values, $|-7| + |-9| = 7 + 9 = 16$. Thus the sum is -16. The reasoning is summarized as

$$-7 + (-9) = -(7 + 9) = -16.$$

□

Example 45. Add $-36 + 49$.

Solution. The numbers have opposite signs, and the number with the larger absolute value (49) determines the sign of the sum (+). The absolute value of the sum is the difference $|49| - |-36| = 49 - 36$. Thus

$$-36 + 49 = +(49 - 36) = +13 = 13.$$

□

2.3.1 Exercises

Add.

1. $-6 + 19$
2. $12 + (-4)$
3. $-34 + (-28)$
4. $266 + (-265)$
5. $-133 + (-93)$
6. $-1001 + 909$

Use an appropriate signed number addition for the following.

11. Find the temperature at noon in Anchorage if the temperature at dawn was -11° F and, by noon, the temperature had risen by 36° F.
12. Find the height (in feet above ground level) of an elevator which started 30 feet below ground level and subsequently rose 70 feet.

2.3.2 Opposites, Identity

There are exactly two numbers at any given non-zero distance from 0, one negative and the other positive. Pairs of numbers such as $\{-5, 5\}$, $\{-\sqrt{2}, \sqrt{2}\}$, $\{-\pi, \pi\}$, which are unequal but *equidistant* from 0, and hence have the same absolute value, are called **opposites**. (0 is the only number which is its own opposite, having absolute value 0.) To find the opposite of a nonzero number, we simply change its sign.

Example 46. (a) The opposite of 11 is -11. (b) The opposite of -17 is 17.

Since there are only two possible signs, the *opposite of the opposite* of a number is the number we started with.

The opposite of the opposite of N is N:

$$-(-N) = N.$$

Example 47. The opposite of the opposite of -5 is $-(-(-5)) = -5$.

Suppose a nonzero number is added to its opposite, for example, $3+(-3)$. Our rule for adding signed numbers with opposite signs doesn't quite work since it is not clear what the sign of the sum should be. But in fact it doesn't matter: the *absolute value* of the sum is 0, which has no sign. Accordingly, $-3 + 3 = 0$. The tug-of-war picture helps here: if two people of exactly the same weight and strength pull in opposite directions, the rope doesn't move at all!

The sum of a number and its opposite is 0:

$$N + (-N) = 0.$$

Example 48. (a) $25 + (-25) = 0$. (b) $-153 + 153 = 0$.

Recall that 0 is the **additive identity** for ordinary addition because, when 0 is added to a number, the result is the identical number, i.e, the number does not change. This remains true for *signed* numbers.

Example 49. $-4 + 0 = -4$.

2.3.3 Exercises

Find the following:

1. The opposite of 65
2. The opposite of -257
3. The sum of 99 and its opposite.
4. The sum of $-\pi$ and its opposite.
5. The opposite of the opposite of -31.
6. The sum of -5 and the opposite of 5.
7. $-258 + (-(-258))$
8. $-91 + (-91)$
9. $-38 + 0$
10. $0 + 55$
11. $4 + 223 + (-223)$

2.3.4 Associativity

Another important property of addition, **associativity**, extends to signed number addition. Associativity of addition means that when three or more numbers are added, it doesn't matter how you associate them into groups for addition: $x + y + z = (x + y) + z = x + (y + z)$. Recall that this property allowed us to add long columns of nonnegative numbers.

We can make use of column addition with signed numbers, too, by associating and adding all the positive numbers, and, separately, associating and adding the *absolute values* of all the negative numbers. Then we add the two subtotals, treating the subtotal associated with the negative numbers as negative. This way, we apply the signed number rule just once, at the end.

Example 50. Add: $43 + (-5) + (-135) + 69 + (-134) + 158 + (-162)$

Solution. We add all the positive numbers,

$$\begin{array}{r} 4\ 3 \\ 6\ 9 \\ +\ 1\ 5\ 8 \\ \hline 2\ 7\ 0 \end{array}$$

and the absolute values of all the negative numbers,

$$\begin{array}{r} 5 \\ 1\ 3\ 5 \\ 1\ 3\ 4 \\ +\ 1\ 6\ 2 \\ \hline 4\ 3\ 6 \end{array}$$

Treating the subtotal associated with the negative numbers as negative, we add $270 + (-436)$. By the rule for adding signed numbers with opposite signs,

$$270 + (-436) = -(436 - 270) = -166.$$

□

If we think of positive numbers as "profits," and negative numbers as "losses," then signed number addition is like "balancing the books." There is a net gain if the total profits are larger than the total losses (in absolute value); otherwise there is a net loss.

2.3.5 Exercises

Perform the additions.

1. $1 + (-1) + 1 + (-1) + 1 + (-1)$
2. $-82 + (-1985) + 82$
3. $44 + (-55) + 288 + 36 + (-191) + (-8)$
4. $123 + (-78) + (-143) + 78 + 15 + (-6) + 50$
5. Find the average noon temperature for the first week of January in Barrow, Alaska if the noon temperatures were: Monday: $-8°$ F, Tuesday: $2°$ F, Wednesday: $-3°$ F, Thursday: $-5°$ F, Friday: $1°$ F, Saturday: $4°$ F, Sunday: $9°$ F.

2.4 Subtracting signed numbers

The phrase "subtract A from B" indicates the operation $B - A$. We extend subtraction to signed numbers by defining it in terms of addition.

> To subtract a signed number A from a signed number B, add B to the *opposite* of A:
>
> $$B - A = B + (-A).$$

With this definition, there is no longer any real need for a separate operation called subtraction. For example, instead of subtracting $17 - 9$, we could simply add $17 + (-9)$ (verify that the results are the same). Despite this, it is convenient to retain the operation of subtraction.

Example 51. (a) Subtract 15 from 9. (b) Subtract 8 from -2.

Solution. (a) $9 - 15 = 9 + (-15)$. Applying the rule for adding signed numbers with opposite signs,

$$9 + (-15) = -(15 - 9) = -6.$$

(b) $-2 - 8 = -2 + (-8)$. Applying the rule for adding signed numbers with the same sign,

$$-2 + (-8) = -(2 + 8) = -10.$$

□

Example 52. Subtract -17 from 12.

Solution. $12 - (-17) = 12 + (-(-17))$. Remembering that $-(-17) = 17$ (opposite of the opposite), we have

$$12 - (-17) = 12 + 17 = 29.$$

□

Example 53. Subtract (-15) from -32.

Solution. $-32 - (-15) = -32 + (-(-15)) = -32 + 15$. Applying the rule for adding signed numbers with opposite signs,

$$-32 - (-15) = -32 + 15 = -(32 - 15) = -17.$$

□

Example 54. Perform the subtraction $3359 - 10080$.

Solution. $3359 - 10080 = 3359 + (-10080) = -(10080 - 3359)$. We perform the subtraction of absolute values vertically.

$$\begin{array}{rrrrrr} & 1 & 0 & 0 & 8 & 0 \\ - & & 3 & 3 & 5 & 9 \\ \hline & & 6 & 7 & 2 & 1 \end{array}$$

Remembering that the sign was negative, $3359 - 10080 = -6721$. □

Formerly "impossible" subtractions, such as

$$7 - 12$$

can now be easily performed.

$$7 - 12 = 7 + (-12) = -(12 - 7) = -5.$$

Examples like this show that subtraction is **not commutative**:

Changing the order of subtraction changes the result to its opposite.

In symbols, for any two signed numbers A and B,

$$B - A = -(A - B).$$

Example 55. $22 - 100 = -78 = -(100 - 22)$.

We often need to find the *difference* of two unequal quantities. By convention, difference is given as a positive quantity. If the two quantities are A and B, their difference is either $A - B$ or $B - A$ (whichever is positive). Intuitively,

$$\text{difference} = \text{larger} - \text{smaller}.$$

Formally, the difference of A and B is defined to be the *absolute value* of $A - B$.

Example 56. Find the difference in temperature between -2° F and 50°F.

Solution. Intuitively,

$$\text{higher temperature} - \text{lower temperature} = 50 - (-2) = 52,$$

so the temperature difference is 52° F. Formally,

$$|-2 - 50| = |-52| = 52.$$

□

Example 57. The summit of Mount Whitney in California is 14505 feet above sea level. Not far away, in Death Valley, the lowest point is 282 feet below sea level. What is the altitude difference between Mount Whitney's summit and the lowest point in Death Valley?

Solution. Assigning a negative altitude to a point below sea level, the difference is

$$14505 - (-282) = 14505 + 282 = 14787 \text{ feet}.$$

□

2.4.1 Exercises

Perform the subtractions.

1. Subtract 31 from 7
2. $98 - 100$
3. $65 - (-640)$
4. Subtract 55 from 24
5. Subtract -53 from 68.6
6. $-888 - (-111)$
7. Subtract 22 from $\frac{1}{5}$
8. $-87 - 23$
9. $-87 - (-23)$
10. A snowball at -5° C is heated until it melts and then boils. If water boils at $100^\circ C$, by how much did the temperature of the snowball rise?
11. A sunken car is salvaged from the bottom of a lake. The elevation of the lake bottom is -66 feet. The car is lifted by a crane to a height 79 feet above lake level. Through what vertical distance was the car lifted?

2.5 Multiplying Signed Numbers

Multiplication of positive numbers was defined in terms of repeated addition. For example,

$$3 \times 4 = 4 + 4 + 4 = 12.$$

There is no problem extending this definition to the product of a negative number by a positive number.

$$3 \times (-4) = (-4) + (-4) + (-4) = -12.$$

Products such as

$$(-2) \times 5$$

pose no problem if, for consistency, we define signed number multiplication to be *commutative* (we do):

$$(-2) \times 5 = 5 \times (-2) = (-2) + (-2) + (-2) + (-2) + (-2) = -10.$$

The general rule is:

> *The product of two numbers with opposite signs is the negative of the product of their absolute values.*

Example 58. Find the products (a) $7 \times (-11)$ and (b) $(-12) \times 5$.

Solution. We take the negative of the product of the absolute values in each case. (a) $7 \times (-11) = -(7 \times 11) = -77$. (b) $(-12) \times 5 = -(12 \times 5) = -60$. □

When it comes to the product of two negative numbers, our intuition fails. It makes no sense to "repeatedly" add a number to itself when the number of repeats is *negative*! How should we define $(-2) \times (-3)$?

It is best to give up on intuition and let consistency rule. Look at the pattern below:

$$\begin{aligned}
4 \times (-3) &= -12 & &= \text{the opposite of } 4 \times 3\\
3 \times (-3) &= -9 & &= \text{the opposite of } 3 \times 3\\
2 \times (-3) &= -6 & &= \text{the opposite of } 2 \times 3\\
1 \times (-3) &= -3 & &= \text{the opposite of } 1 \times 3\\
0 \times (-3) &= 0 & &= \text{the opposite of } 0 \times 3\\
(-1) \times (-3) &= & &= ?\\
(-2) \times (-3) &= & &= ??
\end{aligned}$$

It looks like the pattern "ought" to continue as follows:

$$\begin{aligned}
(-1) \times (-3) &= 3 & &\text{the opposite of } (-1) \times 3\\
(-2) \times (-3) &= 6 & &\text{the opposite of } (-2) \times 3
\end{aligned}$$

We make the following general definition:

The product of two negative numbers is the (positive!) product of their absolute values.

Example 59. Find the product $(-8) \times (-12)$.

Solution. $(-8) \times (-12) = 8 \times 12 = 96$. □

Note that our definition is consistent with what we already know about positive numbers: the product of two *positive* numbers is also the product of their absolute values. The definition now applies whenever numbers *with the same sign* are multiplied.

[OPTIONAL: If you find it hard to accept that the product of two negative numbers is positive, the following discussion might help. An important axiom (fact) of arithmetic states that multiplication "distributes" over addition. That is, for any three numbers, A, B and C,

$$A(B + C) = AB + AC.$$

If signed number arithmetic is to be consistent with the arithmetic of nonnegative numbers, this axiom must continue to hold. In particular, if A and B are positive,

$$(-A)(B + (-B)) = (-A)(B) + (-A)(-B).$$

Since the left hand side equals 0 (why?) so does the right hand side. It follows that $(-A)(-B)$ must be the opposite of $(-A)(B) = -(-AB) = AB$. In other words, it must be true that $(-A)(-B) = AB$.]

Here is a summary of the rules for multiplying signed numbers.

> When two signed numbers are **multiplied**
>
> - if the numbers have the **same** sign, the product is positive.
> - if the numbers have **opposite** signs, the product is negative.
>
> In both cases, the absolute value of the product is the product of the individual absolute values.

In the examples and exercises below, we freely use all three ways of symbolizing multiplication: $\cdot$, $\times$, and juxtaposition. It is easiest to determine the sign of the product first, and then compute with the absolute values.

Example 60. Multiply $(-16) \times 5$.

Solution. $(-16) \times 5 = -80$ (negative since the numbers have opposite signs). □

Example 61. Find the product $(-11) \cdot (-12)$.

Solution. $(-11) \cdot (-12) = 132$ (positive since the numbers have the same signs). □

Example 62. Find the product $(-630)(-205)$.

Solution. The product is positive since the numbers have the same sign. Multiplying the absolute values, we obtain

$$\begin{array}{r} 630 \\ \times \quad 205 \\ \hline 3150 \\ 1260 \\ \hline 129150 \end{array}$$

□

Recall the **zero property**, which states that

$$0 \cdot x = 0 \quad \text{and} \quad x \cdot 0 = 0, \qquad \text{for any number } x.$$

This remains true for signed numbers.

Example 63. $(-17) \cdot 0 = 0$.

Also, 1 continues to be the **multiplicative identity** , that is,

$$1 \cdot x = x \quad \text{and} \quad x \cdot 1 = x, \qquad \text{for any number } x.$$

Example 64. $(-55) \cdot 1 = -55$.

Multiplication, like addition, continues to be **associative** when extended to signed numbers. Thus we can multiply several signed numbers without worrying how we group them for multiplication. One consequence is that the sign of a product of several signed numbers can be quickly determined in advance by simply determining whether the number of negative factors is *even* or *odd*.

The sign of a product of several factors is

- positive if the number of negative factors is *even*,
- negative if the number of negative factors is *odd*.

This is true because every *pair* of negative factors has a positive product. If there is an even number of negative factors, they can be "paired off" (in any convenient order), producing a product of positive numbers which is, of course, positive. But if the number of negative factors is odd, there is always one "unpaired" negative factor, which makes the total product negative.

Example 65.

$(-1)(2)(-3)(-4) = -24$ (an odd number (three) of negative factors makes a negative product),
$(-1)(-2)(-3)(-4) = 24$ (an even number (four) of negative factors makes a positive product).

2.5.1 Exercises

Find the products.

1. $8 \times (-6)$
2. -9×7
3. $(-6) \times (-9)$
4. $(-1) \times 5$
5. $(914)(-1)$
6. $(0)(-888)$
7. $(-1) \times (-1)$
8. $(-1) \times (-1) \times (-1)$
9. $65 \times (-31)$
10. $(-503) \times (-6)$
11. $(-162)(1000)$
12. $1(-1)$
13. $(-3)(-50)(-2)$
14. $(-3)(5)(-7)(1)$
15. $(3)(-10)(2)(-5)$
16. $(-6)(-5)(-4)(-3)(0)$

2.6 Dividing Signed Numbers

The rules for dividing signed numbers are exactly analogous to the rules for multiplying them.

> When two signed numbers are **divided** (and the divisor is nonzero)
>
> - if the dividend and divisor have the **same** sign, the quotient is positive
> - if the dividend and the divisor have **opposite** signs, the quotient is negative.
>
> In both cases, the absolute value of the quotient is the quotient of the individual absolute values.

Notice that we have made no mention of remainders here. To do so would require a new definition of the quotient; this is not worth the trouble. When a division of signed numbers is not exact (i.e, when the remainder is not zero) it is much better to treat the division as a *fraction*. We'll do that in the next chapter. For now, we consider only exact divisions. In this case, as with positive numbers, we can restate division in terms of multiplication, and that explains the rules above. For example, because

$$6 \times 4 = 24,$$

we say that $24 \div 6 = 4$ (or $24 \div 4 = 6$.) Similarly, we say that

$$24 \div -6 = -4,$$

because

$$24 = (-6) \times (-4).$$

Example 66. Express the statement $-72 = -8 \cdot 9$ as an exact division of signed numbers.

Solution. We can write

$$-72 \div (-8) = 9$$

or we can write

$$-72 \div 9 = -8.$$

□

Example 67. The division $63 \div (-7)$ is exact. Find the quotient.

Solution. Since $-63 = 9(-7)$,

$$63 \div (-7) = -9,$$

that is, the quotient is -9. □

We remind you that 0 cannot be the *divisor* in any division problem: repeated subtraction of 0 has no effect on any number (recall the discussion in Section 1.6). This remains true for signed numbers.

Example 68. $(-23) \div 0$ is undefined.

It remains true that 0 can be the *dividend*.

Example 69. $0 \div (-12) = 0$. Note that this can be restated in terms of multiplication as $0 \times (-12) = 0$.

We summarize the properties of 0 with respect to division.

For a nonzero signed number N

- $0 \div N = 0$
- $N \div 0$ is undefined.

$0 \div 0$ is also undefined.

2.6.1 Exercises

Perform the divisions, or state that they are undefined.

1. $(-24) \div (-8)$
2. $(-24) \div 8$
3. $66 \div 0$
4. $30 \div (-6)$
5. $(-30) \div 6$
6. $0 \div (-1000)$
7. $50 \div (-4)$
8. $-95 \div 0$

2.7 Powers of Signed Numbers

Since exponents indicate repeated multiplication, there is no problem applying exponents to signed numbers.

$$(-4)^3 = (-4)(-4)(-4) = -64 \qquad \text{and} \quad (-2)^4 = (-2)(-2)(-2)(-2) = 16.$$

Example 70. Evaluate $(-5)^3$, and $(-5)^4$.

Solution.

$$(-5)^3 = (-5)(-5)(-5) = (25)(-5) = -125.$$

$$(-5)^4 = (-5)(-5)(-5)(-5) = (25)(25) = 625.$$

□

Notice that every pair of negative factors has a positive product, by the rules for multiplying signed numbers with the same sign. It follows that

A power of a negative number is

- negative if the exponent is *odd*,
- positive if the exponent is *even*.

When applying exponents to signed numbers, it is essential to put parentheses around the number, including its sign. For example, the square of -3 is written

$$(-3)^2 = (-3)(-3) = 9.$$

Without parentheses, as in

$$-3^2,$$

the exponent applies only to 3, not to -3, and this is *not* a power of -3. Rather, -3^2 represents the *opposite* of 3^2, so

$$-3^2 = -9.$$

The behavior of 0 in an exponential expression, in particular, the interpretation of 0 as an exponent, extends unchanged to signed numbers. (You may wish to review Section 1.4 on powers of whole numbers in Chapter 1.)

For a nonzero signed number N,

- $N^0 = 1$
- $0^N = 0$

(0^0 is undefined.)

Example 71. $(-3)^0 = 1$. $0^4 = 0$. 0^0 is undefined.

2.7.1 Exercises

Find the value of each expression.

1. 8^2
2. $(-8)^2$
3. -8^2
4. -6^3
5. $(-6)^3$
6. 0^5
7. $-(-3^3)$

8. $-(-2)^4$

9. $(10)^0$

10. $(-23)^0$

11. $(-1)^{59}$

12. 0^0

2.8 Square Roots and Signed Numbers

Recall that a number whose square is N is called a *square root* of N. 6 is a square root of 36, because $6^2 = 36$. -6 is also a square root of 36, since the product of two numbers with the same sign is positive:

$$(-6)^2 = 36.$$

In general,

> Every positive number N has two square roots, which are opposites:
>
> - The positive square root is denoted $\sqrt{N}$.
> - The negative square root is $-\sqrt{N}$.

As usual, 0 is a special case. It has only one square root, itself:

$$\sqrt{0} = 0.$$

Example 72. Between what two consecutive integers does $-\sqrt{19}$ lie?

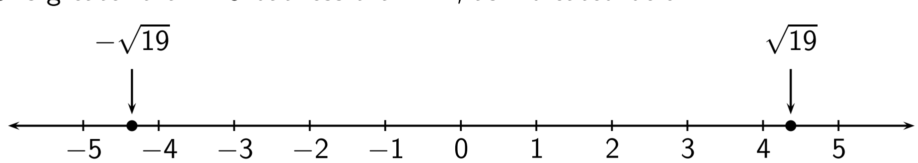
Solution. 19 is greater than 4^2 but less than 5^2. It follows that

$$4 < \sqrt{19} < 5.$$

The "mirror image" of this relationship on the negative half of the number line is

$$-5 < -\sqrt{19} < -4,$$

that is, $-\sqrt{19}$ is greater than -5 but less than -4, as indicated below.



□

It is important to note that $-\sqrt{N}$ is not the same as $\sqrt{-N}$. In fact, the latter expression presents a problem: do negative numbers have square roots? For example, what is $\sqrt{-4}$? This number, if it exists, must have absolute value 2, so it would have to be either 2 or -2. But

$$2^2 = (-2)^2 = 4 \qquad (\text{not } -4).$$

So neither 2 nor -2 is a square root of -4. A similar argument applies to any negative number.

Square roots of negative numbers *are undefined* within the system of signed numbers.

It is possible to expand the set of signed numbers so as to remedy this defect, but we leave that for a more advanced course.

Example 73. $\sqrt{-5}$ is undefined. But $-\sqrt{5}$ is a negative number between -3 and -2:

$$-3 < -\sqrt{5} < -2.$$

2.8.1 Exercises

Find the square roots, or state that they are undefined.

1. $\sqrt{9}$
2. $-\sqrt{25}$
3. $\sqrt{-25}$
4. $-\sqrt{49}$
5. $\sqrt{100}$
6. $\sqrt{0}$

Between what two consecutive integers do the following square roots lie?

11. $\sqrt{7}$
12. $-\sqrt{7}$
13. $-\sqrt{30}$

Insert the appropriate inequality ($<$ or $>$) between the following pairs of numbers.

14. $-\sqrt{10} \qquad -\sqrt{8}$
15. $\sqrt{12} \qquad \sqrt{15}$

Chapter 3

Fractions and Mixed Numbers

Fractions are expressions such as

$$\frac{3}{19} \quad \text{or more generally} \quad \frac{a}{b},$$

where a and b are whole numbers, and $b \neq 0$. For now, we assume that a and b are both positive (we will introduce negative fractions in due course.) The numbers a and b are called **terms**. The term on top is called the **numerator**, and the term on the bottom is called the **denominator**. The horizontal line in the middle is called the **fraction bar**. Sometimes to save space, we write fractions in one line, using a slash instead of the fraction bar, putting the numerator on the left and the denominator on the right:

$$\frac{a}{b} = a/b.$$

When reading a fraction out loud, we attach the suffix *th* or *ths* to the denominator, saying "*a bths*" for a/b.

Example 74. The fractions $\frac{1}{8}$ and $\frac{6}{13}$ are spoken "one eighth," and "six thirteenths," respectively.

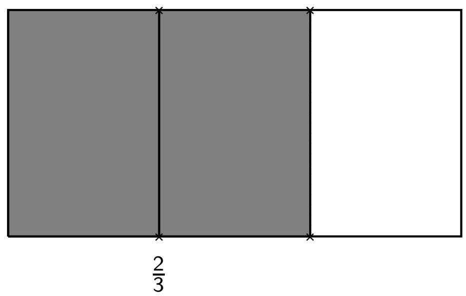
If the denominator is 2 or 3, we say "halves" or "thirds" (not twoths or threeths!), respectively. If the denominator is 4, we sometimes say "fourths," and sometimes, "quarters."

Example 75. 5/2 is spoken "five halves." 4/3 is spoken "four thirds." 3/4 is spoken "three fourths," or "three quarters."

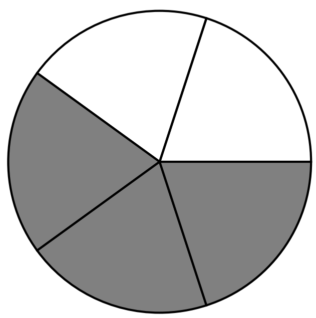
Sometimes, instead of using the *th* suffix, we just say "*a* over *b*." Thus 4/9 can be spoken as "four ninths," or just "four over nine."

3.1 What positive fractions mean

Positive fractions represent **parts of a whole** in a precise way. In the fraction a/b, the denominator b represents the *number of equal parts* into which the whole has been divided, and the numerator represents *how many* of those parts are being taken into account. For example, if a rectangle is divided up into 3 equal parts, and 2 of those parts are shaded, then the shaded portion represents $\frac{2}{3}$ (two-thirds) of the whole rectangle.

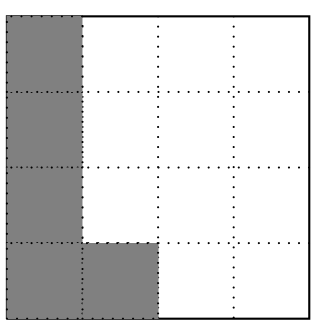


We can divide any convenient figure into equal parts (not just a rectangle), to represent a fraction. For example, the circle below has been divided into 5 equal “wedges,” and three of them are shaded. So the picture represents the fraction $\frac{3}{5}$.



Example 76. Use a square to represent the fraction 5/16.

Solution. Since 16 is a perfect square, it’s easy to make a square 4 units on a side, and divide that into 16 small squares of equal size. Then we shade 5 of them (any 5 will do) to represent 5/16.

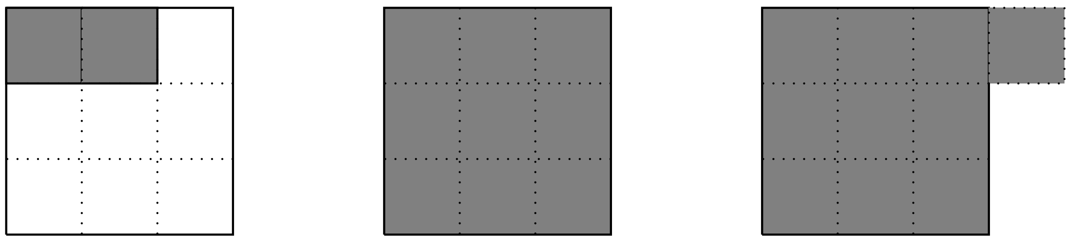


□

3.2 Proper and Improper Fractions

A fraction is called **proper** if its numerator has smaller absolute value than its denominator. A positive proper fraction represents **less than one whole**. An **improper** fraction has a numerator that has absolute value greater than or equal to its denominator. It follows that a positive improper fraction represents a number that is greater than (or equal to) one whole.

Example 77. The figure below represents three fractions with denominator 9: the first one (2/9) is proper, representing a number less than 1. The other two are improper. The one in the middle (9/9) represents the whole number 1. The last (10/9) represents a number that is greater than 1.



$\frac{2}{9}$ $\quad\quad$ $\frac{9}{9} = 1$ $\quad\quad$ $\frac{10}{9} = 1 + \frac{1}{9}$

$9/9 = 1$ illustrates the general fact that the whole number 1 can be represented in infinitely many ways as a fraction – just take any fraction whose numerator and denominator are equal. For example,

$$1 = \frac{2}{2} = \frac{9}{9} = \frac{159}{159}.$$

Intuitively, if an object is divided into equal parts, and *all the parts are taken*, then, in fact, the *whole* (1 whole) has been taken.

For any whole number a (except 0),

$$\frac{a}{a} = 1.$$

Any whole number can be represented as a fraction. For example, to represent the whole number 3, we can think of three whole rectangles, each "divided" into 1 "part," and write

$$3 = \frac{3}{1}.$$

More generally,

For any whole number a (including 0),

$$a = \frac{a}{1}.$$

3.2.1 Zero as Numerator and Denominator

The whole number 0 can be written as a fraction in infinitely many ways:

$$0 = \frac{0}{b}, \quad \text{for any non-zero } b.$$

It is easy to understand why this is true: you can divide something into any number of equal pieces (say, b of them), but if you take *none of them*, you have taken an amount equal to 0 – no matter the value of b. Thus,

$$0 = \frac{0}{1} = \frac{0}{2} = \frac{0}{3} = \ldots$$

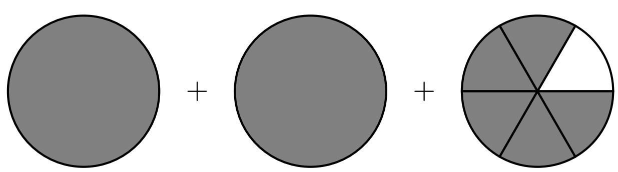
So 0 can certainly be the numerator of a fraction. Can it be the denominator? The answer is no. Here is one way to think about it: does it make sense to divide something into 0 pieces? (1 piece, yes, but 0 pieces?) This is closely related to the fact that 0 cannot be the divisor in a division problem (see Section 1.6.) There is another reason why 0 cannot be the denominator of a fraction. Multiplication and division are mutually inverse operations, meaning that the equation $\frac{a}{b} = c$ is equivalent to the equation $a = b \cdot c$ whenever $b \neq 0$. Suppose we could assign a numerical value to the fraction $1/0$, say, $\frac{1}{0} = 1$. That would mean that $1 = 0 \cdot 1$. But of course, $0 \cdot 1 = 0$, so we arrive at $1 = 0$, an obvious contradiction.

For these reasons, we say that a fraction with denominator 0 is **undefined.**

> For any whole number n, including 0, the fraction $\dfrac{n}{0}$ is undefined.

3.2.2 Exercises

1. What improper fraction does the following picture represent?



Use rectangles, circles, or squares to represent the following fractions.

2. $\frac{3}{2}$
3. $\frac{3}{4}$
4. $\frac{5}{8}$
5. $\frac{11}{6}$
6. $\frac{4}{3}$
7. $\frac{6}{2}$

8. Write five fractions which are equal to 1.

9. Write five fractions which are equal to 0.

10. Write 15 as a fraction.

3.3 Mixed Numbers

We have seen that a positive improper fraction represents either a whole number or a whole number plus a proper fraction. In the latter form, it is called a **mixed number**. Even though it is a sum, the + sign is omitted. Thus

$$3\frac{1}{2}$$

represents 3 "wholes" plus an extra $\frac{1}{2}$.

3.3.1 Converting an improper fraction into a mixed or whole number

Suppose we have an improper fraction

$$\frac{\text{numerator}}{\text{denominator}}.$$

To convert this to a mixed number, we simply perform the division

$$\text{numerator} \div \text{denominator},$$

obtaining a quotient and a remainder. Then we use these to build the mixed number as follows: the quotient is the whole number part; the remainder over the divisor is the fractional part. Note that the fractional part obtained in this way is *always proper*, because the remainder is always smaller than the divisor. Summarizing:

$$\frac{\text{numerator}}{\text{denominator}} = \text{quotient}\frac{\text{remainder}}{\text{divisor}}$$

Example 78. Write the improper fraction 13/3 as a mixed number.

Solution. Dividing 13 by 3 yields a quotient of 4 and a remainder of 1. The corresponding mixed number has whole number part 4 and fractional part 1/3. Thus

$$\frac{13}{3} = 4\frac{1}{3}.$$

□

If the remainder is 0, the mixed number is actually just a whole number.

Example 79. Write the improper fraction 84/7 as a mixed number.

Solution. Dividing 84 by 7 yields a quotient of 12 and a remainder of 0. Thus

$$\frac{84}{7} = 12\frac{0}{7} = 12,$$

a whole number. □

Here are some everyday uses of mixed numbers.

Example 80. Five hikers want to share seven chocolate bars fairly. How many bars does each hiker get?

Solution. For fairness, each of the five hikers should get the same amount: exactly one fifth of the chocolate. There are seven bars, so the amount (in chocolate bars) that each hiker should get is

$$\frac{7}{5} = 1\frac{2}{5}.$$

Each hiker gets $1\frac{2}{5}$ chocolate bars. □

Example 81. Julissa waited 5 minutes for the train on Monday, 2 minutes on Tuesday, and then 4, 8, and 3 minutes on Wednesday, Thursday, and Friday, respectively. What was her average wait time for the week?

Solution. Recall that the average of a set of numbers is their sum, divided by the number of numbers. In this case, we want the average of the five numbers $\{5, 2, 4, 8, 3\}$, which is

$$\frac{5+2+4+8+3}{5} = \frac{22}{5} = 4\frac{2}{5}.$$

Her average wait time was $4\frac{2}{5}$ minutes. (Extra credit: how long is two fifths of a minute, in seconds?) □

3.3.2 Exercises

Convert the following improper fractions into mixed numbers:

1. $\frac{19}{3}$
2. $\frac{11}{2}$
3. $\frac{135}{5}$
4. $\frac{99}{98}$
5. $\frac{77}{5}$

Use mixed numbers to answer the following questions:

6. Three boys share four sandwiches fairly. How many sandwiches does each boy get?
7. What is the average of the set of numbers $\{11, 14, 9, 12\}$?
8. During the first seven days of January, Bill travels $31, 37, 46, 31, 77, 50$ and 40 miles respectively. What is his average daily travel distance?

3.3.3 Converting a mixed or whole number to an improper fraction

Sometimes we need to change a mixed number back into an improper fraction. The key fact, again, is that the fraction

$$\frac{a}{a}$$

is *always equal to* 1, for any number a except 0. For example,

$$\frac{3}{3} = 1.$$

Example 82. Convert the mixed number $2\frac{1}{3}$ to an improper fraction, using the fact that $\frac{3}{3} = 1$.

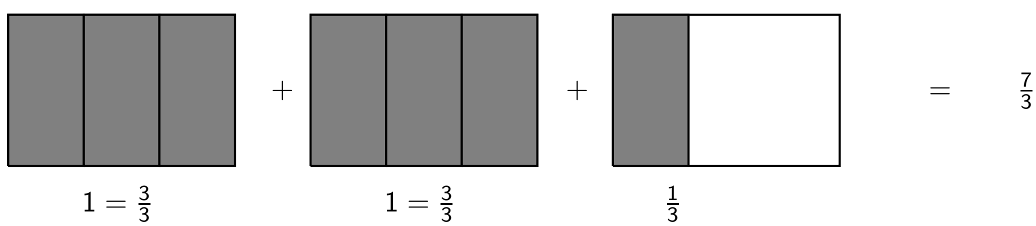
Solution. Think of the whole number 2 in the following way:

$$2 = 1 + 1 = \frac{3}{3} + \frac{3}{3}.$$

It follows that the mixed number $2\frac{1}{3}$ can be written

$$\frac{3}{3} + \frac{3}{3} + \frac{1}{3}.$$

The figure below should make it clear that this sum is equal to the improper fraction $\frac{7}{3}$.



(The figure also demonstrates that *fractions with the same denominator add up to a new fraction, with the same denominator, and a numerator which is the sum of all the old numerators.* We'll say more about adding fractions later.) ☐

The procedure in the previous example is easily turned into a general formula.

> The mixed number $N\frac{p}{q}$ is equal to the improper fraction $\dfrac{N \cdot q + p}{q}$.

Remember to follow the order of operations (multiplication before addition) when evaluating $N \cdot q + p$.

Example 83. Convert the mixed number $8\frac{2}{3}$ into an improper fraction, using the boxed rule.

Solution.

$$8\frac{2}{3} = \frac{8 \cdot 3 + 2}{3} = \frac{24 + 2}{3} = \frac{26}{3}.$$

□

Example 84. Write the whole number 8 as an improper fraction in three different ways.

Solution. We can think of the whole number 8 as the "mixed" number $8\frac{0}{q}$ for any nonzero q. By the boxed rule,

$$8 = \frac{8 \cdot q + 0}{q} = \frac{8 \cdot q}{q}.$$

Picking three nonzero numbers for q, say, $2, 5$ and 10, we get

$$8 = \frac{8 \cdot 2}{2} = \frac{16}{2}, \qquad 8 = \frac{8 \cdot 5}{5} = \frac{40}{5}, \qquad \text{and} \quad 8 = \frac{8 \cdot 10}{10} = \frac{80}{10}.$$

□

3.3.4 Exercises

Convert the mixed numbers into improper fractions:

1. $1\frac{1}{2}$
2. $8\frac{1}{3}$
3. $15\frac{3}{8}$
4. $5\frac{3}{4}$
5. $11\frac{5}{6}$

Using $N = \dfrac{N \cdot q}{q}$ for any nonzero q, write three fractions equal to each given whole number:

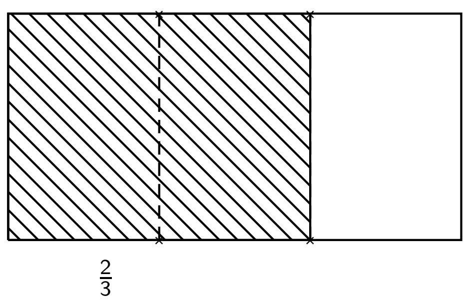
6. 5
7. 11
8. 10

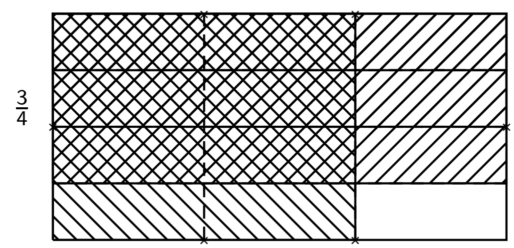
3.4 Multiplication of Fractions

Fractions are numbers in their own right, and we will recall how to add, subtract, multiply and divide them. We begin with multiplication, since it is one of the easiest operations to perform.

We understand what it means to take, let's say, a third of something: divide it into three pieces, and take one of the pieces. Similarly, to take three-fourths of something, divide it into four pieces, and take three of them. What if the "something" is itself a fraction? For example, what is three fourths of $\frac{2}{3}$? It is some fraction, but which one?

We'll try to make sense of this question by representing fractions as shaded portions of rectangles. Represent the fraction $\frac{2}{3}$ using a rectangle divided into 3 equal vertical parts, with 2 of them shaded. Divide the same rectangle into 4 equal horizontal parts, and shade 3 of them (using a different shading). The double-shaded portion represents the desired fraction, $\frac{3}{4}$ of $\frac{2}{3}$. By counting, you can see that it consists of 6 out of 12 smaller rectangles.

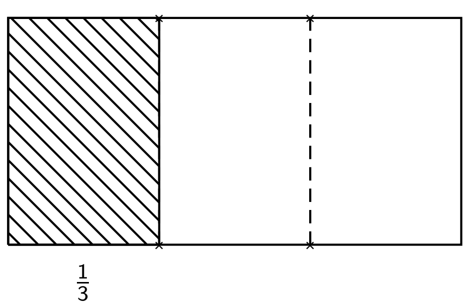


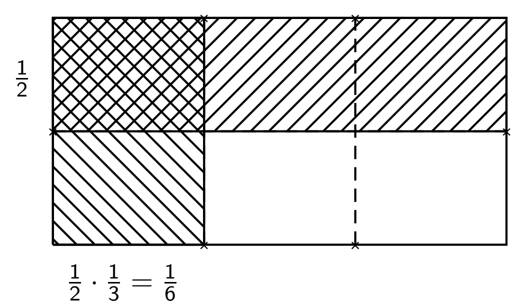


The pictures show that $\frac{3}{4}$ of $\frac{2}{3}$ is the fraction $\frac{6}{12}$.

Example 85. What is one half of one third?

Solution. Start with a rectangle divided into 3 equal vertical parts, and shade one of the parts, to represent the fraction 1/3. Then divide the whole rectangle horizontally into 2 equal parts, and shade one of the horizontal parts using a different shading. The rectangle is now cut into $2 \cdot 3 = 6$ equal parts, and the two shadings overlap in precisely 1 of the 6 parts.





Thus $\frac{1}{2}$ of $\frac{1}{3}$ is the fraction $\frac{1}{6}$. □

The two examples illustrate a simple rule for finding a "fraction of a fraction," or $\frac{a}{b}$ of $\frac{c}{d}$. The new denominator is the *product* of the denominators of the two original fractions, since dividing a rectangle into b vertical parts and also d horizontal parts results in $b \cdot d$ smaller rectangular parts. Each of the

a shaded vertical rectangles is divided into d parts of which c are differently shaded. Hence there are $a \cdot c$ *doubly shaded* smaller rectangles, so that $a \cdot c$ is the new numerator. The rule is

$$\frac{a}{b} \text{ of } \frac{c}{d} \quad = \quad \frac{a \cdot c}{b \cdot d}.$$

We use this rule to define multiplication of fractions, replacing "of" by the multiplication symbol $\cdot$ (or $\times$):

$$\boxed{\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d}}$$

In words: *The product of two fractions is the product of the numerators over the product of the denominators.*

Example 86. Find the product $\frac{3}{8} \cdot \frac{5}{7}$.

Solution. Using the boxed rule,

$$\frac{3}{8} \cdot \frac{5}{7} = \frac{3 \cdot 5}{8 \cdot 7} = \frac{15}{56}.$$

□

It is easy to see that fraction multiplication is just an expanded version of whole number multiplication, since every whole number is a fraction with denominator 1. Thus the boxed rule applies when one (or both!) of the multiplicands is a whole number.

Example 87. What is two thirds of five? That is, find the product $\frac{2}{3} \cdot 5$.

Solution. Writing 5 as the fraction 5/1 and using the fraction multiplication rule, we get

$$\frac{2}{3} \cdot \frac{5}{1} = \frac{2 \cdot 5}{3 \cdot 1} = \frac{10}{3}.$$

□

Example 88. One third of a twelve-member jury are women. How many women are on the jury?

Solution. One third of twelve equals $\frac{1}{3} \cdot \frac{12}{1} = \frac{12}{3} = 4$. There are four women on the jury. □

You may have noticed that in our very first example with rectangles,

$$\frac{3}{4} \cdot \frac{2}{3} = \frac{6}{12},$$

exactly half of the rectangle ends up doubly-shaded (6 is half of 12). So it is correct (and simpler) to say that

$$\frac{3}{4} \cdot \frac{2}{3} = \frac{1}{2}.$$

Does this contradict our rule? No, because 1/2 and 6/12 represent the *same* number. We'll explain this, and examine some other technical details regarding the *representation* of fractions, in the next sections. Then we'll continue with the *arithmetic* (division, addition, subtraction) of fractions.

3.4.1 Exercises

Find the following products. (Just use the fraction product rule – no need to draw rectangles.)

1. $\frac{1}{3} \cdot \frac{5}{7}$
2. One half of one half
3. Two thirds of one third
4. $\frac{3}{4} \cdot \frac{3}{4}$
5. $3 \cdot \frac{5}{8}$
6. $\left(\frac{2}{3}\right)^2$
7. $\frac{1}{2} \cdot \frac{7}{8} \cdot 3$

3.5 Equivalent Fractions

Fractions which look very different can represent the same number. For example, the fractions

$$\frac{2}{4}, \ \frac{5}{10}, \ \frac{6}{12}, \ \text{and} \ \frac{50}{100}$$

all represent the number $\frac{1}{2}$. What property do all these fractions share? Each has a denominator that is exactly *twice* its numerator; the simplest fraction with this property is $\frac{1}{2}$.

Fractions which represent the same number are called **equivalent**, and we use the equal sign to indicate this. Thus, for example,

$$\frac{1}{2} = \frac{50}{100}.$$

There is an easy way to tell when two fractions are equivalent. We give it here because it is so simple and pleasing, but we postpone the explanation until we discuss *proportions*.

$$\frac{a}{b} = \frac{c}{d} \quad \text{if (and only if)} \quad a \cdot d = b \cdot c.$$

In words: two fractions are equivalent if (and only if) their *cross-products* are equal. A cross-product is the product of the numerator of one fraction and the denominator of the other.

Starting with a given fraction, we can generate equivalent fractions easily, using the fact that 1 is the multiplicative identity, and that

$$1 = \frac{c}{c}$$

for any non-zero c. Then

$$\frac{a}{b} = \frac{a}{b} \cdot 1 = \frac{a}{b} \cdot \frac{c}{c},$$

and so we have

The Fundamental Property of Fractions:

If we multiply both the numerator and denominator of a fraction by any nonzero c, then the new fraction is equivalent to the original one, and represents the same number:

$$\frac{a}{b} = \frac{a \cdot c}{b \cdot c}.$$

Example 89. Write two fractions equivalent to $\frac{2}{3}$.

Solution. We use the fact that

$$\frac{2}{3} = \frac{2 \cdot c}{3 \cdot c}$$

for any nonzero c. Picking two values for c, say, 6 and 7, we get two fractions equivalent to 2/3:

$$\frac{2}{3} = \frac{2 \cdot 6}{3 \cdot 6} = \frac{12}{18} \qquad \text{and} \qquad \frac{2}{3} = \frac{2 \cdot 7}{3 \cdot 7} = \frac{14}{21}.$$

Of course, other choices of c would have produced other fractions equivalent to 2/3. ☐

3.5.1 Cancellation and Lowest Terms

The boxed rule above produces equivalent fractions with higher (larger) terms. It is sometimes possible to go the other way, producing lower (smaller) terms. If both numerator and denominator have a **common factor** – a whole number greater than 1 which divides them both with zero remainder – we can "cancel" it by division. The two quotients become the terms of an equivalent fraction with lower terms. For example, the numerator and denominator of $\frac{18}{24}$ have the common factor 6. Thus

$$\frac{18}{24} = \frac{18 \div 6}{24 \div 6} = \frac{3}{4}.$$

In general,

For any non-zero c,

$$\frac{a}{b} = \frac{a \div c}{b \div c}.$$

This method of obtaining lower terms is called **cancellation** or **cancelling out**. It is often indicated as follows:

$$\frac{18}{24} = \frac{\cancel{18}^{\,3}}{\cancel{24}^{\,4}} = \frac{3}{4}.$$

This is useful short-hand, but it has one disadvantage: the common factor (6, in this case) is not made visible. To ensure accuracy, you can show the common factor explicitly, before cancelling it. Thus, explicitly,

$$\frac{18}{24} = \frac{3 \cdot 6}{4 \cdot 6} = \frac{3 \cdot \cancel{6}}{4 \cdot \cancel{6}} = \frac{3}{4}.$$

Example 90. Find lower terms for the fraction $\frac{28}{70}$.

Solution. Since 7 is a common factor of the numerator and denominator, it can be cancelled, yielding

$$\frac{\cancel{28}^{\,4}}{\cancel{70}_{\,10}} = \frac{4}{10} \quad \text{(cancelling 7)}.$$

Still lower terms are possible, since 4 and 10 have the common factor 2:

$$\frac{\cancel{4}^{\,2}}{\cancel{10}_{\,5}} = \frac{2}{5} \quad \text{(cancelling 2)}.$$

□

If the only common factor of the numerator and denominator is 1, then no further reduction is possible, and the fraction is said to be in **lowest terms**. For example, 2 and 5 have no common factor other than 1, so the fraction $\frac{2}{5}$, and also the improper $\frac{5}{2}$, are in lowest terms.

Example 91. Reduce the fraction $\frac{24}{36}$ to lowest terms.

Solution. A common factor of the numerator and denominator is 4, so we can reduce the terms as follows:

$$\frac{24}{36} = \frac{\cancel{24}^{\,6}}{\cancel{36}_{\,9}} = \frac{6}{9} \qquad \text{(cancelling 4)};$$

6 and 9 have the common factor 3, so we can further reduce

$$\frac{6}{9} = \frac{\cancel{6}^{\,2}}{\cancel{9}_{\,3}} = \frac{2}{3} \qquad \text{(cancelling 3)}.$$

It is clear that $\frac{2}{3}$ is in lowest terms, since 2 and 3 have no common factor other than 1.

Note that this reduction to lowest terms could have been accomplished in one step, had we realized, at the outset, that both 24 and 36 are divisible by 12:

$$\frac{24}{36} = \frac{\cancel{24}^{\,2}}{\cancel{36}_{\,3}} = \frac{2}{3} \qquad \text{(cancelling 12)}.$$

□

Example 92. Reduce the fraction $\frac{13}{39}$ to lowest terms.

Solution. A common factor of the numerator and denominator is 13, so that

$$\frac{13}{39} = \frac{1 \cdot 13}{3 \cdot 13} = \frac{1}{3}.$$

□

Example 93. Reduce the fraction $\frac{15}{3}$ to lowest terms.

Solution. $\frac{15}{3} = \frac{5}{1} = 5.$ □

3.5.2 Exercises

Using the rule $\frac{a}{b} = \frac{a \cdot c}{b \cdot c}$ for any nonzero c, write four fractions equivalent to each given fraction:

1. $\dfrac{1}{4}$
2. $\dfrac{3}{4}$
3. $\dfrac{1}{5}$
4. $\dfrac{2}{5}$
5. $\dfrac{1}{8}$
6. $\dfrac{5}{8}$

Using cancellation, reduce each fraction to lowest terms. Convert improper fractions to mixed numbers.

7. $\dfrac{12}{8}$
8. $\dfrac{12}{18}$
9. $\dfrac{20}{45}$
10. $\dfrac{84}{60}$
11. $\dfrac{54}{108}$
12. $\dfrac{360}{120}$

3.6 Prime Factorization and the GCF

Reducing a fraction to lowest terms requires recognizing a common factor (greater than 1) of the numerator and denominator. Doing it *in one step* requires finding and using the *largest* or *greatest* common factor. In this section, we develop a systematic way of finding the **greatest common factor** (GCF) of a set of numbers.

A whole number greater than 1 is **prime** if it has no factors other than itself and 1. The first few prime numbers are

$$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, \ldots .$$

The dots indicate that there are infinitely many larger primes. (This fact is not obvious but was proved more than two thousand years ago by Euclid.) The whole numbers greater than 1 which are *not* prime are called **composite**. (Note that 1 is a special case according to these definitions: it is neither prime nor composite!) The first few composite numbers are

$$4, 6, 8, 9, 10, 12, 14, 15, 16, 18, 20, 21, 22, 24, 25, 26, 27, 28, 30, \ldots$$

How do we know that these numbers are composite? Because each has at least one factor, other than 1 and itself. For example, 15 has factors 3 and 5, that is, $15 = 3 \cdot 5$.

Every composite whole number has a unique **prime factorization**, which is its expression as the product of its prime factors, listed in order of increasing size. For example:

$$\begin{aligned}
4 &= 2 \cdot 2 &&= 2^2 \\
6 &= 2 \cdot 3 \\
8 &= 2 \cdot 2 \cdot 2 &&= 2^3 \\
9 &= 3 \cdot 3 &&= 3^2 \\
10 &= 2 \cdot 5 \\
12 &= 2 \cdot 2 \cdot 3 &&= 2^2 \cdot 3 \\
14 &= 2 \cdot 7 \\
15 &= 3 \cdot 5 \\
16 &= 2 \cdot 2 \cdot 2 \cdot 2 &&= 2^4
\end{aligned} \tag{3.1}$$

To find prime factorizations, we use repeated division with prime divisors. Start by testing the number for divisibility by the smallest prime, 2 (a number is divisible by 2 if its final digit is *even*: $0, 2, 4, 6$ or 8). If it is divisible by 2, we divide by 2 as many times as possible, until we arrive at a quotient which is no longer divisible by 2. We then repeat the procedure, starting with the last quotient obtained, and using the next larger prime, 3 (a number is divisible by 3 if the *sum of its digits* is divisible by 3 – did you know that?). Repeating again (if necessary) with the next larger prime, we eventually arrive at a quotient which is itself a prime number. At that point, we are almost finished. We collect all the primes that were used as divisors (if the same prime has been used more than once, it should be collected as many times), together with the final (prime) quotient. The product of all these numbers is the prime factorization.

Example 94. Find the prime factorization of 300.

Solution. 300, being even, is divisible by 2, so we start by dividing by 2. The steps are as follows:

$$\begin{aligned}
300 \div 2 &= 150; \\
150 \div 2 &= 75; && \text{75 is not divisible by 2; move on to 3} \\
75 \div 3 &= 25; && \text{25 is not divisible by 3; move on to 5} \\
25 \div 5 &= 5; && \text{5 is a prime number; stop.}
\end{aligned}$$

The primes that were used as divisors were 2, 2, 3, 5. Note that 2 was used twice, so it is listed twice. The final prime quotient is 5. The prime factorization is the product of all those prime divisors and the final prime quotient:

$$300 = 2 \cdot 2 \cdot 3 \cdot 5 \cdot 5 \quad \text{or} \quad 2^2 \cdot 3 \cdot 5^2.$$

We can easily check our work by multiplying the prime factors and verifying that the product obtained is the original number. ☐

3.6.1 Exercises

Find the prime factorizations of the following numbers. Check the results by multiplication.

1. 60
2. 48
3. 81
4. 360
5. 85
6. 154
7. Which of the numbers above is divisible by 3?

3.6.2 Finding the GCF

Using prime factorizations, it is easy to find the greatest common factor of a set of numbers.

Example 95. Find the greatest common factor (GCF) of the two-number set $\{a, b\}$, where a and b have the following prime factorizations.

$$a = 2^4 \cdot 3^2 \cdot 7 \qquad \text{and} \quad b = 2 \cdot 3^4 \cdot 11.$$

Solution. First, look for the *common* prime factors of a and b: they are 2 and 3. Note that 7 and 11 are not common, and therefore cannot be factors of the GCF. Now look at the *powers* (exponents) on 2 and 3. A small power of any prime is a factor of any larger power of the same prime. The smallest power of 2 that appears is $2 = 2^1$ (in the factorization of b), and the smallest power of 3 that appears is 3^2 (in the factorization of a). It follows that the GCF is the product of the two smallest powers of 2 and 3. Thus

$$\text{GCF}\{a, b\} = 2^1 \cdot 3^2 = 18.$$

Notice that the actual values of a and b (which we could have determined by multiplication) were not used – only their prime factorizations. □

Here is a summary of the procedure:

To find the GCF of a set of numbers:

1. find the prime factorization of each number;
2. determine the prime factors *common* to all the numbers;
3. if there are no common prime factors, the GCF is 1; otherwise
4. for each common prime factor, find the smallest exponent that appears on it;
5. the GCF is the product of the common prime factors with the exponents found in step 4.

Example 96. (a) Find the GCF of the set $\{60, 135, 150\}$. (b) Find the GCF of the subset $\{60, 150\}$.

Solution. (a) Following the boxed procedure:

1. the prime factorizations are

$$\begin{aligned} 60 &= 2 \cdot 2 \cdot 3 \cdot 5 = 2^2 \cdot 3 \cdot 5 \\ 135 &= 3 \cdot 3 \cdot 3 \cdot 5 = 3^3 \cdot 5 \\ 150 &= 2 \cdot 3 \cdot 5 \cdot 5 = 2 \cdot 3 \cdot 5^2 \end{aligned}$$

2. the common prime factors are 3 and 5;
3. (does not apply to this example);
4. the smallest exponent on 3 is 1 (in the factorizations of 60 and 150); the smallest exponent on 5 is also 1 (in the factorizations of 60 and 135);
5. the GCF is $3^1 \cdot 5^1 = 15$.

(b) For the two-number subset $\{60, 150\}$, the common prime factors are 2, 3 and 5. The smallest exponent on all three factors is 1. So the GCF is $2^1 \cdot 3^1 \cdot 5^1 = 30$.

Can you explain why the GCF in part (b) is bigger than in part (a)? □

3.6.3 Exercises

Find the GCF of each of the following sets of numbers:

1. $\{72, 48\}$
2. $\{72, 48, 36\}$
3. $\{72, 36\}$
4. $\{48, 36\}$
5. $\{36, 15\}$
6. $\{36, 14\}$
7. $\{15, 14\}$

3.6.4 Cancelling the GCF for lowest terms

Knowing that the GCF of $\{60, 150\} = 30$ allows us to reduce the fraction $\frac{60}{150}$ to lowest terms in one step: we simply cancel it out. Thus,

$$\frac{60}{150} = \frac{\cancel{60}^{\,2}}{\cancel{150}_{\,5}} = \frac{2}{5} \qquad \text{(cancelling the GCF, 30).}$$

Recall that this is short-hand for

$$\frac{60 \div 30}{150 \div 30} = \frac{2}{5}.$$

Example 97. Reduce $\frac{168}{252}$ to lowest terms by finding and cancelling the GCF.

Solution. The prime factorizations of the numerator and denominator are

$$168 = 2^3 \cdot 3 \cdot 7$$
$$252 = 2^2 \cdot 3^2 \cdot 7.$$

The smallest exponent on 2 is 2, and the smallest exponents on 3 and 7 are 1; it follows that the GCF is $2^2 \cdot 3 \cdot 7 = 84$. Cancelling the GCF yields lowest terms:

$$\frac{168}{252} = \frac{\cancelto{2}{168}}{\cancelto{3}{252}} = \frac{2}{3} \qquad \text{(cancelling the GCF, 84).}$$

□

3.6.5 Exercises

Reduce each fraction to lowest terms by finding and cancelling the GCF of the numerator and denominator. Convert improper fractions to mixed numbers.

1. $\frac{36}{72}$
2. $\frac{48}{36}$
3. $\frac{14}{48}$
4. $\frac{36}{14}$
5. $\frac{14}{15}$
6. $\frac{72}{84}$
7. $\frac{48}{180}$
8. $\frac{96}{56}$
9. $\frac{105}{147}$
10. $\frac{300}{360}$

3.7 Pre-cancelling when Multiplying Fractions

In the fraction product

$$\frac{21}{5} \cdot \frac{15}{14},$$

we could simply follow the rule that the product of fractions is the product of the numerators over the product of the denominators,

$$\frac{21 \cdot 15}{5 \cdot 14},$$

calculate the products in the numerator and denominator and, finally, reduce the resulting fraction to lowest terms by cancelling out the GCF. But there's a short-cut: Since multiplication is commutative, we can reverse the order of multiplication in the numerator, obtaining

$$\frac{15 \cdot 21}{5 \cdot 14} = \frac{15}{5} \cdot \frac{21}{14}.$$

The fractions on the right-hand side are easily reduced to lowest terms by cancelling the obvious GCFs (5 and 7, respectively)

$$\frac{\cancel{15}^{\,3}}{\cancel{5}^{\,1}} \cdot \frac{\cancel{21}^{\,3}}{\cancel{14}^{\,2}},$$

leaving a simpler product

$$\frac{3}{1} \cdot \frac{3}{2} \quad = \frac{9}{2},$$

which has the further advantage that the solution, $\frac{9}{2}$, is already in lowest terms. The trick of cancelling *before* multiplying – pre-cancellation – saves us from bigger numbers,

$$\frac{21}{5} \cdot \frac{15}{14} = \frac{315}{70},$$

and the extra work of finding the $\text{GCF}\{315, 70\} = 35$ for cancellation:

$$\frac{\cancel{315}^{\,9}}{\cancel{70}^{\,2}} = \frac{9}{2} \quad \text{(cancelling 35)},$$

The general rule is this:

> In a product of fractions, a factor which is common to one of the numerators and one of the denominators can be cancelled before multiplying. The numerator and denominator need not belong to the same fraction.

Notice that pre-cancellation works with any number of factors.

Example 98. Find the product

$$\frac{3}{4} \cdot \frac{8}{5} \cdot \frac{10}{9}.$$

Solution. The numerator of the first fraction (3) has a common factor with the denominator of the third fraction (9), so the product is equal to

$$\frac{\cancel{3}^{1}}{4}\cdot\frac{8}{5}\cdot\frac{10}{\cancel{9}^{3}} = \frac{1}{4}\cdot\frac{8}{5}\cdot\frac{10}{3}.$$

Continuing on, the numerator of the second fraction (8) has a common factor with the denominator of the first fraction (4), so the product is equal to

$$\frac{1}{\cancel{4}^{1}}\cdot\frac{\cancel{8}^{2}}{5}\cdot\frac{10}{3} = 1\cdot\frac{2}{5}\cdot\frac{10}{3}.$$

Finally, the numerator of the third fraction (10) has a common factor with the denominator of the second fraction (5), so (omitting the factor 1) the product is equal to

$$\frac{2}{\cancel{5}^{1}}\cdot\frac{\cancel{10}^{2}}{3} = \frac{2}{1}\cdot\frac{2}{3}.$$

No further cancellation is possible. The final answer is now a simple product

$$\frac{2}{1}\cdot\frac{2}{3} = \frac{4}{3} = 1\frac{1}{3},$$

which is already in lowest terms. These cancellations could have been done in a different order, or (carefully) all at once. □

Mixed numbers are multiplied by simply converting them into improper fractions.

Example 99. Find the product $2\frac{3}{8}\cdot 1\frac{1}{4}\cdot 2\frac{2}{3}$. Express the result as a mixed number.

Solution. Rewriting each mixed number as an improper fraction, we have the product

$$\frac{19}{8}\cdot\frac{5}{4}\cdot\frac{8}{3}.$$

Cancelling 8's, we have

$$\frac{19\cdot 5}{4\cdot 3} = \frac{85}{12} = 7\frac{1}{12}.$$

□

Example 100. A gas tank with a $13\frac{1}{2}$-gallon capacity is only one third full. How much gas is in the tank?

Solution. We need to find the product $\frac{1}{3}\cdot 13\frac{1}{2}$. Converting $13\frac{1}{2}$ to the improper fraction $\frac{27}{2}$, we have

$$\frac{1}{3}\cdot 13\frac{1}{2} = \frac{1}{3}\cdot\frac{27}{2}.$$

Cancelling the common factor 3,

$$\frac{1}{\cancel{3}_{1}}\cdot\frac{\cancel{27}^{9}}{2} = \frac{9}{2}.$$

Converting $\frac{9}{2}$ to a mixed number, we conclude that the tank contains $4\frac{1}{2}$ gallons of gas. □

3.7.1 Exercises

Find the products, using pre-cancellation where possible. Check that the answers are in lowest terms. Express any improper fractions as mixed numbers.

1. $\frac{4}{5} \cdot \frac{7}{12}$
2. $\frac{32}{45} \cdot \frac{9}{16} \cdot \frac{15}{6}$
3. $12 \cdot \frac{5}{8} \cdot \frac{2}{3}$
4. $\frac{9}{22} \cdot \frac{21}{12} \cdot 2\frac{4}{7}$
5. $\frac{2}{3}$ of 24.
6. $\frac{3}{4}$ of $\frac{2}{3}$ of 50.
7. $2\frac{2}{3} \cdot 1\frac{3}{4}$.

Use multiplication to answer the following questions.

8. What is the area of a rectangle with length $2\frac{1}{2}$ feet and width $2\frac{3}{10}$ feet?
9. A $12\frac{1}{2}$-gallon fish tank is only three-fifths full. How many gallons of water must be added to fill it up?
10. $2\frac{2}{3}$ cups of beans are needed to make 4 bowls of chili. How many cups are needed to make 8 bowls? 1 bowl? 3 bowls?

3.8 Adding and Subtracting Fractions

If I eat a third of a pizza for lunch, and another third for dinner, then I have eaten two thirds in total. That is,

$$\frac{1}{3} + \frac{1}{3} = \frac{2}{3}.$$

Similar logic applies whenever we add two (or more) fractions *with the same denominator* – we simply add the numerators, while keeping the denominator fixed:

$$\boxed{\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}.}$$

A very similar rule holds for subtraction of fractions with the same denominator:

$$\frac{a}{c} - \frac{b}{c} = \frac{a-b}{c}.$$

Fractions with the same denominator are called **like** fractions.

Example 101. Here are three examples involving addition or subtraction of like fractions:

$$\frac{1}{5} + \frac{2}{5} = \frac{1+2}{5} = \frac{3}{5}$$

$$\frac{13}{7} - \frac{2}{7} = \frac{13-2}{7} = \frac{11}{7} = 1\frac{4}{7}$$

$$\frac{7}{8} - \frac{5}{8} = \frac{7-5}{8} = \frac{2}{8} = \frac{1}{4}.$$

Notice that we reduced to lowest terms where necessary, and changed improper fractions to mixed numbers.

3.8.1 Exercises

Add or subtract the like fractions as indicated. Reduce the final answers to lowest terms if necessary. Change improper fractions to mixed numbers.

1. $\frac{1}{5} + \frac{3}{5} =$
2. $\frac{7}{5} + \frac{13}{5} =$
3. $\frac{11}{15} + \frac{13}{15} + \frac{8}{15} =$
4. $\frac{20}{51} - \frac{3}{51} =$
5. $\frac{5}{13} - \frac{4}{13} =$
6. $\frac{11}{25} + \frac{9}{25} - \frac{3}{25} =$
7. $\frac{109}{7} - \frac{11}{7} =$
8. $\frac{13}{2} - \frac{1}{2} =$
9. $\frac{10}{7} + \frac{6}{7} - \frac{11}{7} =$
10. $\frac{11}{25} - \frac{2}{25} + \frac{8}{25} =$

3.8.2 Adding and Subtracting Unlike Fractions

Here is the rule for adding any two fractions, like or unlike:

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$

Compared to the rule for multiplying fractions (Section 3.4), it looks a bit strange. But it is easily proved, using (yet again) the fact that $1 = b/b$ and $1 = d/d$, for any nonzero b or d.

$$\begin{aligned} \frac{a}{b} + \frac{c}{d} &= \frac{a}{b} \cdot \frac{d}{d} + \frac{c}{d} \cdot \frac{b}{b} \\ &= \frac{ad}{bd} + \frac{bc}{bd} \\ &= \frac{ad + bc}{bd} \end{aligned}$$

Example 102. Add $\frac{2}{5} + \frac{3}{7}$.

Solution. Using the boxed rule,

$$\frac{2}{5} + \frac{3}{7} = \frac{2 \cdot 7 + 5 \cdot 3}{5 \cdot 7} = \frac{14 + 15}{35} = \frac{19}{35}.$$

□

Example 103. Add $\frac{15}{16} + \frac{19}{24}$.

Solution. Using the boxed rule,

$$\frac{15}{16} + \frac{19}{24} = \frac{15 \cdot 24 + 16 \cdot 19}{16 \cdot 24} = \frac{664}{384}.$$

Cancelling the GCF, 8, we obtain the sum in lowest terms, $\frac{83}{48}$. □

The rule works for like fractions too:

Example 104. Use the boxed rule to show that $\frac{4}{5} + \frac{3}{5} = \frac{7}{5}$.

Solution.

$$\frac{4}{5} + \frac{3}{5} = \frac{20 + 15}{25} = \frac{35}{25} = \frac{7}{5}.$$

Note that an extra factor of 5 was introduced, and then cancelled at the end. (A needless complication!)

□

A disadvantage of the boxed rule is that it produces a fraction which may be far from lowest terms. Another disadvantage is that it doesn't extend easily to sums of more than two fractions. To remedy these, we develop a method, somewhat analogous to pre-cancellation in multiplication, which produces the sum of any set of fractions in a form which is as close as possible to *lowest terms*. The idea is to replace each fraction by an equivalent fraction, each having the same (common) denominator, which is *as small as possible*.

Example 105. Find the sum:

$$\frac{2}{3} + \frac{1}{5}.$$

Solution. Observe that

$$\frac{2}{3} = \frac{2 \cdot 5}{3 \cdot 5} = \frac{10}{15} \qquad \text{and} \qquad \frac{1}{5} = \frac{1 \cdot 3}{5 \cdot 3} = \frac{3}{15}.$$

It follows that

$$\frac{2}{3} + \frac{1}{5} = \frac{10}{15} + \frac{3}{15}.$$

Now we have like fractions, and so

$$\frac{10}{15} + \frac{3}{15} = \frac{10+3}{15} = \frac{13}{15}.$$

Since equivalent fractions represent the same number, we conclude that

$$\frac{2}{3} + \frac{1}{5} = \frac{13}{15}.$$

□

Why (and how) did we choose 15 as the common denominator? And how do we know it is the *smallest* common denominator we could have used? We could have chosen any number that is a *multiple* of both 3 and 5 (the two original denominators in the example). The multiples of 3 are

$$3, 6, 9, 12, 15, 18, 21, \text{etc.},$$

and the multiples of 5 are

$$5, 10, 15, 20, 25, 30, \text{etc.},$$

and it is easy to see that the *smallest* number that is a multiple of both – the *least common* multiple – is 15.

3.8.3 The LCM

The method of listing small multiples is often the simplest way to find the **least common multiple**, or LCM, of a set of whole numbers.

Example 106. Find the LCM$\{6, 10, 15\}$.

Solution. The multiples of 6 are

$$6, 12, 18, 24, 30, 36, \dots,$$

the multiples of 10 are

$$10, 20, 30, 40, \dots,$$

and the multiples of 15 are

$$15, 30, 45, \dots.$$

It is evident that the smallest number which is a multiple of *all three* numbers is 30. □

The LCM can also be found using prime factorizations. This is useful when the numbers are rather large.

Example 107. Find the LCM of the two-number set $\{a, b\}$, where a and b have the following prime factorizations.

$$a = 2^4 \cdot 3^2 \cdot 7 \quad \text{and} \quad b = 2 \cdot 3^4 \cdot 11.$$

Solution. The LCM must be a multiple of both numbers, so that it must be divisible by the highest power of every prime factor that appears in any one of the factorizations. The prime factors of a and b are 2, 3, 7 and 11. The highest powers that appear are

$$2^4, \quad 3^4, \quad 7^1, \quad 11^1.$$

The LCM is the product of these powers:

$$\text{LCM}\{a, b\} = 2^4 \cdot 3^4 \cdot 7 \cdot 11.$$

This is a rather large number, but it is the smallest which is a multiple of both a and b. Notice that the actual values of a and b, and their LCM (which we could calculate by multiplication) were not needed – only their prime factorizations. ☐

Here is a summary of the procedure:

> To find the LCM of a set of numbers:
>
> 1. find the prime factorization of each number;
> 2. for each prime factor, find the largest exponent that appears on it in any of the factorizations;
> 3. the LCM is the product of the prime factors with the exponents found in step 2.

Compare this procedure with the procedure for finding the GCF of a set of numbers. There are similarities and significant differences.

3.8.4 Exercises

Find the LCM of the following sets of numbers. Use the prime factorization method for larger numbers.

1. $\text{LCM}\{25, 10\}$
2. $\text{LCM}\{48, 60\}$
3. $\text{LCM}\{10, 15, 25\}$
4. $\text{LCM}\{8, 12\}$
5. $\text{LCM}\{9, 6, 12\}$
6. $\text{LCM}\{60, 168\}$

7. LCM$\{51, 34, 17\}$
8. LCM$\{15, 12\}$
9. LCM$\{18, 8\}$
10. LCM$\{3, 4, 5\}$
11. LCM$\{4, 14\}$
12. LCM$\{2, 5, 9\}$
13. LCM$\{3^2 \cdot 5^2 \cdot 11, \quad 3^4 \cdot 7^2\}$

3.8.5 The LCD

To add unlike fractions so that the sum is in lowest terms, we use the LCM of their denominators. This is such a useful number that it has a special name: the LCD or **Least Common Denominator**.

Example 108. Find the LCD of the fractions

$$\frac{1}{8}, \quad \frac{3}{10}, \quad \text{and} \quad \frac{1}{18}.$$

Solution. The LCD of the fractions is the LCM of their denominators,

$$\text{LCM}\{8, 10, 18\}.$$

Looking at the prime factorizations

$$8 = 2^3, \qquad 10 = 2 \cdot 5, \qquad 18 = 2 \cdot 3^2,$$

and taking the highest power of each prime that occurs, we see that the LCM is

$$2^3 \cdot 3^2 \cdot 5 = 360.$$

This is LCD of the fractions. □

Example 109. Find the sum of the unlike fractions $\frac{1}{8} + \frac{3}{10} + \frac{1}{18}$.

Solution. The LCD is the LCM from the previous example: 360. Now we observe that

$$360 \quad = \quad 8 \cdot 45 \quad = \quad 10 \cdot 36 \quad = \quad 18 \cdot 20.$$

Thus

$$\frac{1}{8} = \frac{1 \cdot 45}{8 \cdot 45}, \qquad \frac{3}{10} = \frac{3 \cdot 36}{10 \cdot 36} \qquad \text{and} \qquad \frac{1}{18} = \frac{1 \cdot 20}{18 \cdot 20}.$$

It follows that

$$\begin{aligned} \frac{1}{8} + \frac{3}{10} + \frac{1}{18} &= \frac{1 \cdot 45}{8 \cdot 45} + \frac{3 \cdot 36}{10 \cdot 36} + \frac{1 \cdot 20}{18 \cdot 20} \\ &= \frac{45 + 108 + 20}{360} \\ &= \frac{173}{360}. \end{aligned} \tag{3.2}$$

Since the 173 is not divisible by 2, 3, or 5, the fraction is in lowest terms. □

Practically the same method works for subtracting unlike fractions.

Example 110. Find the difference $\frac{14}{25} - \frac{2}{10}$, using the LCD. Reduce to lowest terms, if necessary.

Solution. The LCD is LCM$\{25, 10\} = 50$. Now $50 = 25 \cdot 2 = 10 \cdot 5$. So

$$\frac{14}{25} = \frac{14 \cdot 2}{25 \cdot 2} = \frac{28}{50}, \qquad \text{and} \qquad \frac{2}{10} = \frac{2 \cdot 5}{10 \cdot 5} = \frac{10}{50}.$$

Thus, the difference of the two fractions is

$$\frac{28}{50} - \frac{10}{50} = \frac{28 - 10}{50} = \frac{18}{50}.$$

The latter fraction is not in lowest terms, since the GCF of 18 and 50 is 2. Cancelling the GCF, we get

$$\frac{18}{50} = \frac{\cancel{18}^{\,9}}{\cancel{50}_{\,25}} = \frac{9}{25}.$$

□

Example 111. Subtract $\frac{1}{3} - \frac{1}{4}$, and reduce to lowest terms if necessary.

Solution. The LCD is 12. Changing both fractions to equivalent fractions with denominator 12, we get

$$\frac{1}{3} = \frac{1 \cdot 4}{3 \cdot 4} = \frac{4}{12} \quad \text{and} \quad \frac{1}{4} = \frac{1 \cdot 3}{4 \cdot 3} = \frac{3}{12}.$$

Thus,

$$\frac{1}{3} - \frac{1}{4} = \frac{4}{12} - \frac{3}{12} = \frac{4 - 3}{12} = \frac{1}{12}.$$

The latter fraction is already in lowest terms, so we are done. □

Example 112. Find the sum $\frac{4}{5} + \frac{3}{4}$, reduce to lowest terms, and express the answer as a mixed number.

Solution. The LCD$\{4, 5\} = 20$, so that

$$\frac{4}{5} + \frac{3}{4} = \frac{4 \cdot 4}{5 \cdot 4} + \frac{3 \cdot 5}{4 \cdot 5} = \frac{16}{20} + \frac{15}{20} = \frac{31}{20} = 1\frac{11}{20}.$$

□

3.8.6 Exercises

Add or subtract the following fractions as indicated, reduce to lowest terms if necessary, and change improper fractions to mixed numbers.

1. $\frac{1}{5} + \frac{3}{6}$
2. $\frac{7}{5} + \frac{13}{3}$
3. $\frac{15}{1} + \frac{2}{3}$

4. $\dfrac{1}{5}+\dfrac{0}{2}$

5. $\dfrac{2}{3}+\dfrac{3}{4}+\dfrac{4}{5}$

6. $\dfrac{11}{15}+\dfrac{13}{25}$

7. $\dfrac{17}{51}-\dfrac{3}{50}$

8. $\dfrac{5}{3}-\dfrac{4}{13}$

9. $\dfrac{11}{5}-\dfrac{2}{25}$

10. $\dfrac{11}{7}-\dfrac{11}{17}$

11. $\dfrac{3}{2}-\dfrac{1}{12}$

12. $\dfrac{6}{7}-\dfrac{11}{17}$

13. $\dfrac{11}{20}-\dfrac{2}{25}$

3.9 Comparison of Fractions

When two fractions have the same denominator, it is clear that the larger fraction is the one with the larger numerator. This suggests a simple way to compare any set of fractions: replace each fraction by an equivalent fraction having the LCD. Then the largest fraction is the one with the largest numerator, the second-largest is the one with the second-largest numerator, etc.

Example 113. Which is larger, $\dfrac{2}{3}$ or $\dfrac{3}{4}$?

Solution. The LCD is 12,

$$\frac{2}{3}=\frac{8}{12}, \qquad \text{and} \qquad \frac{3}{4}=\frac{9}{12}.$$

The larger of the two fractions with denominator 12 is the one with the larger numerator, $\frac{9}{12}$. It follows that $\frac{3}{4}$ is larger than $\frac{2}{3}$. ☐

Numbers arranged in **increasing order** are written from left to right, separated by the $<$ symbol. (This is the left-to-right ordering on the number line.) We can also arrange numbers in **decreasing order**, from left to right, separated by the $>$ ("greater than") symbol.

Example 114. Arrange the fractions $\dfrac{3}{4}, \dfrac{2}{3},$ and $\dfrac{7}{9}$ in increasing order.

Solution. LCD$\{3,4,9\} = 36$, and

$$\frac{3}{4} = \frac{27}{36} \qquad \frac{2}{3} = \frac{24}{36} \qquad \frac{7}{9} = \frac{28}{36}.$$

We are asked to arrange the fractions in *increasing* order, so we list them in increasing order of their numerators, separated by the inequality symbol $<$. Thus,

$$\frac{24}{36} < \frac{27}{36} < \frac{28}{36}.$$

Now, going back to the equivalent fractions in lowest terms, we have

$$\frac{2}{3} < \frac{3}{4} < \frac{7}{9}.$$

□

Example 115. Arrange the numbers $1\frac{3}{5}, 1\frac{3}{4}, 1\frac{6}{7}$, and $\frac{11}{12}$ in decreasing order.

Solution. Recall that a proper fraction is less than 1, so $\frac{11}{12}$ is obviously the smallest of the four numbers (the others being mixed numbers greater than 1). To compare the other three numbers, it is enough to compare their fractional parts, because each has the same whole number part (1). To compare $\frac{3}{5}$, $\frac{3}{4}$, and $\frac{6}{7}$, we find the LCM$\{5,4,7\} = 140$, and write equivalent fractions with denominator 140:

$$\frac{3}{5} = \frac{84}{140} \quad \frac{3}{4} = \frac{105}{140} \quad \frac{6}{7} = \frac{120}{140}.$$

In decreasing order, the numerators are $120 > 105 > 84$. The corresponding ordering of the fractions is

$$\frac{6}{7} > \frac{3}{4} > \frac{3}{5},$$

and it follows that the corresponding order of the mixed numbers (each with whole number part 1) is

$$1\frac{6}{7} > 1\frac{3}{4} > 1\frac{3}{5}.$$

Adjoining the smallest (proper) fraction $\frac{11}{12}$ at the end, we have, finally, the decreasing order

$$1\frac{6}{7} > 1\frac{3}{4} > 1\frac{3}{5} > \frac{11}{12}.$$

□

3.9.1 Exercises

1. Arrange in decreasing order: $\frac{4}{9}, \frac{3}{8}$, and $\frac{1}{3}$.
2. Arrange in increasing order: $\frac{3}{5}, \frac{3}{4}$, and $\frac{5}{7}$.
3. Is $\frac{5}{12}$ of an inch more or less than $\frac{7}{16}$ of an inch?
4. Is $9\frac{5}{8}$ inches closer to 9 or 10 inches?
5. A stock price changed from $5\frac{5}{8}$ to $5\frac{3}{4}$. Did the price go up or down?

3.10 Division of Fractions

Our intuition is not very good when we think of dividing two fractions. How many times does $\frac{2}{3}$ "go into" $\frac{3}{4}$, for example? Equivalently, what fraction results from the division problem

$$\frac{3}{4} \div \frac{2}{3}?$$

It is not at all obvious. We will derive a simple rule, but we'll need an auxiliary concept.

3.10.1 Reciprocals

Two non-zero numbers are **reciprocal** if their product is 1. Thus, if x and y are nonzero numbers, and if

$$x \cdot y = 1,$$

then x *is the reciprocal of* y, and, also, y *is the reciprocal of* x.

The rule for multiplying fractions, together with obvious cancellations, shows that

$$\frac{a}{b} \cdot \frac{b}{a} = \frac{\cancelto{1}{a} \cdot b}{b \cdot \cancelto{1}{a}} = \frac{1 \cdot \cancelto{1}{b}}{\cancelto{1}{b} \cdot 1} = \frac{1}{1} = 1.$$

This means that

> The reciprocal of the fraction $\frac{a}{b}$ is the fraction $\frac{b}{a}$.
> (both a and b nonzero)

Since every whole number n can be written as the fraction $\frac{n}{1}$, we have the following special case:

> The reciprocal of the whole number n (nonzero)
> is the fraction $\frac{1}{n}$.

Example 116. The reciprocal of 5 is $\frac{1}{5}$. The reciprocal of $\frac{1}{9}$ is 9. The reciprocal of $\frac{3}{8}$ is $\frac{8}{3}$, or, expressed as a mixed number, $2\frac{2}{3}$.

We note some important facts and special cases:

- the *reciprocal of the reciprocal* of a number is the number itself. For example, the reciprocal of 2/3 is 3/2, and in turn, the reciprocal of 3/2 is 2/3.
- 1 is the only (positive) number that is its own reciprocal (since $1 = \frac{1}{1}$.)
- the reciprocal of a number less than 1 is a number greater than 1, and vice versa (since the reciprocal of a *proper* fraction is an *improper* fraction).
- 0 has no reciprocal (since division by 0 is undefined).

3.10.2 Exercises

Find the reciprocals of the following numbers. Change improper fractions to mixed or whole numbers.

1. $\dfrac{3}{7}$
2. 11
3. $\dfrac{14}{5}$
4. $\dfrac{1}{25}$
5. $\dfrac{5}{6}$

3.10.3 Division is Multiplication by the Reciprocal of the Divisor

Consider again the division problem

$$\frac{3}{4} \div \frac{2}{3}.$$

Using the LCD, this is equivalent to

$$\frac{9}{12} \div \frac{8}{12}$$

We want to know how many times 8 things of a certain size (1/12) go into 9 things of the same size. The actual denominator doesn't really matter, only that it is the same for both fractions. In this form it seems obvious that the answer is the quotient of the numerators, $\frac{9}{8}$.

Now, how were the numbers 9 and 8 obtained from the original fractions? Well, in writing equivalent fractions with the LCD, we computed the numerators as follows: $9 = 3 \cdot 3$ and $8 = 4 \cdot 2$. We have shown that

$$\frac{3}{4} \div \frac{2}{3} = \frac{3 \cdot 3}{4 \cdot 2} = \frac{3}{4} \cdot \frac{3}{2}.$$

The original division turned into multiplication by the *reciprocal of the divisor*, and the LCD wasn't actually needed.

We have derived a simple rule:

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$$

In words: the quotient of two fractions is the product of the dividend and the *reciprocal of the divisor*.

Example 117. Perform the division $\frac{5}{12} \div \frac{5}{6}$.

Solution. We convert the division into multiplication by the reciprocal of $\frac{5}{6}$.

$$\begin{aligned}\frac{5}{12} \div \frac{5}{6} &= \frac{5}{12} \cdot \frac{6}{5} \\ &= \frac{30}{60} \\ &= \frac{1}{2} \qquad \text{(cancelling 30).}\end{aligned}$$

□

Example 118. What is $\frac{3}{7}$ divided by 6?

$$\begin{aligned}\frac{3}{7} \div 6 &= \frac{3}{7} \div \frac{6}{1} \\ &= \frac{3}{7} \cdot \frac{1}{6} \\ &= \frac{3}{42} \\ &= \frac{1}{14} \qquad \text{(cancelling 3).}\end{aligned}$$

We can divide mixed numbers, too, by simply converting them into improper fractions first.

Example 119. Perform the division $2\frac{8}{9} \div 1\frac{1}{2}$. Express the result as a mixed number.

Solution. Converting mixed numbers to improper fractions, we have

$$2\frac{8}{9} \div 1\frac{1}{2} = \frac{26}{9} \div \frac{3}{2}.$$

Now we convert the division into multiplication by the reciprocal of $\frac{3}{2}$:

$$= \frac{26}{9} \cdot \frac{2}{3}.$$

No pre-cancellation is possible, so

$$\frac{26 \cdot 2}{9 \cdot 3} = \frac{52}{27} = 1\frac{25}{27},$$

where, at the last step, we converted the improper fraction into a mixed number. □

Example 120. How many books, each $1\frac{3}{4}$ inch thick, can fit on a 35 inch bookshelf?

Solution. The question calls for division, since we need to know how many times $1\frac{3}{4}$ "goes into" 35.

$$35 \div 1\frac{3}{4} = \frac{35}{1} \div \frac{7}{4} = \frac{35}{1} \cdot \frac{4}{7}.$$

Cancelling 7, we get

$$\frac{5}{1} \cdot \frac{4}{1} = \frac{20}{1} = 20.$$

So, exactly 20 books will fit on the shelf. □

3.10.4 Exercises

Perform the divisions. Reduce to lowest terms, and change improper fractions to mixed numbers.

1. $\frac{1}{5} \div \frac{3}{2}$
2. $15 \div \frac{2}{3}$
3. $\frac{8}{15} \div \frac{16}{21}$

4. $10\frac{1}{2} \div 5\frac{5}{6}$
5. $\frac{7}{8} \div \frac{21}{4}$
6. $\frac{100}{34} \div \frac{50}{17}$
7. $100 \div 33$
8. $\frac{100}{3} \div 5$

Use division to answer the following questions.

9. How many $1\frac{1}{2}$ inch pieces can be cut from a board that is 36 inches long?
10. Seeds are sold in $2\frac{1}{2}$-ounce packets. A gardener needs 32 ounces of seed. How many packets should he buy to be sure he has enough?

3.11 Mixed Numbers and Mixed Units

Mixed numbers can be added and subtracted by first converting them into improper fractions, and very often this is what we do. But a problem such as

$$75\frac{1}{2} + 59\frac{3}{4}$$

is not well-suited to this method. If we convert to improper fractions, the numerators of both fractions would be quite large and unwieldy.

3.11.1 Vertical Addition and Subtraction

We can align mixed numbers vertically so that the fractional and whole number parts line-up, and then separately add the fraction and whole number columns. We must ensure that the resulting mixed number is in proper form (the fractional part must be a *proper* fraction), and this involves a procedure analogous to "carrying" in whole number addition.

Example 121. Add $2\frac{1}{2} + 6\frac{3}{4}$.

Solution. We first align the whole number and fractional places vertically:

$$\begin{array}{rrr} & 2 & \frac{1}{2} \\ + & 6 & \frac{3}{4} \\ \hline \end{array}$$

To add the fractional parts, we write them both with their LCD, which is 4.

$$\begin{array}{rrr} & 2 & \frac{2}{4} \\ + & 6 & \frac{3}{4} \\ \hline \end{array}$$

Then we add the columns separately.

$$\begin{array}{rrr} & 2 & \frac{2}{4} \\ + & 6 & \frac{3}{4} \\ \hline & 8 & \frac{5}{4} \end{array}$$

The resulting mixed number, $8\frac{5}{4}$, is not in proper form, because $\frac{5}{4} = 1\frac{1}{4}$. So we leave the proper fractional part, $\frac{1}{4}$, in the fractions column, and carry the whole number part, 1, over to the whole number column. (We indicate the carrying by writing $\{1\}$ above the whole number column.) Then we recompute the whole number sum, obtaining the final answer in proper form:

$$\begin{array}{rrr} & \{1\} & \\ & 2 & \frac{2}{4} \\ + & 6 & \frac{3}{4} \\ \hline & 9 & \frac{1}{4} \end{array}$$

□

To subtract mixed numbers, we sometimes need a "borrowing" procedure, analogous to the procedure we use when subtracting whole numbers.

Example 122. Subtract $7 - 3\frac{2}{5}$.

Solution. Note that the whole number 7 is a mixed number with 0 fractional part, and we can represent 0 as a fraction using any convenient denominator. Here the convenient choice is 5. So we write $7 = 7\frac{0}{5}$. Aligning whole number parts vertically, we have

$$\begin{array}{rrr} & 7 & \frac{0}{5} \\ - & 3 & \frac{2}{5} \\ \hline & & \end{array}$$

Now we see that the subtraction in the fractions place is not possible (we can't take 2 fifths from 0 fifths), so we need to borrow 1 from the whole numbers column. $1 = \frac{n}{n}$ for any convenient n (except 0), and in this example, it is convenient to write $1 = \frac{5}{5}$. Borrowing $1 = \frac{5}{5}$ from 7 (reducing it to 6), and adding $1 = \frac{5}{5}$ to $\frac{0}{5}$ at the top of the fractions column, we have

$$\begin{array}{rrr} & 6 & \frac{5}{5} \\ - & 3 & \frac{2}{5} \\ \hline & 3 & \frac{3}{5} \end{array}$$

□

Both of the previous examples could have been done by first converting the mixed numbers into improper fractions. For example,

$$7 - 3\frac{2}{5} = \frac{35}{5} - \frac{17}{5} = \frac{18}{5} = 3\frac{3}{5}.$$

Perhaps you find this easier. But here is an example that would involve much more work if we were to use that method.

Example 123. Subtract $153\frac{2}{15} - 67\frac{4}{9}$.

Solution. The $\text{LCD}\{15, 9\} = 45$, and since $\frac{2}{15} = \frac{6}{45}$ and $\frac{4}{9} = \frac{20}{45}$, we write, in vertical form,

$$\begin{array}{rrr} & 153 & \frac{6}{45} \\ - & 67 & \frac{20}{45} \\ \hline \end{array}$$

The subtraction in the fractions column is not possible, so we borrow 1 from 153 (in the form $\frac{45}{45}$) and add it to $\frac{6}{45}$, obtaining

$$\begin{array}{rrr} & 152 & \frac{51}{45} \\ - & 67 & \frac{20}{45} \\ \hline \end{array}$$

Now the subtraction in the fractions column is possible, and we obtain

$$\begin{array}{rrr} & 152 & \frac{51}{45} \\ - & 67 & \frac{20}{45} \\ \hline & 85 & \frac{31}{45} \end{array}$$

The fractional part is in lowest terms, and we are done. □

If we had done the previous example by first converting the mixed numbers into improper fractions, we would have had to work with rather large numerators:

$$\frac{6891}{45} - \frac{3035}{45}$$

Not impossible, but the other method is easier!

3.11.2 Exercises

Add or subtract the mixed numbers as indicated. Express the final answers as mixed numbers in proper form.

1. $1\frac{1}{3} + \frac{1}{2}$

2. $3\frac{2}{5} + 7\frac{1}{3} + 1\frac{3}{5}$

3. $1\frac{1}{5} + 10$

4. $75\frac{3}{4} + 91\frac{2}{3}$

5. $12\frac{11}{15} - 2\frac{13}{20}$

6. $3\frac{7}{8} - 2\frac{1}{4}$

7. $1\frac{2}{3} - 1\frac{4}{13}$

8. $3\frac{3}{5} + 1\frac{1}{2} + \frac{3}{10}$

9. $1\frac{1}{7} - \frac{3}{7}$

10. $21\frac{3}{5} - 19\frac{3}{4}$

11. $3\frac{6}{7} - \frac{11}{12}$

12. $5\frac{11}{20} - 4\frac{4}{5}$

3.11.3 Measurements in Mixed Units

When a number results from a measurement, it is given in terms of a *unit of measure*, or *unit* for short. For example, if you use a standard ruler to measure the length of a table, the length is given in units of feet (ft). All 1-foot rulers are divided into 12 smaller units called inches (in), so that

$$1 \text{ ft} = 12 \text{ in.}$$

More precise measurements can be given in terms of the **mixed units** feet-and-inches. For example, the length of your table might be 3 feet 6 inches or 3 ft 6 in.

Other familiar examples of mixed units are hours-and-minutes, or hours-minutes-and-seconds. The abbreviations are hr, min and sec. Of course,

$$1 \text{ hr} = 60 \text{ min} \quad \text{and} \quad 1 \text{ min} = 60 \text{ sec.}$$

Mixed units are just like mixed numbers, because the smaller units are simple fractions of the larger ones:

$$1 \text{ in} = \frac{1}{12} \text{ ft} \qquad 1 \text{ min} = \frac{1}{60} \text{ hr} \qquad 1 \text{ sec} = \frac{1}{60} \text{ min}.$$

It follows that measurements in mixed units can be added and subtracted just like mixed numbers. The notion of proper form remains since, for example, a measurement of

$$4 \text{ ft } 17 \text{ in}$$

makes sense, but is *properly* written as

$$5 \text{ ft } 5 \text{ in},$$

since any measurement of over 12 inches is at least 1 foot. Borrowing and carrying are sometimes needed, and are easily accomplished, as the next example shows.

Example 124. Hugo and Hector go to separate movies that start at the same time. Hugo's movie is 2 hours and 5 minutes long, while Hector's is 1 hour and 43 minutes. How long will Hector have to wait, after his movie is over, for Hugo to come out?

Solution. Hector's waiting time is the difference in the lengths of the movies,

$$\begin{array}{rrr} & 2 \text{ hr} & 5 \text{ min} \\ - & 1 \text{ hr} & 43 \text{ min} \\ \hline \end{array}$$

The subtraction in the minutes column can't be done, so we borrow 1 hr = 60 min from the hours column, reducing the number of hours at the top of the hours column from 2 to 1, and increasing the number of minutes at the top of the minutes column from 5 min to 5 min + 60 min = 65 min.

$$\begin{array}{rrr} & 1 \text{ hr} & 65 \text{ min} \\ - & 1 \text{ hr} & 43 \text{ min} \\ \hline & 0 \text{ hr} & 22 \text{ min} \end{array}$$

Hector will have to wait 22 minutes. □

Here's an example involving length measurements.

Example 125. Two boards are laid end-to-end. One of the boards measures 6 ft 8 in, and the other measures 11 ft 10 in. What is the total length of the boards? Give your answer in proper form.

Solution. We need to calculate the mixed-unit sum

$$\begin{array}{rrr} & 6 \text{ ft} & 8 \text{ in} \\ + & 11 \text{ ft} & 10 \text{ in} \\ \hline \end{array}$$

Adding the inches column, we obtain 18 in = 1 ft 6 in. So we put down 6 in in the inches column, and carry 1 ft to the top of the feet column:

$$\begin{array}{rrr} & \{1 \text{ ft}\} & \\ & 6 \text{ ft} & 8 \text{ in} \\ + & 11 \text{ ft} & 10 \text{ in} \\ \hline & & 6 \text{ in} \end{array}$$

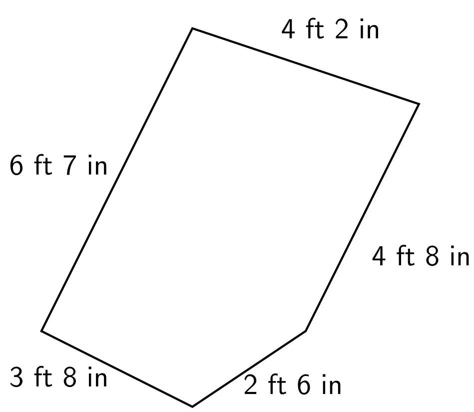
We add up the feet column to obtain the final answer:

$$\begin{array}{rrr} & \{1 \text{ ft}\} & \\ & 6 \text{ ft} & 8 \text{ in} \\ + & 11 \text{ ft} & 10 \text{ in} \\ \hline & 18 \text{ ft} & 6 \text{ in} \end{array}$$

The total length, in proper form, is 18 ft 6 in. □

3.11.4 Exercises

1. A carpenter cuts a 6 ft 7 in piece from a 9 ft 4 in board. What is the length of the leftover piece?
2. Find the perimeter of the polygon:



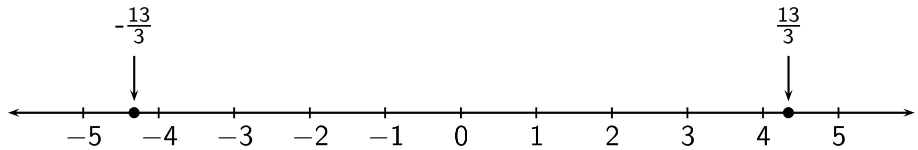
3. A film critic watched three movies. The first was $2\frac{1}{2}$ hrs, the second 2 hrs 15 min, and the last 1 hr 45 min. (a) What was the total length of the movies? (b) What was the average length of the movies?

3.12 Signed fractions

Every point on the number line represents a number. In this regard, there is nothing special about fractions. Each nonzero fraction has an opposite lying at the same distance from, but on the other side of 0.

Example 126. Find the opposite of $\dfrac{13}{3}$ and locate it on a number line.

Solution. The negative fraction $-\dfrac{13}{3}$ lies at a distance of $\left|-\dfrac{13}{3}\right| = 4\dfrac{1}{3}$ units to the left of 0:



□

One of the advantages of expressing an improper fraction as a mixed number is the ease with which you can locate it on the number line: a positive mixed number lies between its whole number part and the next larger whole number. By mirror symmetry, a negative mixed number lies between its (negative) whole number part and the next *smaller* whole number. Thus

$$-5 < -\frac{13}{3} < -4 \qquad \text{and} \qquad 4 < \frac{13}{3} < 5.$$

Some words about notation: A fraction bar indicates division, that is, $\dfrac{x}{y} = x \div y$. By the rule for division of signed integers, if either x or y (but not both) is negative, the fraction represents a negative number. Now suppose x and y are positive. Then the fractions

$$\frac{-x}{y} = \frac{x}{-y} = -\frac{x}{y}$$

all represent the same (negative) number. It is customary to avoid the form in the middle (with a negative number in the denominator).

Example 127. Find the reciprocal of $\dfrac{-2}{3}$.

Solution. Formally, the reciprocal is $\dfrac{3}{-2}$. In keeping with custom, we avoid a negative denominator and write instead

$$-\frac{3}{2} \quad \text{or} \quad \frac{-3}{2}.$$

□

Note well: if x and y are positive,

$$\frac{-x}{-y}$$

does NOT represent a negative number! (Why?)

You will be glad to know that the sign rules for integers apply, without change, to signed fractions. There are no new rules to learn! It is just a question of combining the fraction rules and the sign rules. For convenience, we repeat the rules for adding signed numbers from Section 2.3. We have simply substituted the word "fraction" for the word "number." Other than that, the rules are exactly the same.

When two signed fractions are added

- if the fractions have **opposite** signs,
    1. the *sign of the sum* is the sign of the fraction with the larger absolute value;
    2. the *absolute value of the sum* is the difference between the two individual absolute values (larger − smaller).
- if the fractions have the **same** sign,
    1. the *sign of the sum* is the common sign of the summands;
    2. the *absolute value of the sum* is the sum of the individual absolute values.

Example 128. Add $4\frac{3}{5}+\left(-7\frac{1}{3}\right)$.

Solution. The numbers have opposite signs, and the number with the larger absolute value is negative, so, by the sign rules, the sum is the negative of the difference of the absolute values:

$$\begin{aligned} 4\frac{3}{5}+\left(-7\frac{1}{3}\right) &= -\left(7\frac{1}{3}-4\frac{3}{5}\right) \\ &= -\left(7\frac{5}{15}-4\frac{9}{15}\right) \qquad \text{(using the LCD = 15)} \\ &= -\left(6\frac{20}{15}-4\frac{9}{15}\right) \qquad \text{(borrowing } 1=\frac{15}{15} \text{ from 7)} \\ &= -2\frac{11}{15}. \end{aligned}$$

□

We remind you that "subtract A from B" indicates the operation $B-A$, and that subtraction is equivalent to "adding the opposite:"

$$B-A=B+(-A).$$

Example 129. Subtract $2\frac{7}{8}$ from $1\frac{5}{6}$.

Solution. Adding the opposite of $2\frac{7}{8}$ to $1\frac{5}{6}$, we get

$$\begin{aligned} 1\frac{5}{6} + \left(-2\frac{7}{8}\right) &= -\left(2\frac{7}{8} - 1\frac{5}{6}\right) \\ &= -\left(2\frac{21}{24} - 1\frac{20}{24}\right) \qquad (\text{LCD} = 24) \\ &= -1\frac{1}{24}. \end{aligned}$$

□

Recall that the product or quotient of two numbers with opposite signs is always negative, and that the product or quotient of two numbers with the same sign is always positive.

Example 130. Find the product $\frac{3}{4} \cdot \left(-\frac{6}{5}\right)$.

Solution. The numbers have opposite signs, so the product is negative.

$$-\left(\frac{3}{\cancel{4}_{2}} \cdot \frac{\cancel{6}^{3}}{5}\right) = -\frac{9}{10} \qquad \text{(pre-cancelling 2).}$$

□

Example 131. Find the product $\left(-2\frac{1}{2}\right)\left(-3\frac{1}{4}\right)$.

Solution. The numbers have the same sign so the product is positive, and we compute with the absolute values. Converting the mixed numbers to improper fractions for multiplication,

$$\frac{5}{2} \cdot \frac{13}{4} = \frac{65}{8} = 8\frac{1}{8}.$$

□

Example 132. Express the fraction $-\frac{33}{5}$ as a mixed number.

Solution. The fraction represents division of numbers with opposite signs, so the result is negative. For the division $33 \div 5$, the quotient is 6 and the remainder is 3, so

$$-\frac{33}{5} = -6\frac{3}{5}.$$

□

Example 133. Find $\left(-2\frac{5}{9}\right) \div \left(-\frac{1}{6}\right)$.

Solution. Because the numbers have the same sign, the quotient is positive. We convert the mixed number to an improper fraction and multiply by the reciprocal of the divisor.

$$\begin{aligned}\left(-2\frac{5}{9}\right) \div \left(-\frac{1}{6}\right) &= \left(\frac{-23}{9}\right) \cdot \left(\frac{-6}{1}\right) \\ &= \left(\frac{23}{\cancel{9}^{\,3}}\right) \cdot \left(\frac{\cancel{6}^{\,2}}{1}\right) \\ &= \frac{46}{3} \\ &= 15\frac{1}{3}.\end{aligned}$$

□

When adding chains of fractions with various signs, it is good 'bookkeeping' to add all the positive fractions, and, separately, add the *absolute values* of all the negative fractions. Then, perform a single subtraction ('profits' − 'losses'), using the signed number rule just once, at the end.

Example 134. Add: $2\frac{1}{2} + \frac{2}{3} + \left(-1\frac{3}{4}\right) + \frac{4}{5} + \left(-\frac{5}{6}\right)$.

Solution. Collecting 'Profits' − 'Losses,'

$$\begin{aligned}&= 2\frac{1}{2} + \frac{2}{3} + \frac{4}{5} - \left(1\frac{3}{4} + \frac{5}{6}\right) \\ &= \frac{5}{2} + \frac{2}{3} + \frac{4}{5} - \left(\frac{7}{4} + \frac{5}{6}\right) \\ &= \frac{119}{30} - \frac{31}{12} \\ &= \frac{83}{60}.\end{aligned}$$

(Verify the details!) □

Recall that an odd power of a negative number is negative, while an even power of a negative number is positive. These and other rules of exponents apply unchanged to fractions. (See Section 2.7.)

Example 135.

$$\left(-\frac{2}{3}\right)^3 = -\frac{8}{27}$$

$$\left(-\frac{2}{3}\right)^2 = \frac{4}{9}$$

$$\left(-\frac{2}{3}\right)^0 = 1.$$

Positive fractions, like all positive numbers, have two square roots, one positive, the other negative. Square roots of negative fractions, like square roots of all negative numbers, are undefined. As before, the symbol $\sqrt{\ }$ indicates the *positive* square root. For example, $\sqrt{\frac{4}{9}} = \frac{2}{3}$ because $\left(\frac{2}{3}\right)^2 = \frac{4}{9}$. Note that the square root of a positive fraction is not always a fraction:

$$\sqrt{\frac{2}{5}} = \frac{\sqrt{2}}{\sqrt{5}}.$$

This is not a quotient of whole numbers (i.e., a fraction), since neither 2 nor 5 is a perfect square.

3.12.1 Exercises

Perform the operations. If the result is undefined, say so.

1. $\frac{1}{2} + \left(-\frac{1}{3}\right) + \frac{6}{7}$
2. $\left(-6\frac{3}{8}\right) + 4\frac{3}{4} + \left(-2\frac{1}{2}\right) + \frac{7}{8}$
3. $\left(-1\frac{2}{3}\right) + \left(-2\frac{5}{6}\right) + \left(-8\frac{1}{3}\right)$
4. $5 + \left(-3\frac{1}{2}\right) + 4\frac{1}{2} + \left(-2\frac{1}{2}\right) + (-4) + \left(-\frac{1}{2}\right) + 7$
5. $\left(\frac{-7}{9}\right) \times 0$
6. $\left(\frac{-7}{9}\right) \div 0$
7. $\left(\frac{6}{7}\right)\left(-\frac{7}{6}\right)$
8. $\left(-1\frac{6}{7}\right)\left(-1\frac{1}{2}\right)$
9. $\left(-\frac{1}{2}\right)\left(-\frac{2}{3}\right)\left(-\frac{3}{8}\right)\left(\frac{4}{5}\right)$
10. $0 \div (-1000)$
11. $\left(\frac{4}{5}\right) \div \left(-\frac{25}{32}\right)$
12. $\left(-4\frac{1}{2}\right) \div \left(-1\frac{7}{8}\right)$
13. $-10\frac{3}{4} \div 5$
14. $100 \div \frac{-1}{4}$

15. $\left(-\frac{2}{5}\right)^3$

16. $\left(2\frac{1}{2}\right)^2$

17. $\left(-\frac{1}{10}\right)^4$

18. $\sqrt{\frac{81}{4}}$

19. $-\sqrt{\frac{121}{144}}$

20. $-\sqrt{\frac{25}{36}}$

21. $\sqrt{\frac{-25}{36}}$

22. $\sqrt{\frac{-64}{-49}}$

23. $-\sqrt{\frac{1}{16}}$

24. Find the average noon temperature for the first week of January in Barrow, Alaska if the noon temperatures were: Monday: -8° F, Tuesday: 2° F, Wednesday: -3° F, Thursday: -5° F, Friday: 1° F, Saturday: 4° F, Sunday: 9° F.

3.13 Combined operations

Now we put together everything we have learned so far. Recall that when there is more than one operation to be performed in a calculation, we agree to follow the **order of operations**:

1. operations within grouping symbols first;
2. exponents and roots next;
3. multiplications and divisions (in order of appearance) next;
4. additions and subtractions (in order of appearance) last.

We remind you that "*in order of appearance*" means in order *from left to right*, and that "grouping symbols" include parentheses, brackets, braces (curly brackets), the square root symbol, and the horizontal fraction bar as in $\frac{a+b}{c-d}$.

Example 136. Calculate $2 - \frac{3}{5} + \frac{1}{2}$.

Solution. Subtraction comes first in the left-to-right order. Rewriting 2 as $\frac{10}{5}$, we have

$$\begin{aligned}\left(\frac{10}{5}-\frac{3}{5}\right)+\frac{1}{2} &= \frac{7}{5}+\frac{1}{2}\\ &= \frac{14}{10}+\frac{5}{10} \qquad\qquad (\text{LCD} = 10)\\ &= \frac{19}{10}\\ &= 1\frac{9}{10}.\end{aligned}$$

□

Example 137. Calculate $1-\left(\frac{3}{5}+\frac{1}{2}\right)$.

Solution. The parentheses force us to do the addition first. Rewriting $\frac{3}{5}$ and $\frac{1}{2}$ with the LCD 10, we have

$$\begin{aligned}1-\left(\frac{6}{10}+\frac{5}{10}\right) &= 1-\frac{11}{10}\\ &= \frac{10}{10}-\frac{11}{10}\\ &= -\frac{1}{10}.\end{aligned}$$

□

Example 138. Calculate $\left(-\frac{2}{3}\right)^2+1\frac{2}{3}\cdot\frac{1}{10}$.

Solution. We evaluate the expression with an exponent first: $(-\frac{2}{3})^2=\frac{4}{9}$. Then we do the multiplication (converting the mixed number to an improper fraction). Addition comes last.

$$\begin{aligned}\left(-\frac{2}{3}\right)^2+1\frac{2}{3}\cdot\frac{1}{10} &= \frac{4}{9}+1\frac{2}{3}\cdot\frac{1}{10}\\ &= \frac{4}{9}+\frac{5}{3}\cdot\frac{1}{10}\\ &= \frac{4}{9}+\frac{5}{30}\\ &= \frac{4}{9}+\frac{1}{6} \qquad\qquad (\text{cancelling } 5)\\ &= \frac{8}{18}+\frac{3}{18} \qquad\qquad (\text{LCD} = 18)\\ &= \frac{11}{18}.\end{aligned}$$

□

Example 139. Calculate $1\frac{1}{4}\cdot\frac{2}{3}+1\frac{1}{6}\div\left(-1\frac{1}{2}\right)$.

Solution. The order of operations is multiplication, division, addition. We first convert mixed numbers into improper fractions, and change the division into multiplication by the reciprocal.

$$\begin{aligned}
\frac{5}{4}\cdot\frac{2}{3}+\frac{7}{6}\div\left(-\frac{3}{2}\right) &= \frac{10}{12}+\frac{7}{6}\cdot\left(-\frac{2}{3}\right)\\
&= \frac{10}{12}+\left(-\frac{14}{18}\right)\\
&= \frac{5}{6}+\left(-\frac{7}{9}\right) && \text{(lowest terms)}\\
&= \frac{15}{18}+\left(-\frac{14}{18}\right) && \text{(LCD} = 18\text{)}\\
&= \frac{15}{18}-\frac{14}{18}\\
&= \frac{1}{18}.
\end{aligned}$$

□

Example 140. Calculate $\{1+[2+\left(1\frac{1}{4}\cdot\frac{2}{3}-1\frac{1}{6}\div 1\frac{1}{2}\right)]\}$.

Solution. In this example we have so-called *nested* grouping symbols, that is, grouping symbols inside other grouping symbols. In this case, we must work "from the inside out." This means we work within the innermost grouping symbols first. From the previous example we know that

$$1\frac{1}{4}\cdot\frac{2}{3}-1\frac{1}{6}\div 1\frac{1}{2}=\frac{1}{18},$$

thus we get

$$\{1+[2+\left(1\frac{1}{4}\cdot\frac{2}{3}-1\frac{1}{6}\div 1\frac{1}{2}\right)]\}$$

$$=\{1+[2+\left(\frac{1}{18}\right)]\}=\{1+[\frac{37}{18}]\}=\frac{55}{18}=3\frac{1}{18}.$$

□

3.13.1 Exercises

Calculate, using the correct order of operations. Express improper fractions as mixed numbers.

1. $\frac{3}{4}+\frac{2}{3}\div\frac{4}{9}$
2. $2-\left(\frac{3}{4}-3\frac{1}{2}\right)$
3. $\frac{3}{4}-\left(3\frac{1}{2}-2\right)$
4. $\left(8\frac{1}{4}-1\frac{2}{3}\right)\div\frac{1}{3}$

5. $7\frac{1}{2} \div \frac{3}{5} + 1\frac{7}{8} \cdot 2\frac{2}{5}$

6. $\left(\frac{3}{4}\right)^2 + 2\frac{4}{5}\left(-1\frac{1}{4}\right)$

7. $\frac{1}{2} - \left[\frac{1}{3} - \left(\frac{1}{4} - \frac{1}{5}\right)\right]$

8. Find the perimeter of a rectangle with length $2\frac{1}{2}$ ft and width $1\frac{3}{4}$ ft.

9. Julia has two closets. The floor of one is a square $3\frac{1}{2}$ ft on a side. The floor of the other is a rectangle of width $3\frac{1}{2}$ ft and length $4\frac{1}{2}$ ft. She wants to put square tiles on the floors of the closets. How much will it cost if the tiles are \$5 per square foot?

Chapter 4

Decimals and Percents

Decimals are mixed numbers in which the proper fractional part has a denominator which is 10, or 100, or 1000, or, more generally, a *power* of 10. Here are some powers of 10:

$$\begin{aligned}
10^0 &= 1 \\
10^1 &= 10 \\
10^2 &= 100 \\
10^3 &= 1000 \\
10^4 &= 10000 \\
10^n &= 1 \text{ followed by } n \text{ 0's.}
\end{aligned}$$

The number above and to the right of 10 is the *exponent*, or *power*, and indicates the number of times that 10 appears as a factor. Thus

$$10^3 = 10 \times 10 \times 10 = 1000,$$

and so forth.

Here are some examples of mixed numbers in which the denominator of the fractional part is a power of ten, together with their representations as decimals:

$$\begin{aligned}
1\frac{3}{10} &= 1.3 \\
21\frac{37}{100} &= 21.37 \\
13\frac{21}{1000} &= 13.021
\end{aligned}$$

The *decimal point* (which looks like the period at the end of a sentence) separates the whole number part from the proper fractional part. The digits to the right of the decimal point represent the fractional part according to the following rule:

- the digits to the right of the decimal point constitute the numerator;
- the *number* of digits to the right of the decimal point is the power of 10 which constitutes the denominator.

There is no need to show the fraction bar. As we shall see, this makes computation very easy. There are some subtleties to watch out for, however. In the example

$$13\frac{21}{1000} \quad = \quad 13.021,$$

we had to use 021 to represent the numerator 21 because we need *three* digits to show that the denominator is 10^3.

4.1 Decimal place values

In Chapter 1 we showed that any whole number can be written using just the digits $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$, by using the *place value* system. Recall that

$$4267 \text{ stands for } \quad 4 \textit{ thousands} \;+\; 2 \textit{ hundreds} \;+\; 6 \textit{ tens} \;+\; 7 \textit{ ones}.$$

Using powers of 10, we can write this more compactly:

$$4267 = 4 \times 10^3 \;+\; 2 \times 10^2 \;+\; 6 \times 10^1 \;+\; 7 \times 10^0.$$

The decimal point allows us to adjoin more places (to the right of the decimal point), with place values less than 1. Thus

$$23.59 \text{ stands for } \quad 2 \textit{ tens} \;+\; 3 \textit{ ones} \;+\; 5 \textit{ tenths} \;+\; 9 \textit{ hundredths}.$$

The first place to the right of the decimal point has the place value $\frac{1}{10}$, or one *tenth*; the second place to the right of the decimal point has the place value $\frac{1}{100}$, or one *hundredth*, and so forth. More generally,

> The nth place to the right of the decimal point has the place value $\dfrac{1}{10^n}$.

This is a continuation of the pattern we established in Chapter 1: as we move to the right, place values decrease ten-fold (by a factor of *one tenth*). If we adopt the convention that

$$10^{-n} = \frac{1}{10^n},$$

then the sequence of place values, from left to right, is just the sequence of decreasing powers of ten:

$$\ldots, 10^3, \quad 10^2, \quad 10^1, \quad 10^0 \quad \text{(decimal point)} \quad 10^{-1}, \quad 10^{-2}, \quad 10^{-3}, \quad \ldots$$

..., *thousands, hundreds, tens, ones* (decimal point) *tenths, hundredths, thousandths,* ...

Example 141. Name the digit in tens place, and the digit in the hundredths place, in the decimal 304.671.

Solution. The tens (10^1) place is the second place to the left of the decimal point. The digit in that place is 0. The hundredths (10^{-2}) place is the second place to the right of the decimal point. The digit in that place is 7. □

Example 142. What is the digit in the ten-thousandths place of the decimal 0.00286?

Solution. One ten-thousandth is

$$\frac{1}{10000} = \frac{1}{10^4},$$

so the ten-thousandths place is the fourth place to the right of the decimal point. The digit in that place is 8. □

4.1.1 Exercises

1. 19.01 stands for
2. 0.946 stands for
3. 98.6 stands for
4. Name the digit in the hundred-thousandths place in the decimal 1.870457.
5. Express the following mixed numbers as decimals:
   (a) $2\frac{23}{100}$
   (b) $\frac{237}{1000}$
   (c) $19\frac{7}{10}$
   (d) $\frac{12}{10000}$.
6. Express the following decimals as mixed numbers or proper fractions:
   (a) 14.3
   (b) 28.61
   (c) 100.1
   (d) 4.003

4.2 Significant and Insignificant 0's

Decimals are read aloud in the following way:

- read the whole number part;
- read "and" for the decimal point;
- read the digits to the right of the decimal point as if they represented a whole number;
- append the place-name of the last decimal place.

Example 143. (a) 1.13 is read aloud as "one and thirteen hundredths."
(b) 31.045 is read aloud as "thirty-one and forty-five thousandths".
(c) 48.006 is read aloud as "forty-eight and six thousandths."

In the decimal 48.006, the fractional part .006 stands for 0 tenths + 0 hundredths + 6 thousandths. Reading aloud, we simply say "six thousandths." Nonetheless, the two 0's in .006 are absolutely necessary. If we had written .6 instead of .006, we would have named a different fraction, namely, $\frac{6}{10}$, rather than $\frac{6}{1000}$. On the other hand, in the decmial 006.23, the first two 0's are unnecessary, since the whole number 006 is just 6. In one instance the 0's are crucial, and in the other, they are unnecessary.

How do we know which 0's are significant, and which are insignificant? We can precede any whole number or whole number part of a decimal by as many 0's as we want, because these 0's do not change the value of the number (as in $006 = 6$). Similarly, we can *follow* the fractional part of a decimal with as many 0's as we want, because these 0's do not change the value of the number. For example, $.98000 = .98$ because $.98000 = 9 \times 10^{-1} + 8 \times 10^{-2} + 0 \times 10^{-3} + 0 \times 10^{-4} + 0 \times 10^{-5}$. In these cases, the 0's are *insignificant*. In contrast, *internal* 0's hold a place and cannot be omitted.

Example 144. 2.3004 is not equal to 2.304, or to 2.34, because the fractional parts, $\frac{3004}{10000}$, $\frac{304}{1000}$, and $\frac{34}{100}$ are not equal. The 0's in the hundredths and thousandths places of 42.3004 are *significant*, and therefore cannot be omitted.

> In a decimal, a 0 digit is *insignificant*, and may be omitted, if (and only if)
>
> - it precedes the left-most non-zero digit of the whole number part; or
> - it follows the right-most non-zero digit of the fractional part.
>
> All other 0's are *significant*, and cannot be omitted.

Example 145. In the decimal 0304.90800, the 0 between 3 and 4, and the 0 between 9 and 8 are significant; the others are insignificant. Thus

$$0304.90800 = 304.908.$$

Example 146. Does 23400 contain any insignificant 0's?

Solution. 23400 is a whole number. The decimal point is not shown, but is "understood" to be at the rightmost end. It follows that the two 0's in 23400 are internal, and therefore, significant. □

It is sometimes convenient to preserve or adjoin non-significant 0's. For example, when a decimal represents a *proper* fraction, the whole number part is 0, but that 0, to the left of the decimal point, is often retained for emphasis or clarity, even though it is insignificant. Thus the proper fraction $\frac{67}{100} = .67$ is often written as 0.67.

4.3 Comparing Decimals

Insignificant 0's are also useful in comparing decimals as to size. Recall that when fractions have the same (common) denominator, the one with the largest numerator represents the largest number. It is very easy to find a common denominator for two or more decimals, because the number of decimal *places* to the right of the decimal point is all that is needed to determine the denominator: if there are n places to the right of the decimal point, the denominator is 10^n. Given a set of decimals, we just find the one with the largest number of decimal *places* to the right of the decimal point, and use that number to determine the common denominator. There is no computation involved: we simply "pad out" the shorter decimals with insignificant 0's to the right of the decimal place until all have the same number of decimal places. Then it is easy to tell which decimal has the largest numerator, and hence, which is largest.

Example 147. Arrange the decimals in descending order, from largest to smallest: 0.102, 0.09876, 0.2.

Solution. Of the three given decimals, 0.09876 is the longest, with 5 places to the right of the decimal point. So we pad out all three to 5 places, using insignificant 0's: Thus

$$\begin{aligned} 0.102 &= 0.10200 \qquad (4.1)\\ 0.09876 &= 0.09876 \\ 0.2 &= 0.20000 \end{aligned}$$

In this form, the decimals have a common denominator of $10^5 = 100000$:

$$0.102 = \frac{10200}{100000} \quad 0.09876 = \frac{9876}{100000} \quad 0.2 = \frac{20000}{100000}.$$

Now, it is easy to see which has the largest numerator, which the second-largest, and which the smallest. Hence we have

$$0.2 > 0.102 > 0.09876.$$

□

4.3.1 Exercises

Eliminate the insignificant 0's in the following decimals.

1. 210304.0900
2. 00206.006070
3. 210.00
4. 21030900

Arrange each group of decimals in descending order, from largest to smallest:

5. 0.2, .009, .121.
6. 1.31, 1.9, 1.224.
7. 0.106, 0.5, 0.61.
8. 9.104, 9.14, 9.137, 9.099.

4.4 Rounding-off

The numbers

$$0.1, \quad 0.11, \quad 0.111, \quad 0.1111, \quad 0.11111$$

get closer and closer together (on the number line) as we move from left to right. After a few steps, they are almost too close together to visualize. The second number was obtained from the first by adding $\frac{1}{100}$, and the last from the second-to-last by adding just $\frac{1}{100000}$, a very small quantity indeed. Such small quantities can be important in the sciences, but even in circumstances where precision is important, there is always a limit beyond which small differences become negligible – not worth worrying about. (For example, when your bank calculates the interest on your savings account, it calculates, but then ignores, amounts that are less than one half of one cent.) Being precise, but not overly precise, is the purpose of *rounding-off* numbers.

To round off a number, we must first decide how precise to be. That means choosing the place whose value we consider the smallest worth worrying about. For example, in negotiating an annual salary, we would probably not argue about amounts less than $100, but in negotiating an hourly wage, we would be willing to argue about pennies. In the first instance, we would round off our dollar amounts to the nearest hundred, and in the second, to the nearest hundredth (cent).

Example 148. \$175 is closer to \$200 than to \$100, so, rounding to the *nearest hundred*, we say that an annual salary of \$36,175 is approximately \$36,200. We write

$$36175 \approx 36200 \quad \text{(to the nearest hundred).}$$

The symbol $\approx$ means "approximately equal to."

Example 149. 16.84 is closer to 16.80 than to 16.90, so, *to the nearest tenth*,

$$16.84 \approx 16.8.$$

Notice that we dropped the insignificant 0 in 16.80.

Example 150. 0.135 is exactly halfway between 0.130 and 0.140, so, if we want to round *to the nearest hundredth*, it is not clear whether we should round "up" to 0.140, or "down" to 0.130. Here, we have to make up a rule. The convention is that "halfway is almost home," so we round up.

$$0.135 \approx 0.14 \quad \text{(to the nearest hundredth),}$$

where, again, we have dropped the insignificant 0 in 0.140.

Below is a summary of the procedure for rounding off a number expressed as a decimal to a given place, called the *round-off place*.

> To round off a decimal to a given place (the *round-off place*):
>
> 1. Preserve the digit in the round-off place if the right-neighboring digit is less than 5; otherwise, increase the digit in the round-off place by 1;
> 2. Replace all digits to the right of the round-off place by 0's;
> 3. Eliminate insignificant 0's (except a 0 in the round-off place).

Example 151. Round 26.03 to the nearest tenth.

Solution. The round-off place is the *tenths* place, and the right-neighboring digit (3) is less than 5, so we preserve the digit in the round-off place (0) and replace all digits to the right (there is only one) with 0. We get 26.00. Here the exception in step 3 of the procedure applies: both the 0's are insignificant, but we keep the 0 in the round-off place (otherwise it might seem as if we had rounded to the nearest unit). Thus

$$26.03 \approx 26.0 \quad \text{(to the nearest tenth)}$$

□

Example 152. Round 51 to the nearest hundred.

Solution. The round off place (hundreds) doesn't appear, but we can show it by adjoining an insignificant 0 on the left: $51 = 051$. The right neighbor of the round-off digit is 5, so, by the "halfway" convention, we round the 0 in the round-off place up to 1, and replace all digits to the right with 0. We get

$$51 \approx 100 \quad \text{(to the nearest hundred).}$$

Notice that the two 0's in 100 are significant, and cannot be eliminated! □

One final remark about step 2 of the procedure. If the digit in the round-off place is 9, and we need to round up (to 10), we must carry the 1 to the left-neighboring place.

Example 153. Round 6.597 to the nearest hundredth.

Solution. The 9 in the round-off place has right-neighbor 7, which is greater than 5. So we increase 9 to 10, and carry the 1 to the (left-neighboring) tenths place:

$$6.597 \approx 6.60 \quad \text{(to the nearest hundredth)}$$

Notice we preserved the insignificant 0, since it is in the round-off place. □

4.4.1 Exercises

Round off each number twice: (a) to the nearest ten; (b) to the nearest hundredth.

1. 304.0900
2. 96.075
3. 115.497
4. 100.0055
5. 7.009

4.5 Adding and Subtracting Decimals

One of the advantages of decimals over ordinary fractions is ease of computation. The four operations (addition, subtraction, multiplication and division) are done almost exactly as with whole numbers. The only question is where to put the decimal point.

Addition and subtraction are the easiest: we line up the numbers vertically, with the ones places aligned on top of eachother. This puts the decimal points in alignment as well. The decimal point in the sum is placed directly below the decimal points in the numbers being added. Carrying and borrowing are done as with whole numbers.

Example 154. Find the sum of 96.8 and 342.03.

Solution. Line up the numbers vertically so that the ones places (and hence the neighboring decimal points) are directly on top of eachother.

$$\begin{array}{ccccccc} & & 9 & 8 & . & 8 & \\ + & 3 & 4 & 2 & . & 0 & 3 \\ \hline \end{array}$$

For clarity, you can "pad out" the top number with an insignificant 0 so that both numbers have the same number of decimal places. (This is not strictly necessary: just remember that an empty place is occupied by a 0.) Place a decimal point, vertically aligned with the others, where the sum will go.

$$\begin{array}{ccccccc} & & 9 & 8 & . & 8 & 0 \\ + & 3 & 4 & 2 & . & 0 & 3 \\ \hline & & & & . & & \end{array}$$

As before, add the digits in each column starting from the right-most one, carrying when necessary:

$$\begin{array}{rrrrcrr} & & 9 & 8 & . & 8 & 0 \\ + & 3 & 4 & 2 & . & 0 & 3 \\ \hline & 4 & 4 & 0 & . & 8 & 3 \end{array}$$

The sum is 440.83. □

Example 155. Find the sum of 804.09, 56.384, 107, and 0.205.

Solution. Line up the numbers vertically at the decimal points (equivalently, at the ones place). In the whole number 107, the decimal point is understood to be directly to the right of the ones place. Add the columns from right to left, imagining 0's in the empty places, and carrying when necessary.

$$\begin{array}{rrrrcrrr} & 8 & 0 & 4 & . & 0 & 9 & \\ & & 5 & 6 & . & 3 & 8 & 4 \\ & 1 & 0 & 7 & . & & & \\ + & & & 0 & . & 2 & 0 & 5 \\ \hline & 9 & 6 & 7 & . & 6 & 7 & 9 \end{array}$$

□

Subtraction follows the same vertical alignment rule.

Example 156. Find the difference $50 - 4.706$.

Solution. The larger number, 50, goes on top. We fill it out with insignificant 0's so that both the minuend and the subtrahend have the same number of decimal places.

$$\begin{array}{rrrcrrr} & 5 & 0 & . & 0 & 0 & 0 \\ - & & 4 & . & 7 & 0 & 6 \\ \hline & & & . & & & \end{array}$$

Borrowing from the *tens* place of the subtrahend, we can write

$$\begin{aligned} 50.000 &= 5 \textit{ tens } + \ 0 \textit{ ones } + \ 0 \textit{ tenths } + \ \text{ etc.} \\ &= 4 \textit{ tens } + \ 9 \textit{ ones } + \ 9 \textit{ tenths } + \ 9 \textit{ hundredths } + \ \{10\} \textit{ thousandths} \\ &= 49.99\{10\} \end{aligned}$$

Then our subtraction becomes

$$\begin{array}{rrrcrrr} & 4 & 9 & . & 9 & 9 & \{10\} \\ - & & 4 & . & 7 & 0 & 6 \\ \hline & 4 & 5 & . & 2 & 9 & 4 \end{array}$$

□

If we encounter negative decimal fractions, the usual sign rules apply (see Section 3.12).

Example 157. Find the sum $-3.6 + 4.9$.

Solution. The numbers have opposite signs, and the number with the larger absolute value (4.9) determines the sign of the sum (+). The absolute value of the sum is the difference $|\ 4.9\ | - |\ -3.6\ | = 4.9 - 3.6$. Thus

$$-3.6 + 4.9 = +(4.9 - 3.6) = +1.3 = 1.3.$$

□

Example 158. Add: $43.6 + (-5.8) + (-135) + 69.5 + (-134) + 158.7 + (-162.3)$

Solution. We add all the positive numbers,

$$\begin{array}{r} 43.6 \\ 69.5 \\ +\ 158.7 \\ \hline 271.8 \end{array}$$

and the absolute values of all the negative numbers,

$$\begin{array}{r} 5.8 \\ 135. \ \ \\ 134. \ \ \\ +\ 162.3 \\ \hline 437.1 \end{array}$$

Treating the subtotal associated with the negative numbers as negative, we add $271.8 + (-437.1)$. By the rule for adding signed numbers with opposite signs,

$$271.8 + (-437.1) = -(437.1 - 271.8) = -165.3.$$

□

Example 159. Subtract: $3.359 - 10.08$.

Solution. $3.359 - 10.08 = 3.359 + (-10.08) = -(10.08 - 3.359)$. We first subtract the absolute values, and affix the negative sign at the end.

$$\begin{array}{r} 10.080 \\ -\ \ 3.359 \\ \hline 6.721 \end{array}$$

Remembering that the sign was negative, $3.359 - 10.08 = -6.721$. □

Example 160. Last week, a business received checks from clients in the amounts of \$350.65, \$461.00 and \$900.78, and paid bills in the amounts of \$261.50, \$551.00 and \$78.70. What was the net profit or loss for the week?

Solution. Bills paid are losses, so we treat them as negative numbers; checks received are profits, counted as positive numbers. The total profits are

$$\begin{array}{r} 350.65 \\ 461.00 \\ +\ 900.78 \\ \hline 1{,}712.43 \end{array}$$

and the total losses are

$$\begin{array}{r} 261.50 \\ 551.00 \\ +\ 78.70 \\ \hline 891.20 \end{array}$$

Affixing a negative sign to the total losses, we calculate the net profit (or loss) as the sum

$$1712.43 + (-891.20) = +(1712.43 - 891.20) = 821.23.$$

There was a net profit of \$821.23 for the week. □

4.5.1 Exercises

Find the sums or differences:

1. $680.48 + 56.09$
2. $804.09 + 5.8409$
3. $58.09 - 32.1$
4. $4.09 + 0.38409$
5. $14.093 - 6.39$
6. $100 - 23.441$
7. $830 - 16.61$
8. $80 - 56.384$
9. $-13.38 + (-9.03)$
10. $-1001.36 + 909$
11. $-8.2 + (-198.5) + 8.2$
12. $44 + (-5.5) + 28.8 + 36 + (-19.1) + (-8)$
13. $18.50 + (-21.25) + (-69.95) + 13.50 + 79.99 + (-86.50)$
14. Find the sum of 64.09, 15.3, 4, and 9.09.
15. The difference of 20 and the sum of 4.6 and 0.07

16. $.65 - (-6.4)$

17. Subtract 5.5 from 2.4

18. Subtract -53 from 68.6

19. $-8.88 - (-1.11)$

20. Subtract 2.2 from $\frac{1}{5}$

21. Find the net profit (or loss) of a business that received checks in the amounts of \$453.05, \$865.50 and \$300.25, and paid bills in the amounts of \$561.50, \$449.25, \$798.75 and \$75.25.

22. A sunken car is salvaged from the bottom of a lake. The elevation of the lake bottom is -66.2 feet. The car is lifted by a crane to a height 79.5 feet above lake level. Through what vertical distance was the car lifted?

4.6 Multiplying and Dividing Decimals by Powers of 10

What happens when we multiply a whole number by 10? *Ones* become *tens*, *tens* become *hundreds*, *hundreds* become *thousands*, etc. For example, in the multiplication

$$234 \times 10 = 2340,$$

the 4 *ones* in 234 turn into the 4 *tens* in 2340, the 3 *tens* turn into 3 *hundreds* and so forth. (In the product, there are no longer any *ones*, and that fact must be recorded with a significant 0 in the *ones* place.)

Remember that a whole number is a decimal, with the decimal point understood to be immediately to the right of the ones place. Making the decimal point explicit in our example,

$$234.0 \times 10 = 2340.$$

We can describe what happens this way: when we multiply by 10, all the digits shift one place to the left, including the insignificant 0 which was understood to be in the *tenths* place of the whole number 234. (In this description, we imagine the decimal point remaining fixed.) Similarly, if we multiply a whole number by 100, all the digits shift *two* places to the left, including the *two* insignificant 0's which are understood to be in the *tenths* and *hundredths* places. Thus, $597 \times 100 = 59\ 700$, or, more explicitly,

$$597.00 \times 100 = 59\ 700.$$

In general, if we multiply a whole number by any positive power of 10, say, by 10^n, all the digits shift n places to the left, including the n insignificant 0's understood to be in the n places to the right of the decimal point (while the decimal point remains fixed). Thus, for example,

$$281 \times 10^4 = 2\ 810\ 000.$$

If there are non-zero digits to the right of the decimal place, they shift in exactly the same way. Here are some examples:

$$
\begin{aligned}
38.623 \times 100 &= 3862.3 \\
0.6 \times 10 &= 6 \\
0.0031 \times 1000 &= 3.1 \\
12.09 \times 10^4 &= 120\,900 \\
100 \times 10^2 &= 10000.
\end{aligned}
$$

Note that insignificant 0's have been omitted from the products.

It is often convenient to imagine that, when multiplying by a power of 10, the digits remain fixed while the decimal point moves (to the *right*). (Electronic calculators work this way, using a "floating" decimal point.) This description leads to a very easy rule for multiplying a decimal by a positive power of 10:

> To multiply a decimal by 10^n, move the decimal point n places to the right.

If we *divide* a decimal by 10, *hundreds* becomes *tens*, *tens* become *ones*, *ones* become *tenths*, *tenths* become *hundredths*, etc. The whole discussion above can be repeated, except that, in this case of division, digits shift *to the right*, or, equivalently, the decimal point moves *to the left*. The easy rule is

> To divide a decimal by 10^n, move the decimal point n places to the left.

Here are some examples:

$$
\begin{aligned}
623 \div 10 &= 62.3 \\
0.023 \div 100 &= 0.00023 \\
480 \div 10 &= 48 \\
37.5 \div 10^3 &= 0.0375
\end{aligned}
$$

Example 161. Divide $3.738 \div (-100)$.

Solution. The quotient is negative because the numbers have opposite signs. Thus

$$3.738 \div (-100) = -(3.738 \div 100) = -0.03738.$$

□

With the convention that

$$10^{-n} = \frac{1}{10^n},$$

division by 10^n can be thought of as multiplication by 10^{-n}. For example,

$$5 \div 100 = 5 \times 10^{-2} = 0.05$$
$$6.5 \div 10 = 6.5 \times 10^{-1} = 0.65$$
$$86.37 \div 10000 = 86.37 \times 10^{-4} = 0.008637$$
$$8 \div 100000 = 8 \times 10^{-5} = 0.00008$$

The left movement of the decimal point is indicated by the negative exponent.

A **caution** about negative exponents:

> 10^{-n} is **not** a negative number!

Example 162. Show that -10^3 and 10^{-3} are not equal.

Solution. The first one

$$-10^3 = -1000$$

is a negative number with a relatively large absolute value, while the second one

$$10^{-3} = 0.001$$

is a positive number with a relatively small absolute value. □

4.6.1 Exercises

Multiply or divide each decimal by the indicated power of 10.

1. 6080.48×10
2. $6080.48 \div 10$
3. 19×10^{-1}
4. 804.09×10^2
5. $804.09 \div 10^2$
6. 0.0908×10^3
7. 0.0908×10^{-3}
8. $0.0908 \times (-10^3)$
9. 38×10^{-3}
10. 260804.09×10^4
11. $260804.09 \div 10^5$
12. 48.3×10^{-3}
13. What is the value, in dollars, of 5 million pennies?

4.7 Multiplication of general decimals

We already know how to multiply and divide decimals when one of them is a power of 10. If neither factor is a power of ten, there is more to the process than just moving the decimal point. Consider the following example.

Example 163. Multiply 0.3×0.07.

Solution. Writing this as a product of ordinary fractions, we get

$$0.3 \times 0.07 = \frac{3}{10} \times \frac{7}{100} = \frac{3 \times 7}{10 \times 100} = \frac{21}{1000}.$$

The last fraction can be written $21 \div 10^3$, or 21×10^{-3} which, from the results of the last section, is equal to 0.021. Thus,

$$0.3 \times 0.07 = 0.021.$$

□

Multiplication of two decimals always involves a whole number multiplication for the numerator ($3 \times 7 = 21$ in the example) and a multiplication of powers of 10 for the denominator ($10 \times 100 = 10^1 \times 10^2 = 1000 = 10^3$ in the example). By looking back at the previous section, it is easy to see the product of two powers of 10 is itself a power of 10. Which power of 10? The rule is simple:

$$\boxed{10^n \times 10^k = 10^{(n+k)}}$$

This implies that the number of decimal places in the product of two (or more) decimals is the sum of the numbers of decimal places in the factors. Thus any set of decimals can be multiplied by following a two step procedure:

To multiply two or more decimals:

- Multiply the decimals as if they were whole numbers, ignoring the decimal points (this gives the numerator of the decimal fraction);
- Add the number of decimal places in each factor (this gives the denominator of the decimal fraction by specifying the number of decimal places).

Example 164. Find the product 21.02×0.004.

Solution. Temporarily ignoring the decimal points, we multiply $2102 \times 4 = 8408$. Now 21.02 has two decimal places, and 0.004 has three. So the product will have $2+3=5$ decimal places. In other words,

$$21.02 \times 0.004 = 8408 \div 10^5 = 0.08408.$$

□

Example 165. Find the product of 12, 0.3, and 0.004.

Solution. Ignoring the decimal points, the product is $12 \times 3 \times 4 = 144$. The numbers of decimal places in the three decimals are, left to right, 0, 1 and 3, which add up to 4. Thus,

$$12 \times 0.3 \times 0.004 = 144 \div 10^4 = 0.0144.$$

□

Example 166. Find the product $(6.30)(-2.05)$.

Solution. The product is negative since the numbers have opposite signs. Temporarily ignoring the decimal points, and the signs (i.e., working with the absolute values) we have

$$\begin{array}{r} 6\,3\,0 \\ \times \quad\quad 2\,0\,5 \\ \hline 3\,1\,5\,0 \\ 1\,2\,6\,0 \\ \hline 1\,2\,9\,1\,5\,0 \end{array}$$

Inserting a total of $4 = 2 + 2$ decimal places counting from the right, and recalling that our product is negative, we conclude that

$$(6.30)(-2.05) = -12.915.$$

□

4.7.1 Exercises

Find the following products.

1. 68.4×23
2. -804×6.2
3. 26.09×0.004
4. 4.09×93
5. $100 \times (-9.9)$
6. 14.093×6.39
7. 64.9×0.345
8. 0.0001×0.001
9. 1000×0.053
10. $6 \times (-0.9) \times 0.02 \times (-0.001)$

4.8 Division of a decimal by a whole number

If the divisor is a whole number, the division procedure is almost exactly like the long division procedure for whole numbers. The only difference is this: a decimal point is placed where the quotient will go, vertically aligned with the decimal point in the dividend. For example, to set up the division 12.0342÷31, we write, as before

```
        _____________
31     | 1 2 . 0 3 4 2
```

Then we put a decimal point vertically aligned with the decimal point in the divisor, where the quotient will go:

```
             .
        _____________
31     | 1 2 . 0 3 4 2
```

Now the long division is performed with no further attention paid to the decimal point. In this example, since 31 is bigger than 12, we put 0 in the units place of the quotient. Continuing, we estimate that 31 goes into 120 3 times, so 3 is placed in the next (tenths) place in the quotient, and 3×31 is subtracted from 120 to yield 27. (Note that the decimal point between 2 and 0 is ignored at this point.)

```
           0 . 3
        _____________
31     | 1 2 . 0 3 4 2
        − 9   3
        _____________
           2   7
```

We bring down the next digit in the dividend, which is 3, and start over with a new dividend of 273. We estimate that 31 goes into 273 8 times. $8 \times 31 = 248$, which is subtracted from 273 to yield 25.

```
        0 . 3 8
31   | 1 2 . 0 3 4 2
     - 9 3
       2 7 3
     - 2 4 8
           2 5
```

We bring down the next digit in the dividend, which is 4, and start over with a new dividend of 254, etc. The remaining steps are as follows.

```
        0 . 3 8 8 2
31   | 1 2 . 0 3 4 2
     - 9 3
       2 7 3
     - 2 4 8
           2 5 4
         - 2 4 8
              6 2
            - 6 2
                0
```

Thus, $12.0342 \div 31 = 0.3882$, with 0 remainder. We can check the result by multiplication, of course: $31 \times 0.3882 = 12.0342$, as it should.

In this example, the remainder was 0. In fact, because of the decimal point, we never need to write remainders again. The reason is that we can always adjoin insignificant 0's to to the end of the dividend, bring them down, and so continue the division process.

Example 167. Perform the division $12.4 \div 5$.

Solution. We set up the long division process as before, with a decimal point where the quotient will go, directly above the decimal point in the dividend.

```
           .
5    |  1 3 . 4
```

After two steps, we arrive at

```
      2 . 6
   ___________
5 |  1 3 . 4
   - 1 0
   ___________
       3   4
     - 3   0
   ___________
           4
```

Now, instead of stopping with a remainder of 4, we adjoin an insignificant 0 to the dividend, bring it down, and continue the process:

```
      2 . 6
   ___________
5 |  1 3 . 4 0
   - 1 0
   ___________
       3 4
     - 3 0
   ___________
         4 0
```

5 goes into 40 exactly 8 times:

```
      2 . 6 8
   ___________
5 |  1 3 . 4 0
   - 1 0
   ___________
       3 4
     - 3 0
   ___________
         4 0
     -   4 0
   ___________
           0
```

The 0 remainder signals the end of the process. Thus $13.4 \div 5 = 2.68$. □

Since we can adjoin as many insignificant 0's as we need, it might seem that we can always continue until we get a 0 remainder. But this doesn't always happen. Sometimes, we never get a 0 remainder, but instead, we get a sequence of non-zero remainders that repeats, forever. In this case, we get what is know as a *repeating* decimal.

Example 168. Perform the division $3.2 \div 11$.

Solution. We set up the long division process as usual.

$$
\begin{array}{r|l}
 & . \\ \cline{2-2}
11 & 3.2
\end{array}
$$

11 goes into 32 twice with a remainder of 10:

$$
\begin{array}{r|l}
 & .2 \\ \cline{2-2}
11 & 3.2 \\
 & -\,2\;2 \\ \cline{2-2}
 & \,1\;0
\end{array}
$$

Adjoin an insignificant 0 to the dividend, bring it down, and perform another step: 11 goes into 100 9 times with a remainder of 1.

$$
\begin{array}{r|l}
 & .29 \\ \cline{2-2}
11 & 3.20 \\
 & -\,2\;2 \\ \cline{2-2}
 & \,1\;00 \\
 & -\;99 \\ \cline{2-2}
 & \;1
\end{array}
$$

Now if we adjoin another insignificant 0 to the dividend, and bring it down, we see that 11 goes into 10 0 times. We record that with a 0 digit in the quotient, adjoin another insignificant 0 to the dividend, and bring it down. Now we see that (as above) 11 goes into 100 9 times with a remainder of 1.

```
          . 2 9 0 9
11  |   3 . 2 0
   − 2  2
   -----------------
        1  0 0
   −       9 9
   -----------------
            1 0 0
      −       9 9
   -----------------
                1
```

It is evident that these last two steps will now repeat again and again, forever. The decimal will never terminate, since a 0 remainder will never occur. Still, it is easy to describe the quotient: it will consist of 29 after the decimal point, followed by an endless string of 09's. To indicate this, we put a bar over the repeated string:

$$0.2909090909090909\cdots = 0.29\overline{09}.$$

Thus, $3.2 \div 11 = 0.29\overline{09}$. □

We emphasize that it may take a while for the digits to start repeating, and the bar is only placed over the repeating part. For example, the repeating decimal

$$0.796812341234123412341234\ldots$$

is written, using the bar notation, as

$$0.7968\overline{1234}.$$

The repeated string can be quite long, or just a single digit. Here are some examples, which you should verify by carrying out the division process:

$$3 \div 7 = 0.\overline{428571}$$
$$2 \div 11 = 0.\overline{09}$$
$$5 \div 3 = 1.\overline{6}$$

Sometimes, we don't care whether a decimal terminates or repeats. For example, if we know in advance that we can round off our answer to a given decimal place, we carry out the division only as far as the right neighboring place and stop (this gives us all we need to know to round off).

Example 169. Suppose you bought 900 buttons for \$421. To the nearest cent, how much did you spend for each button?

Solution. We need to perform the division $421 \div 900$. Since we are rounding to the nearest *hundredth* (cent), we need only carry out the division as far as the *thousandths* place. Since the thousandths place is the the third place to the right of the decimal point, we adjoin three insignificant 0's to the dividend, setting up the division as $421.000 \div 900$. We omit the details, but you can verify that the quotient, carried out to three decimal places, is 0.467, which, rounded to the nearest hundredth, is 0.47. Therefore, the buttons cost approximately 47 cents each.

If you carried out the division process a little further, you noticed that the quotient is actually a repeating decimal:

$$421 \div 900 = 0.46777777\cdots = 0.46\overline{7}.$$

□

4.8.1 Exercises

Perform the indicated divisions. Use the bar notation for repeating decimals. If you round off, indicate to what place.

1. $91 \div 20$
2. $14.68 \div 5$
3. $23 \div 90$
4. $86 \div 71$
5. $6.02 \div 9$
6. $4.19 \div 13$
7. $804.09 \div 215$
8. $353 \div 37$
9. $17 \div 19$
10. $38.8 \div 40$

4.9 Division of a decimal by a decimal

What do we do when the divisor is a decimal? For example, how do we perform the division $8.61 \div 2.5$? The answer is simple. Just multiply both the dividend and divisor by a power of 10 sufficiently large to make the divisor into a whole number, and then proceed as before. In this example, if we multiply both numbers by 10^1, the divisor 2.5 turns into the whole number 25 and the dividend turns into 86.1. Recall that two fractions are *equivalent*, that is, represent the same number, if one is obtained from the other by multiplying both numerator and denominator by the same nonzero number. In this case, the convenient choice of nonzero multiplier is $10^1 = 10$, because it turns the denominator (divisor) into a whole number. Thus,

$$\frac{8.61}{2.5} = \frac{8.61 \times 10}{2.5 \times 10} = \frac{86.1}{25}.$$

In other words, the division problem $8.61 \div 2.5$ has the same quotient as the division problem $86.1 \div 25$. We do the latter problem exactly as in the previous section.

```
         3 . 4 4 4
25  | 8 6 . 1 0 0
    - 7 5
      1 1  1
    - 1 0  0
          1 1 0
        - 1 0 0
            1 0 0
          - 1 0 0
                0
```

Since the remainder is 0, the quotient is a terminating decimal. $8.61 \div 2.5 = 3.444$. (Question: why didn't we write $3.\overline{4}$?)

Example 170. Perform the division $32.067 \div 6.41$ and round off the quotient to the nearest hundredth.

Solution. There are two decimal places in the divisor, 6.41, so multiplying it by $10^2 = 100$ will give us a whole number: $6.41 \times 10^2 = 641$. Doing the same to the dividend yields the equivalent division problem,

$$3206.7 \div 641.$$

Since we are rounding off to the nearest hundredth, we will need to carry out the division to the thousandths place. We therefore add two insignificant 0's to the dividend, and perform the division as follows:

$$\begin{array}{r|l} & 5.002 \\ \hline 641 & 3206.700 \\ - & 3205 \\ \hline & 1\,700 \\ & -\,1\,282 \\ \hline & 418 \end{array}$$

Since the digit in the thousandths place of the quotient is less than 5, we preserve the digit in the hundredths place. The quotient is ≈ 5.00 (to the nearest hundredth). □

Example 171. Divide $16 \div (-0.25)$.

Solution. The quotient is negative because the signs are opposite. We can either perform the long division

$$0.25\overline{)16} \qquad \text{equivalent to} \quad 25\overline{)1600},$$

or convert 0.25 to the fraction $\frac{1}{4}$ and multiply by its reciprocal. The latter seems easier:

$$16 \div (-0.25) = -\left(16 \div \frac{1}{4}\right) = -(16 \times 4) = -64.$$

□

4.9.1 Exercises

Write each division as an equivalent division with a whole number divisor.

1. $680.4 \div 1.01$
2. $0.48 \div 11.4$
3. $804.09 \div 9.18$
4. $9.8 \div 0.0215$

Perform the divisions.

5. $4.19 \div 0.5$

6. $4.09 \div 0.21$

7. $353 \div 2.5$

8. $0.004 \div 0.002$

9. $29.997 \div 0.01$

4.10 Order of magnitude, Scientific notation

In the sciences, we often see very large and very small numbers. For example, the distance that light travels in one year (known as a "light-year") is approximately

$$5,870,000,000,000 \quad \text{miles.}$$

At the other extreme, the mass of an electron is approximately

$$0.000\ 000\ 000\ 000\ 000\ 000\ 000\ 000\ 000\ 91 \quad \text{grams.}$$

The **order of magnitude** of a number N is defined to be the unique integer m such that the absolute value of

$$N \times 10^{-m}$$

is a number between 1 and 10 (including 1 but excluding 10.)

Example 172. Find the orders of magnitude of (a) the distance light travels in a year; (b) the mass of an electron.

Solution. (a) The decimal point in $5,870,000,000,000$ must be moved left 12 places (count them!) to obtain a number between 1 and 10 (namely, 5.87). This is equivalent to multiplying by 10^{-12}. Thus, one light year has order of magnitude 12. (b) The decimal point in

$$0.000\ 000\ 000\ 000\ 000\ 000\ 000\ 000\ 000\ 91$$

must be moved right 28 (!) decimal places to obtain a number between 1 and 10 (namely, 9.1). This is equivalent to multiplying by $10^{28} = 10^{-(-28)}$. So the tiny mass of an electron has negative order of magnitude -28. □

'Small' positive numbers (*less than* 1) have *negative* order of magnitude; 'large' numbers (greater than or equal to 10) have positive order of magnitude. Numbers between 1 and 10 but not equal to 10 have order of magnitude 0. (10 itself has order of magnitude 1.)

Any number – huge, tiny or anywhere in between – can be written compactly using the order of magnitude and the number between 1 and 10 (and not equal to 10) found in obtaining it. Thus

$$\text{A light-year} = 5.87 \times 10^{12} \quad \text{miles}$$
$$\text{The mass of an electron} = 9.1 \times 10^{-28} \quad \text{kilograms.}$$

Numbers written like this are in **scientific notation**.

Example 173. Express the following numbers in scientific notation:

$$0.0023 \quad 1.0 \quad 5 \quad 10 \quad 102 \quad 8,560 \quad 8,000,000.$$

Proof. The orders of magnitude are, respectively, -3, 0, 0, 1, 2, 3, 6. So we have

$$\begin{aligned} 0.0023 &= 2.3 \times 10^{-3} \\ 1.0 &= 1.0 \times 10^{0} \\ 5 &= 5 \times 10^{0} \\ 10 &= 1 \times 10^{1} \\ 102 &= 1.02 \times 10^{2} \\ 8,560 &= 8.56 \times 10^{3} \\ 8,000,000 &= 8 \times 10^{6}. \end{aligned}$$

□

Example 174. Express the following numbers in scientific notation:

$$11 \times 10^{4} \quad \text{and} \quad 0.06 \times 10^{-8}$$

Solution. Although these numbers look like they are *already* in scientific notation, they are not. Both multipliers, 11 and 0.06, do not meet the criterion: they are not numbers between 1 and 10. This is easily remedied. We first express the multipliers in scientific notation, and then, to get the proper order of magnitude, we use the exponent sum rule

$$10^{n} \times 10^{k} = 10^{n+k}$$

for any exponents n, k. For the first number

$$11 \times 10^{4} = 1.1 \times 10^{1} \times 10^{4} = 1.1 \times 10^{5}.$$

For the second number,

$$0.06 \times 10^{-8} = 6 \times 10^{2} \times 10^{-8} = 6 \times 10^{-6}.$$

The order of magnitude -6 was obtained from the signed number addition $2 + (-8)$. □

Associativity of multiplication makes it easy and convenient to multiply numbers in scientific notation. The powers of 10 can be associated and multiplied using exponent sum rule; the numbers between 1 and 10 are multiplied in the ordinary way. The only slight complication is that the resulting number may not be in scientific notation. That is easily remedied, as the previous example shows.

Example 175. Express the product $(1.5 \times 10^{12})(8 \times 10^{-5})$ in scientific notation.

Solution. Associating the powers of 10 and the ordinary numbers, we get

$$\begin{aligned}(1.5 \times 10^{12})(8 \times 10^{-5}) &= 1.5 \times 8 \times 10^{12} \times 10^{-5} \\ &= 12 \times 10^{12-5} \\ &= 12 \times 10^{7}.\end{aligned}$$

This is not yet in scientific notation, since 12 is not a number between 1 and 10. But

$$12 = 1.2 \times 10^{1},$$

so the product, in scientific notation, is

$$\begin{aligned}12 \times 10^{7} &= 1.2 \times 10^{1} \times 10^{7} \\ &= 1.2 \times 10^{8}.\end{aligned}$$

□

Note how tedious this example would have been without the use of scientific notation. You would need a large sheet of paper, and many rows of 0's to compute

$$1,500,000,000,000 \times 0.00008 = 120,000,000.$$

4.10.1 Exercises

Find the order of magnitude of each number.

1. 89
2. 1,056,503
3. 0.99
4. 0.000052
5. 8
6. 12
7. 1,181,000,000,000

Express each number in scientific notation.

8. 89
9. 1,056,503
10. 0.99
11. 0.000052
12. 8
13. 12

14. $1,181,000,000,000$

15. 35×10^{-9}

16. 0.0061×10^{7}

17. 100×1000

Express the products in scientific notation.

18. $(5 \times 10^{13})(4 \times 10^{8})$

19. $(1.1 \times 10^{-9})(9.5 \times 10^{15})$

20. $(6.03 \times 10^{23})(1 \times 10^{-9})$

4.11 Percents, Conversions

A *percent* is a fraction in which the denominator is exactly 100. So it is a special type of decimal fraction. The word "percent" comes from the Latin phrase *per centum* meaning "out of 100," and is symbolized by %. For example,

$$97\% \quad \text{means} \quad \frac{97}{100} \quad \text{or} \quad 0.97.$$

Percents need not be whole numbers, and they need not represent proper fractions. For example,

$$150\% = \frac{150}{100} = 1.5$$
$$0.5\% = \frac{0.5}{100} = 0.005$$

Fractions, decimals, and percents represent the same quantities in different ways, and we need to know how to convert one to another.

A decimal can be converted to a percent by moving the decimal point two places to the right and adjoining the % symbol. This is the same as multiplying the decimal fraction by 100, which exactly cancels the denominator, leaving just the numerator (the percent). Thus

$$0.68 = 68\%$$
$$2.05 = 205\%$$
$$0.708 = 70.8\%$$
$$1.4 = 140\%$$
$$0.0067 = 0.67\%$$

To convert a percent back into a decimal, we simply divide by 100, which, as we know, is equivalent to moving the decimal point two places to the left. Thus

$$92\% = 0.92$$
$$0.2\% = 0.002$$
$$138\% = 1.38$$
$$71.02\% = 0.7102$$

To convert a percent to a fraction (or mixed number), first convert the percent to a decimal, as above, then express the decimal as a fraction (with a visible denominator), and finally, reduce the fraction to lowest terms, if needed.

Example 176. Convert 10.8% into a fraction in lowest terms.

Solution. We first write 10.8% as a decimal.

$$10.8\% = 0.108.$$

Then, we write the decimal as a fraction with a visible denominator. In this case, because there are 3 decimal places, the denominator is 1000.

$$0.108 = \frac{108}{1000}.$$

Finally, the GCF of the numerator and denominator is 4, so we cancel 4 from both of these numbers, obtaining a fraction in lowest terms:

$$\frac{108}{1000} = \frac{\cancelto{27}{108}}{\cancelto{250}{1000}} = \frac{27}{250}.$$

□

Here are some common percents with their decimal and fractional equivalents (in lowest terms):

$$10\% = 0.1 = \frac{1}{10}, \quad 20\% = 0.2 = \frac{1}{5}, \quad 25\% = 0.25 = \frac{1}{4}, \quad 50\% = 0.5 = \frac{1}{2}.$$

To convert a fraction to a decimal, we just perform long division.

Example 177. Convert $\frac{1}{4}$ to a decimal.

Solution. Performing the division $1 \div 4$, we obtain

$$\begin{array}{r|l} & \;0.25 \\ \hline 4 & \;1.00 \\ & -\;\;8 \\ \hline & \;\;\;20 \\ & -20 \\ \hline & \;\;\;\;0 \end{array}$$

Thus $\frac{1}{4} = 0.25$. □

To convert a fraction to a percent, first convert it to a decimal, and then convert the decimal to a percent.

Example 178. Convert $\frac{1}{40}$ to a percent.

Solution. Performing the division $1 \div 40$, we obtain 0.025 (verify this – it is quite similar to the last example). Then, we convert 0.025 to a percent by multiplying by 100 (equivalently, by moving the decimal point two places to the right). We obtain

$$\frac{1}{40} = 2.5\%.$$

□

4.11.1 Exercises

Convert the following percents to decimals.

1. 43%
2. 608%
3. 56.04%
4. 4.09%

Convert the following decimals to percents.

5. 14.09
6. 0.00679
7. 1.384
8. 0.384

Convert the following decimals or percents to fractions (or mixed numbers) in lowest terms.

9. 44%
10. 2%
11. 0.15
12. 0.25
13. 40%
14. 5%
15. 98%
16. 7.2%
17. 18%

Convert the following fractions, first to decimals, then to percents. Round percents to the nearest tenth of a percent.

18. $\frac{1}{8}$

19. $\frac{1}{6}$

20. $\frac{2}{5}$

21. $\frac{3}{8}$

22. $\frac{3}{4}$

23. $\frac{1}{13}$

24. $\frac{1}{12}$

25. $\frac{5}{12}$

4.12 Fractional parts of numbers

We now have several ways of indicating a fractional part of a number. For example, the phrases

$$\begin{array}{ccc} \frac{1}{4} & \text{of} & 90 \\ 0.25 & \text{of} & 90 \\ 25\% & \text{of} & 90 \end{array}$$

are all different ways of describing the number $\frac{1}{4} \times 90 = 22.5$. The word "of" indicates multiplication. In the case of percent, the multiplication is done after first converting the percent to a fraction or a decimal.

The fractional part taken need not be a *proper* fractional part. That is, we could end up with more than we started with.

Example 179. Find 125% of 500.

Solution. Converting the percent to the decimal 1.25, and then multiplying by 500, we get

$$1.25 \times 500 = 625.$$

□

Example 180. Sales tax in New York State is $8\frac{1}{4}\%$. What is the sales tax on a shirt priced at \$25?

Solution. $8\frac{1}{4} = 8.25$, and 8.25%, as a decimal, is .0825. So the sales tax on a \$25 shirt is

$$.0825 \times 25 = 2.0625 \approx \$2.06.$$

Note that we rounded off to the nearest cent. □

Example 181. Josh spends $\frac{2}{5}$ of his income on rent. If he earns \$1250 per month, how much does he have left over, after paying his rent?

Solution. The left over part is $\frac{3}{5}$, since $1 - \frac{2}{5} = \frac{5}{5} - \frac{2}{5} = \frac{3}{5}$. So he has

$$\frac{3}{5} \times 1250 = \frac{3}{\cancel{5}_{1}} \times \frac{\cancel{1250}^{250}}{1} = \$750$$

left over after paying rent. □

4.12.1 Exercises

1. Find 16% of 75
2. Find $\frac{3}{8}$ of 60
3. Find .05 of 280
4. Find 150% of 105
5. Find $\frac{1}{2}$% of 248
6. A $500 television is being sold at a 15% discount. What is the sale price?
7. Angela gets a 5% raise. Her original salary was $36,000 per year. What is her new salary?
8. A car loses $\frac{2}{5}$ of its value over a period of years. If the car originally sold for $12,500, what it would it sell for now?
9. On a test, Jose answers $\frac{7}{8}$ of the problems correctly. If there were 24 problems on the test, how many did he get wrong?
10. Medical expenses can be deducted from a person's income tax if they exceed 2% of total income. If Maribel's medical expenses were $550, and her total income was $28,000, can she deduct her medical expenses?

Chapter 5

Ratio and Proportion

In this chapter we develop another interpretation of positive fractions, as comparisons between two quantities.

5.1 Ratio

If a team wins ten games and loses five, we say that the **ratio** of wins to losses is 2 : 1, or "2 to 1." Where did the numbers 2 : 1 come from? We simply made a fraction whose numerator is the number of games the team won, and whose denominator is the number of games they lost, and reduced it to lowest terms: $\frac{\text{wins}}{\text{losses}} = \frac{10}{5} = \frac{2}{1}$.

We can do the same for any two quantities, a and b, as long as $b \neq 0$.

> The ratio of a to b ($b \neq 0$) is the fraction $\dfrac{a}{b}$, reduced to lowest terms.

Example 182. Find (a) the ratio of 9 to 18, (b) the ratio of 21 to 12, (c) the ratio of 64 to 4.

Solution. (a) The ratio of 9 to 18 is

$$\frac{9}{18} = \frac{1}{2} \quad \text{or} \quad 1:2.$$

(b) The ratio of 21 to 12 is

$$\frac{21}{12} = \frac{7}{4} \quad \text{or} \quad 7:4.$$

(c) The ratio of 64 to 4 is

$$\frac{64}{4} = \frac{16}{1} \quad \text{or} \quad 16:1.$$

□

Notice that in (c) we left the denominator 1 (rather than just writing 16) to maintain the idea of a *comparison* between two numbers.

We can form ratios of non-whole numbers. When the ratio is expressed in lowest terms, however, it is always the ratio of two whole numbers, as small as possible. This is the main point of a ratio comparison: if the given two numbers were *small whole numbers*, how would they compare?

Example 183. What is the ratio of $3\frac{3}{4}$ to $1\frac{1}{2}$?

Solution. Converting the mixed numbers to improper fractions, and rewriting the division as multiplication by the reciprocal, we obtain

$$3\frac{3}{4} \div 1\frac{1}{2} = \frac{15}{4} \times \frac{2}{3} = \frac{\cancelto{5}{15}}{\cancelto{2}{4}} \times \frac{\cancelto{1}{2}}{\cancelto{1}{3}} = \frac{5}{2}.$$

The ratio is 5 : 2. □

Example 184. What is the ratio of 22.5 to 15?

Solution. We could perform the decimal division $22.5 \div 15$, but it is easier to simplify the equivalent fraction

$$\frac{225}{150} = \frac{3}{2}.$$

The ratio is 3 : 2. □

When finding the ratio of two measurable quantities, we must be sure the quantities are expressed in the *same units*. Otherwise the ratio will be a skewed or false comparison.

Example 185. Find the ratio of 15 dollars to 30 cents.

Solution. If we form the ratio $\frac{15}{30} = \frac{1}{2}$, we imply that a person walking around with \$15 in his pocket has half as much money as a person walking around with only 30 cents! The proper comparison is obtained by converting both numbers to the same units. In this case, we convert dollars to cents. The correct ratio is

$$\frac{1500 \text{ cents}}{30 \text{ cents}} = \frac{50}{1}.$$

□

Example 186. Manuel gets a five minute break for every hour he works. What is the ratio of work time to break time?

Solution. Since 1 hour = 60 minutes, the ratio of work time to break time is

$$\frac{55 \text{ minutes}}{5 \text{ minutes}} = \frac{11}{1}.$$

□

Recall that percent (%) means "per hundred." So a percent can be viewed as a ratio whose second term (denominator) is 100.

Example 187. At a certain community college, 55% of the students are female. Find (a) the ratio of female students to the total number of students, and (b) the ratio of female students to male students.

Solution. 55% indicates the fraction

$$\frac{55}{100} = \frac{11}{20},$$

so the ratio of female students to the total number of students is 11 : 20. For (b), we first deduce that 45% of the students are male (since 45% = 100% − 55%). So the ratio of female students to male students is

$$\frac{55}{45} = \frac{11}{9} \quad \text{or} \quad 11:9.$$

□

Example 188. A printed page $8\frac{1}{2}$ inches in width has a margin $\frac{5}{8}$ inches wide on either side. Text is printed between the margins. What is the ratio of the width of the printed text to the total margin width?

Solution. The total margin width is $\frac{5}{8} + \frac{5}{8} = \frac{5}{4} = 1\frac{1}{4}$ inches (taking into account both left and right margins.) The width of the printed text is the difference

$$\text{total page width} - \text{total margin width} = 8\frac{1}{2} - 1\frac{1}{4} = 7\frac{1}{4} \text{ inches.}$$

The ratio of the width of the printed text to total margin width is

$$7\frac{1}{4} \div 1\frac{1}{4} = \frac{29}{4} \div \frac{5}{4} = \frac{29}{4} \cdot \frac{4}{5} = \frac{29}{5},$$

or 29 : 5. □

5.1.1 Exercises

Find the ratios.

1. 14 to 4
2. 30 to 32
3. 56 to 21
4. $1\frac{5}{8}$ to $3\frac{1}{4}$
5. $2\frac{1}{12}$ to $1\frac{1}{4}$
6. 14.4 to 5.4
7. 1.69 to 2.6
8. 3 hours to 40 minutes
9. 8 inches to $5\frac{1}{2}$ feet
10. In the late afternoon, a 35 foot tree casts an 84 foot shadow. What is the ratio of the tree's height to the shadow's length?

5.2 Proportions

A **proportion** is a statement that two ratios are equal. Thus

$$\frac{40}{20} = \frac{10}{5}$$

is a proportion, because both ratios are equivalent to the ratio 2 : 1 (= the fraction $\frac{2}{1}$).

> A proportion is a statement of the form
>
> $$\frac{a}{b} = \frac{c}{d}$$
>
> where $b, d \neq 0$.

5.2.1 The cross-product property

There is a very useful fact about proportions: If the proportion $\frac{a}{b} = \frac{c}{d}$ is true, then the "cross-products," ad and bc, are equal, and, conversely, if the cross-products are equal, then the proportion must be true.

Example 189. Verify that the cross-products are equal in the (true) proportion

$$\frac{40}{20} = \frac{10}{5}.$$

Solution. $40 \times 5 = 10 \times 20 = 200$. □

The general cross-product property is stated below for reference:

> $$\frac{a}{b} = \frac{c}{d} \quad \text{if and only if} \quad ad = bc.$$

To prove the cross-product property, we need to say a little bit about **equations**. An equation is a mathematical statement of the form $X = Y$. If the equation $X = Y$ is true, and N is any nonzero number, then the following equations are also true, and have exactly the same solution(s):

$$N \times X = N \times Y \qquad \text{and} \qquad \frac{X}{N} = \frac{Y}{N}.$$

Now go back to the proportion $\dfrac{a}{b} = \dfrac{c}{d}$, and multiply both sides by the nonzero number bd. We get

$$bd\frac{a}{b} = bd\frac{c}{d}.$$

Cancelling b on the left and d on the right yields the cross-product property, $ad = bc$. Conversely, if we have four numbers a, b, c, d, $(b,\ d \neq 0)$, and we know that $ad = bc$, then, dividing both sides by the nonzero number bd gives us the proportion $\frac{a}{b} = \frac{c}{d}$.

Example 190. Decide if the given proportions are true or false, using the cross-product property. (a) $\frac{3}{11} = \frac{2}{7}$; (b) $\frac{4}{10} = \frac{6}{15}$.

Solution. (a) The cross-products are $3 \cdot 7 = 21$ and $2 \cdot 11 = 22$. They are not equal, so the proportion is false. (b) The cross-products are $4 \cdot 15$ and $10 \cdot 6$, both equal to 60. So the proportion is true. ☐

5.2.2 Solving a proportion

If one of the four terms of a proportion is missing or unknown, it can be found using the cross-product property. This procedure is called **solving** the proportion. In the proportion below, x represents an unknown term (any other letter would do).

$$\frac{x}{3} = \frac{34}{51}.$$

There is a unique x which makes the proportion true, namely, the one which makes the cross-products, $51x$ and $34(3)$, equal. The equation

$$51x = 34(3) = 102$$

can be divided by 51 (the number which multiplies x) on both sides, giving

$$\frac{51x}{51} = \frac{102}{51}.$$

Cancellation yields

$$\frac{\cancelto{1}{51}x}{\cancelto{1}{51}} = \frac{\cancelto{2}{102}}{\cancelto{1}{51}} = 2.$$

It follows that

$$x = 2$$

which is the solution of the proportion.

It doesn't matter which of the four terms is missing; the proportion can always be solved by a similar procedure.

Example 191. Solve the proportion

$$\frac{9}{100} = \frac{36}{y}$$

for the unknown term y.

Solution. The cross-products must be equal, so

$$9y = 3600.$$

Dividing both sides of the equation by 9 (the number which multiplies y), and cancelling, yields

$$\frac{\cancelto{1}{9}y}{\cancelto{1}{9}} = \frac{\cancelto{400}{3600}}{\cancelto{1}{9}}$$

$$y = 400.$$

The unknown term is 400. We check our solution by verifying that the cross-products

$$9(400) = 36(100) = 3600$$

are equal. ☐

Neither the terms nor the solution of a proportion are necessarily whole numbers.

Example 192. Solve the proportion

$$\frac{10}{B} = \frac{15}{4}$$

for B and check the solution.

Solution. Setting the cross products equal,

$$40 = 15B.$$

Dividing both sides of the equation by 15 (the number multiplying B), and simplifying, yields

$$\frac{\cancelto{8}{40}}{\cancelto{3}{15}} = \frac{15B}{15}$$
$$\frac{8}{3} = B.$$

The unknown term is $\frac{8}{3}$ or $2\frac{2}{3}$.

Substituting $2\frac{2}{3}$ for B in the original proportion, we check that the cross-products are equal.

$$10(4) \stackrel{?}{=} \left(2\frac{2}{3}\right)(15)$$
$$40 \stackrel{?}{=} \frac{8}{\cancelto{1}{3}} \frac{\cancelto{5}{15}}{1}$$
$$40 = 40.$$

□

Example 193. Solve the proportion

$$\frac{\left(\frac{2}{3}\right)}{5} = \frac{x}{15}.$$

Solution. Set the cross-products equal, and solve for x:

$$\frac{2}{3} \cdot 15 = 5x$$
$$\frac{2}{\cancelto{1}{3}} \cdot \cancelto{5}{15} = 5x$$
$$10 = 5x$$
$$2 = x.$$

□

Example 194. Solve the proportion

$$\frac{42}{70} = \frac{x}{1.5}.$$

Solution. We could immediately set the cross-products equal, but it is simpler to first reduce the fraction $\frac{42}{70}$ to its lowest terms, using – by the way– a proportion: $\frac{42}{70} = \frac{3}{5}$. Stringing two proportions together

$$\frac{3}{5} = \frac{42}{70} = \frac{x}{1.5}$$

lets us skip over the middle fraction. It is evident that the solution to our original proportion is the solution to the simpler proportion

$$\frac{3}{5} = \frac{x}{1.5}.$$

Setting the cross-products equal

$$4.5 = 5x,$$

we obtain the solution $x = \frac{4.5}{5} = 0.9$.

□

We summarize the procedure for solving a proportion.

To solve a proportion,

1. Reduce the numerical ratio (not containing the unknown) to lowest terms, if necessary;
2. Set the cross-products equal;
3. Divide both sides of the resulting equation by the number multiplying the unknown term.

To check the solution,

1. In the original proportion, replace the unknown term with the solution you obtained;
2. Verify that the cross-products are equal.

Equivalence of fractions, and hence, most of fraction arithmetic, is based on proportion. To add

$$\frac{1}{2} + \frac{1}{3}$$

for example, we first solve the two proportions

$$\frac{1}{2} = \frac{x}{6} \qquad \text{and} \qquad \frac{1}{3} = \frac{y}{6}$$

(a task we have performed up to now without much comment) so that the fractions can be written with the LCD (6). Then

$$\frac{1}{2} + \frac{1}{3} = \frac{x+y}{6}.$$

Practice: solve the two proportions for x and y.

5.2.3 Exercises

Solve the following proportions.

1. $\dfrac{1}{5} = \dfrac{3}{x}$
2. $\dfrac{15}{y} = \dfrac{2}{3}$
3. $\dfrac{100}{5} = \dfrac{20}{y}$
4. $\dfrac{A}{9} = \dfrac{5}{3}$
5. $\dfrac{11}{B} = \dfrac{1}{2}$
6. $\dfrac{5}{3} = \dfrac{c}{6}$
7. $\dfrac{s}{3} = \dfrac{4}{13}$
8. $\dfrac{1.2}{7} = \dfrac{A}{0.84}$
9. $\dfrac{P}{100} = \dfrac{75}{125}$
10. $\dfrac{3\frac{1}{5}}{x} = \dfrac{4}{2\frac{1}{2}}$

5.3 Percent problems

Any problem involving percent can be stated (or restated) in the form

"A is P percent of B"

where one of the numbers A, B or P is unknown. We can make this into a mathematical equation by making the following "translations:"

$$\begin{aligned} \text{"is"} \quad &\longleftrightarrow \quad = \\ \text{"}P\text{ percent"} \quad &\longleftrightarrow \quad \frac{P}{100} \\ \text{"of"} \quad &\longleftrightarrow \quad \times \end{aligned}$$

This gives us the equation

$$A = \frac{P}{100} \times B.$$

If we divide both sides of the equation by B, we obtain the proportion in the box below.

The statement "A is P percent of B" is equivalent to the proportion

$$\frac{A}{B} = \frac{P}{100}.$$

The letter B is used to suggest "base amount" – that is, B is the amount *from which* (or *of which*) a percentage is taken. Notice that B follows "of" in the verbal statement. It could be that A (the percentage taken) is greater than B, but only if the percentage P is greater than 100.

In the following examples, the phrase involving "what," as in "what number?" or "what percent?" helps determine which of A, B of P is the unknown.

Example 195. 6 is what percent of 300?

Solution. The percent is unknown, and the base amount B is 300 (since it follows the word "of"). We can restate the question as: "6 is P percent of 300." The corresponding proportion is

$$\frac{6}{300} = \frac{P}{100}$$

and the solution (verify it!) is $P = 2$. Therefore, 6 is 2% of 300. □

Example 196. What number is 8% of 150?

Solution. The base amount B is 150, since it follows the word "of," the percentage is $P = 8$, and therefore A is the unknown amount. The question can be restated as: "A is 8% of 150." The corresponding proportion is

$$\frac{A}{150} = \frac{8}{100}$$

and the solution is $A = 12$ (verify!). It follows that 12 is 8% of 150.

(Note that this result could have been obtained without using a proportion, since 8% of 150 simply means $.08 \times 150 = 12$.) □

Example 197. 58% of what number is 290?

Solution. The base amount B is unknown, and A is 290. The question can be restated as "58% of B is 290." The corresponding proportion is

$$\frac{290}{B} = \frac{58}{100}$$

and the solution is $\frac{29000}{58} = 500$. So 58% of 500 is 290. □

Everyday questions involving percent are not always as straightforward as in the previous examples. But with a little thought, they can be converted into such statements.

Example 198. Approximately 55% of the students enrolled at BCC are female. If there are 2970 female students, what is the total enrollment at BCC?

Solution. According to the given information, 2970 is 55% of the total enrollment, which is the base amount, B. The question can be restated as "2970 is 55% of B" and the corresponding proportion is therefore

$$\frac{2970}{B} = \frac{55}{100}.$$

Using lowest terms of the fraction $\frac{55}{100} = \frac{11}{20}$, we solve

$$\begin{aligned} \frac{2970}{B} &= \frac{11}{20} \\ 59400 &= 11B \\ B &= \frac{59400}{11} = 5400. \end{aligned}$$

The total enrollment is 5400 students. To check, verify that 55% of 5400 is 2970. □

Example 199. Full-time tuition at a university increased from \$2,850 to \$3,000. What was the percent increase in the tuition? (Round to the nearest tenth of a percent.)

Solution. The tuition increased by \$3000 − 2850 = \$150. The percent increase is the ratio

$$\frac{\text{amount of increase}}{\text{original tuition}} = \frac{150}{2850},$$

expressed as a percent. Equivalently, we want to answer the question "\$150 is what percent of \$2850?" We solve the proportion

$$\begin{aligned} \frac{150}{2850} &= \frac{P}{100} \qquad (5.1) \\ \frac{3}{57} &= \frac{P}{100} \\ 57P &= 300 \\ P &\approx 5.26 \end{aligned}$$

Rounded to the nearest tenth of a percent, the tuition increase was approximately 5.3%. □

5.3.1 Exercises

1. 12 is 20% of what number?
2. 12 is 30% of what number?
3. 12 is 40% of what number?
4. 90 is what percent of 225?
5. 90 is what percent of 300?
6. 90 is what percent of 375?
7. What is 125% of 600?
8. What is 175% of 600?

9. 250 is what percent of 325? (Round to the nearest tenth of a percent.)

10. 108 is 80% of what number?

11. A baseball team won 93 games, or 62% of the games it played. How many games did the team play?

12. New York State sales tax is 8.25%. If the sales tax on a DVD player is \$16.50, what is the (before-tax) price of the player?

13. Marina's annual salary last year was \$56, 000. This year she received a raise of \$4, 480. By what percent did her salary increase?

14. A town's population decreased from 13, 000 to 12, 220. By what percent did the population decrease?

5.4 Rates

There are lots of real-world quantities which compare in a fixed ratio. For example, for any given car, the ratio of miles driven to gallons of gas used,

$$\frac{\text{miles}}{\text{gallon}} \qquad \text{or} \qquad \text{"miles per gallon"}$$

is essentially unchanging, or fixed. If we know, say, that 5 gallons of gas was needed to drive 150 miles, we can predict the amount that will be needed to drive any other distance, by solving a simple proportion.

Example 200. Maya used 5 gallons of gas to drive 150 miles. How many gallons will she need to drive 225 miles?

Solution. We solve the proportion

$$\frac{150 \text{ miles}}{5 \text{ gallon}} = \frac{225 \text{ miles}}{x \text{ gallons}},$$

where x represents the number of gallons she will need. Before setting the cross-products equal, reduce the fraction on the left side to $\frac{30}{1}$. Then

$$\begin{aligned} 30x &= 225 \qquad (5.2) \\ x &= \frac{225}{30} = 7\frac{1}{2}. \end{aligned}$$

She will need $7\frac{1}{2}$ gallons of gas to drive 225 miles. □

Miles per gallon is an example of a **rate**, or comparison of unlike quantities by means of a ratio. In the example above, the miles per gallon rate for Maya's car was

$$\frac{150 \text{ miles}}{5 \text{ gallons}} = \frac{150}{5} = \frac{30}{1} \qquad \text{or 30 miles per gallon.}$$

Other examples of rates are: dollars per hour (pay rate for an hourly worker), dollars per item (price of an item for sale), calories per minute (energy use by an athlete). You can undoubtedly think of many others.

Example 201. A runner burns 375 calories running 3.5 miles. (a) How many calories will she burn in running a marathon (approximately 26 miles)? (b) What is her rate of energy use (calories per mile)? Round off to the nearest whole unit.

Solution. (a) The ratio

$$\frac{\text{calories}}{\text{miles}}$$

is assumed fixed and, from the given information, is equal to

$$\frac{375}{3.5}.$$

Let x denote the number of calories the runner burns in running a marathon. Then

$$\begin{aligned} \frac{375}{3.5} &= \frac{x \text{ calories}}{26 \text{ miles}} \qquad \text{(setting cross-products equal)} \\ (375)(26) &= 3.5x \\ x &= \frac{(375)(26)}{3.5} \approx 2786. \end{aligned}$$

She will burn approximately 2786 calories running the marathon. (b) Her rate of energy use is

$$\frac{375 \text{ calories}}{3.5 \text{ miles}} \approx 107 \quad \text{calories per mile.}$$

□

Example 202. A painting crew can paint three apartments in a week. If a building contains 40 equal size apartments, how long will it take the crew to paint all the apartments?

Solution. The ratio

$$\frac{\text{apartments painted}}{\text{time in weeks}}$$

is given as $\frac{3}{1}$. If y is the number of weeks needed to paint 40 apartments, then

$$\frac{3}{1} = \frac{40}{y},$$

and $y = \frac{40}{3} = 13\frac{1}{3}$ weeks. □

5.4.1 Exercises

Use proportions to solve the following problems.

1. On a map, $\frac{3}{4}$ inch represents 14 miles. If two cities are 42 miles apart, how far apart are they on the map?

2. A truck burns $2\frac{1}{2}$ quarts of oil on an 1800 mile trip. How many quarts will be burned on a cross-country trip of 3240 miles?

3. An investment of \$2,000 earns \$48 in interest over a year. How much would need to be invested to earn \$200 in interest? (Round to the nearest dollar.)

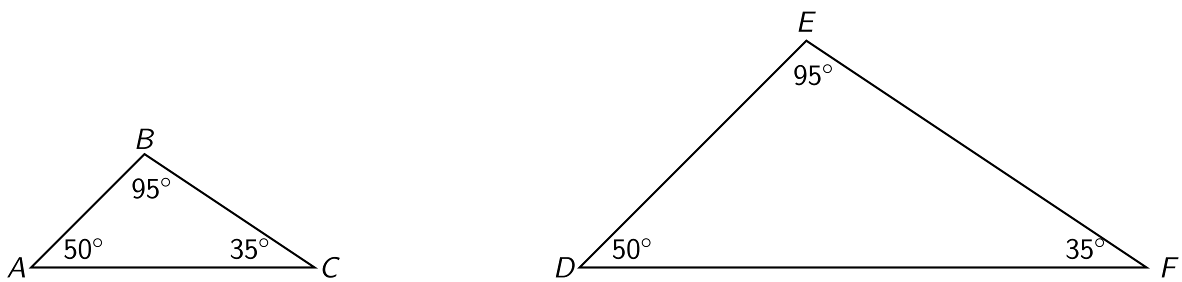
4. In a sample of 600 bottles, 11 were found to be leaking. Approximately how many bottles would you expect to be leaking in a sample of $20,000$ bottles?

5. The ratio of the weight of lead to the weight of an equal volume of aluminum is 21 : 5. If a bar of aluminum weighs 15 pounds, how much would a bar of lead of the same size weigh?

5.5 Similar triangles

Two triangles which have the same shape but possibly different sizes are called **similar**. Having the same shape means that the three angles of one triangle are equal to the three corresponding angles in the other.

The triangles below are similar, because the angle at A in the small triangle is equal to the angle at D in the big triangle ($50°$); the angle at B in the small triangle is equal to the angle at E in the big triangle ($95°$); and similarly the angle at C in the small triangle is equal to the angle at F in the large triangle ($35°$).



Given any triangle, we can obtain a similar one by enlarging (or reducing) all the side lengths in a fixed ratio. In the example above, the bigger triangle was obtained from the smaller using the enlargement ratio 2 : 1. If $\overline{AB}$ stands for the length of the side AB, etc., then

$$\overline{DE} = 2 \times \overline{AB}, \qquad \overline{EF} = 2 \times \overline{BC}, \qquad \text{and} \quad \overline{DF} = 2 \times \overline{AC}.$$

Equivalently,

$$\frac{\overline{DE}}{\overline{AB}} = \frac{\overline{EF}}{\overline{BC}} = \frac{\overline{DF}}{\overline{AC}} = \frac{2}{1}.$$

In words: sides which are opposite equal angles (called **corresponding sides**) have length ratio equal to the ratio 2 : 1. We must be careful to compare side lengths in the proper order. In this case, we used the order larger : smaller, since that is the order in the ratio 2 : 1. We could have used the reverse order, smaller : larger, but only if we also used the reversed ratio 1 : 2.

If we know, say, that $\overline{BC}$ is 12 feet, we can solve the proportion

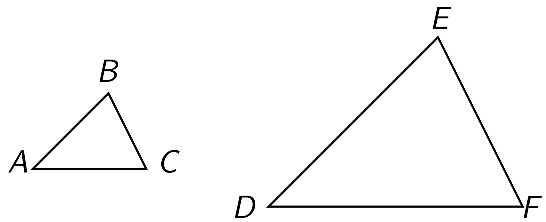
$$\frac{2}{1} = \frac{\overline{EF}}{12}$$

and determine that $\overline{EF}$ is 24 feet.

This is the key fact about similar triangles.

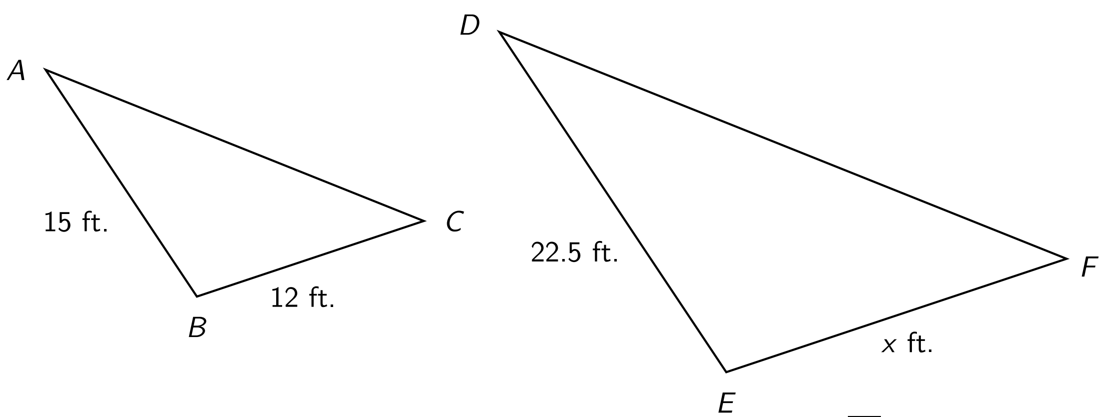
Corresponding sides of similar triangles are proportional:

If: $\angle A = \angle D, \quad \angle B = \angle E, \quad \text{and} \quad \angle C = \angle F,$



$$\text{Then: } \frac{\overline{BC}}{\overline{EF}} = \frac{\overline{AC}}{\overline{DF}} = \frac{\overline{AB}}{\overline{DE}}.$$

Example 203. The triangles below are similar, with $\angle D = \angle A$, $\angle E = \angle B$ and $\angle F = \angle C$. $\overline{AB} = 15$ feet, $\overline{BC} = 12$ feet, and $\overline{DE} = 22.5$ feet. Find $\overline{EF}$.



Solution. The ratios of the corresponding sides opposite $\angle F = \angle C$ $\left(\frac{\overline{DE}}{\overline{AB}}\right)$ and the corresponding sides opposite $\angle D = \angle A$ $\left(\frac{\overline{EF}}{\overline{BC}}\right)$ are equal:

$$\frac{\overline{DE}}{\overline{AB}} = \frac{\overline{EF}}{\overline{BC}}.$$

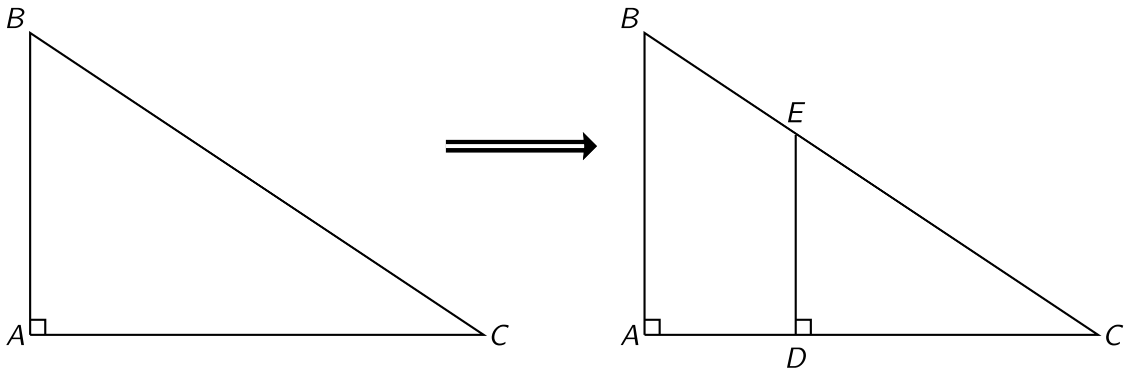
Filling in the given information, and using x to represent the unknown side length $\overline{EF}$, we have

$$\frac{22.5}{15} = \frac{x}{12}$$
$$(22.5)(12) = 15x$$
$$270 = 15x$$
$$x = \frac{270}{15} = 18.$$

$\overline{EF} = 18$ feet. □

We now describe two ways of obtaining a pair of similar triangles. Both involve the notion of **parallel line segments**. Two line segments are parallel if they do not cross, even if extended infinitely in either direction (think of straight train tracks).

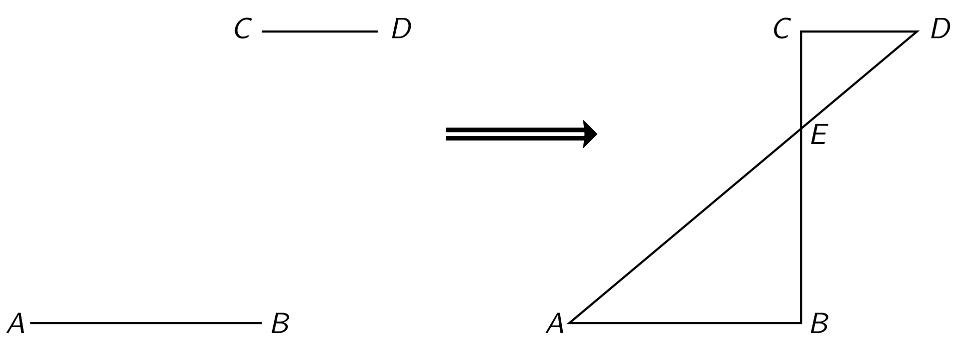
1. If two sides of a triangle are joined by a line segment *parallel to the third side*, the resulting (smaller) triangle is similar to the original triangle.



Starting with triangle ABC on the left, we drew the segment $\overline{DE}$, parallel to $\overline{AB}$, creating (smaller) triangle DEC, similar to ABC. Both triangles are visible on the right (the smaller overlapping the larger), so there was really no need to draw the first picture. By similarity, the corresponding sides are proportional:

$$\frac{\text{smaller triangle}}{\text{larger triangle}} \qquad \frac{\overline{DE}}{\overline{AB}} = \frac{\overline{CD}}{\overline{CA}} = \frac{\overline{CE}}{\overline{CB}}.$$

2. If a pair of *parallel* line segments is "cross-connected" as shown, the two triangles formed are similar.



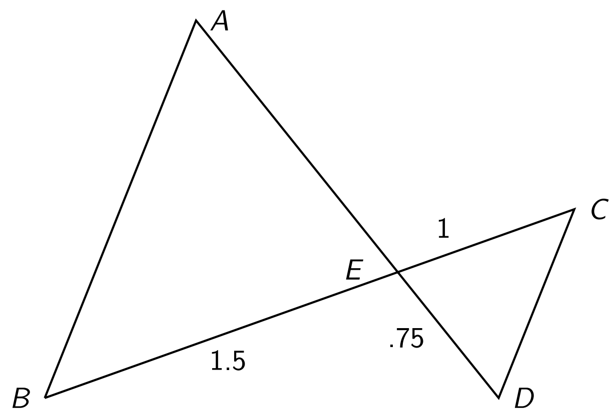
Starting with the parallel segments $\overline{AB}$ and $\overline{CD}$, we connected D to A and C to B by means of straight lines (called **transversals**) which cross at point E. The upper and lower angles at E are equal. In addition, $\angle D = \angle A$ and $\angle C = \angle B$. (These facts are visually plausible; they would be proved in a geometry class.) It follows that triangles ABE and CDE are similar and hence

$$\frac{\text{upper triangle}}{\text{lower triangle}} \qquad \frac{\overline{CD}}{\overline{AB}} = \frac{\overline{EC}}{\overline{EB}} = \frac{\overline{ED}}{\overline{EA}}.$$

In the examples and exercises, it will be convenient to use the following symbolism:

$$\| \quad \longleftrightarrow \quad \text{"is parallel to"}$$
$$\sim \quad \longleftrightarrow \quad \text{"is similar to"}$$
$$\triangle \quad \longleftrightarrow \quad \text{"triangle"}$$

Example 204. In the picture below, $\overline{AB} \parallel \overline{CD}$. Some of the side lengths are given in inches. Find the length of the side $\overline{AE}$.



Solution. Because $\overline{AB} \parallel \overline{CD}$, $\triangle ABE \sim \triangle CDE$. Corresponding sides are proportional, hence

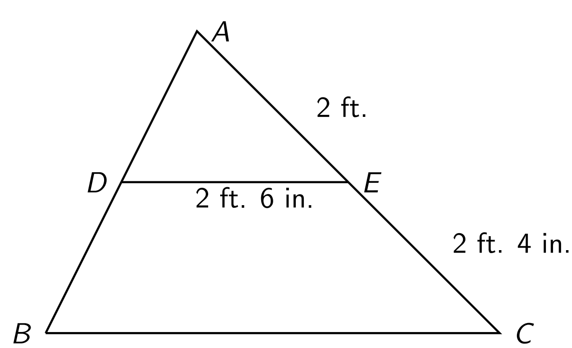
$$\frac{\text{smaller triangle}}{\text{bigger triangle}} \qquad \frac{\overline{CD}}{\overline{AB}} = \frac{\overline{EC}}{\overline{EB}} = \frac{\overline{ED}}{\overline{EA}}.$$

In particular, using $\overline{EC} = 1$, $\overline{EB} = 1.5$, and $\overline{ED} = .75$, and x for the length of $\overline{AE}$,

$$\frac{1}{1.5} = \frac{.75}{x}$$

or $x = (1.5)(.75) = 1.125$. The length of $\overline{AE}$ is 1.125 inches. □

Example 205. In the following figure, $\overline{DE} \parallel \overline{BC}$. Find the length of $\overline{BC}$.



Solution. $\triangle ABC \sim \triangle ADE$ because $\overline{DE} \parallel \overline{BC}$. Corresponding sides are proportional, hence

$$\frac{\overline{AC}}{\overline{AE}} = \frac{2\text{ ft.} + 2\text{ ft. } 4\text{ in.}}{2\text{ ft.}} = \frac{\overline{BC}}{\overline{DE}} = \frac{\overline{BC}}{2\text{ ft. } 6\text{ in.}}.$$

Converting all measurements to inches, and letting x represent the length of $\overline{BC}$ in inches,

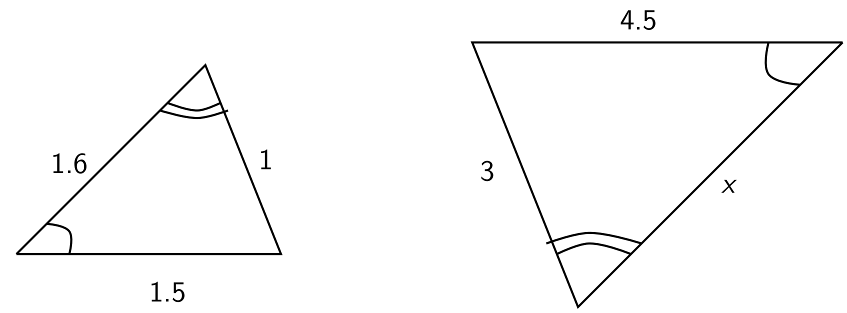
$$\frac{52}{24} = \frac{x}{30}$$

which gives $x = 65$ inches, or 5 ft. 5 in. ☐

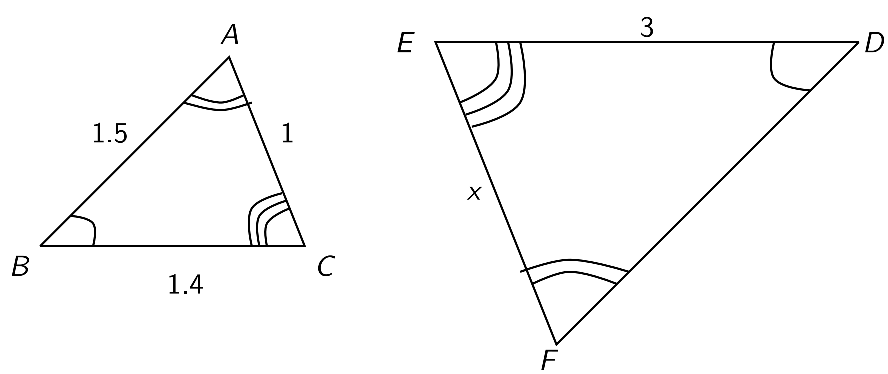
5.5.1 Exercises

In the first two exercises, the triangles are similar, and similarly marked angles are equal. Find the length x in each case.

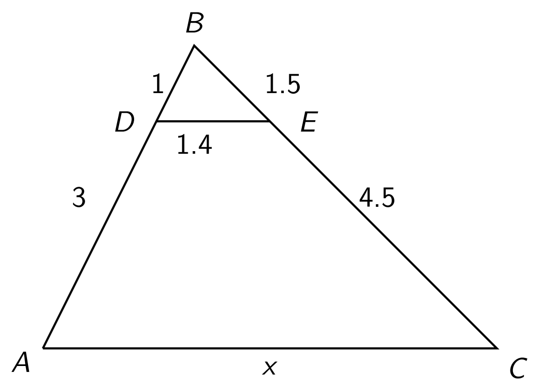
1.



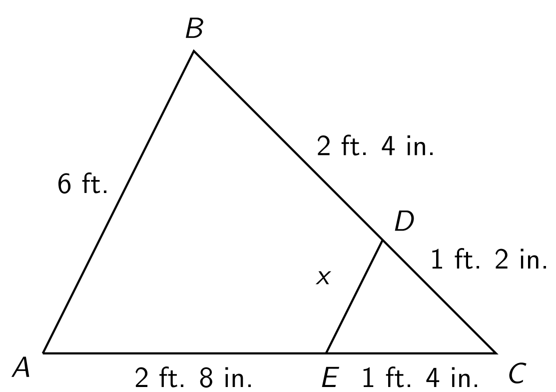
2.



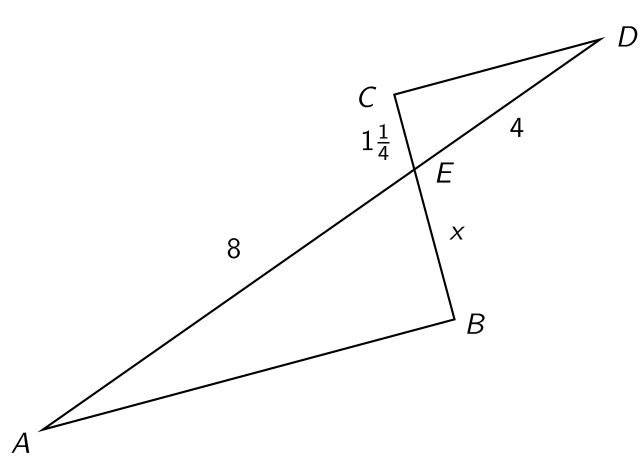
3. In the figure below, $\overline{AC} \parallel \overline{DE}$. Find the side length x.



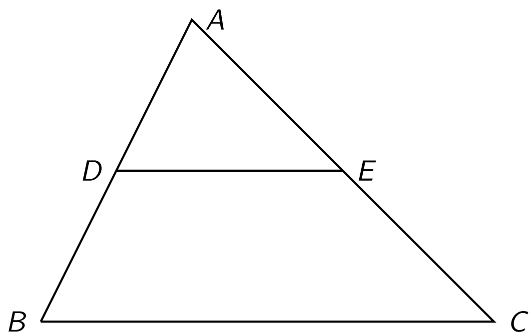
4. In the figure below, $\overline{AB} \parallel \overline{DE}$. Find the side length x.



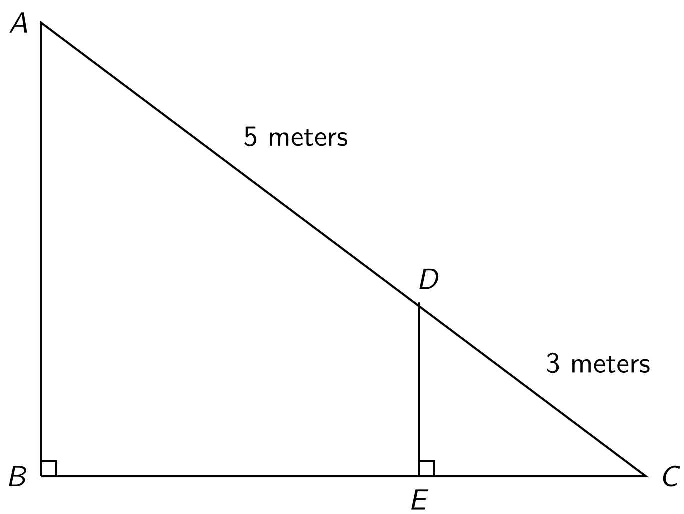
5. In the figure below, $\overline{AB} \parallel \overline{CD}$. Find the side length x.



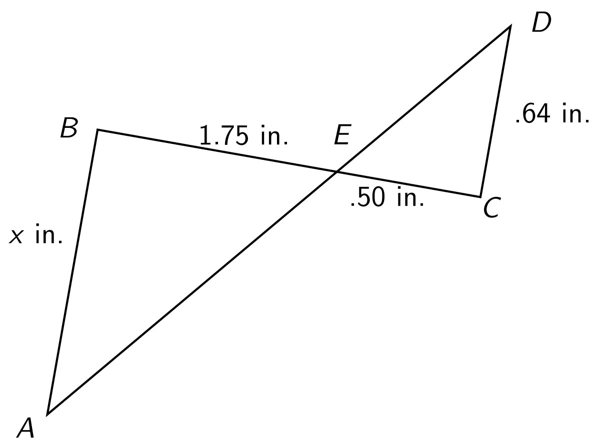
6. In the figure, $\overline{DE} \parallel \overline{BC}$. D cuts $\overline{AB}$ exactly in half. If $\overline{BC}$ is 12 centimeters long, how long is $\overline{DE}$?



7. In the figure below, $\overline{AB} \parallel \overline{DE}$, and length measures are given in meters. How long is $\overline{AB}$ if $\overline{DE}$ has a length of $2\frac{2}{5}$ meters?



8. In the figure below, $\overline{AB} \parallel \overline{CD}$. Find the side length x.



9. A 6 foot man casts an 8 foot shadow on the ground. How long is the shadow of a nearby 32 foot tree? Draw a figure involving similar triangles which illustrates the situation.

Chapter 6

Toward Algebra

6.1 Evaluating Expressions

A mathematical **expression** is any meaningful combination of numbers, letters, operation symbols (such as $+$, $-$, $\times$, $\div$, $\sqrt{\ }$), and grouping symbols (such as parentheses, brackets, fraction bars and the extended square root symbol like $\sqrt{\quad\quad}$). For example, the expression

$$\frac{x+y}{x-y}$$

indicates a fraction whose numerator is the sum of two unspecified numbers, x and y, and whose denominator is their difference. Letters in a mathematical expression are called **variables** because they represent unspecified numbers that can take various different values. Expressions can be **evaluated** (assigned a numerical value) if numerical values are assigned to all the variables appearing in the expression.

Example 206. Evaluate the expression $\dfrac{x+y}{x-y}$ if $x = 2$ and $y = -6$.

Solution. We replace each letter by its assigned numerical value, enclosed in parentheses,

$$\frac{x+y}{x-y} = \frac{(2)+(-6)}{(2)-(-6)},$$

and simplify the resulting expression

$$\frac{(2)+(-6)}{(2)-(-6)} = \frac{-4}{8} = -\frac{1}{2}.$$

□

Assigning different values to the letters in an expression usually changes the value of the expression.

Example 207. Evaluate the expression $\dfrac{x+y}{x-y}$ if $x = -3$ and $y = 3$.

Solution. Replacing x and y by their assigned values,

$$\frac{x+y}{x-y} = \frac{(-3)+(3)}{(-3)-(3)} = \frac{0}{-6} = 0.$$

□

The reason for replacing letters with their assigned values *in parentheses* is to avoid pitfalls such as the ones highlighted in the next two examples.

Example 208. Evaluate x^2 if $x = -4$.

Solution. Forgetting parentheses, we would write $x^2 = -4^2 = -16$, which is wrong, since the square of any nonzero number is positive. The correct evaluation is

$$x^2 = (-4)^2 = 16.$$

□

Example 209. Evaluate ab if $a = 3$ and $b = -4$.

Solution. Without parentheses, we might think $ab = 3 - 4 = -1$, mistakenly turning multiplication into subtraction. The correct evaluation is

$$ab = (3)(-4) = -12.$$

□

To evaluate complicated expressions consistently, we must follow the **order of operations**, which is restated below for convenience.

1. operations within grouping symbols first;
2. exponents and roots next;
3. multiplications and divisions (in order of appearance) next;
4. additions and subtractions (in order of appearance) last.

Recall that "*in order of appearance*" means in order *from left to right*, and that grouping symbols include parentheses, brackets, braces (curly brackets), the square root symbol, and the fraction bar.

Example 210. Evaluate the expression $x - 3y$ if $x = 4$ and $y = -\frac{1}{2}$.

Solution. Substituting the assigned values, and multiplying first according to the order of operations,

$$\begin{aligned} x - 3y &= (4) - 3\left(-\frac{1}{2}\right) \\ &= 4 + \frac{3}{2} \\ &= \frac{8}{2} + \frac{3}{2} \quad = \quad \frac{11}{2} \quad = \quad 5\frac{1}{2}. \end{aligned}$$

□

Example 211. Evaluate $a - b - c$ and $a - (b - c)$ if $a = 2$, $b = -11$ and $c = 10$.

Solution. The first expression is evaluated as follows:

$$\begin{aligned} a-b-c &= (2)-(-11)-(10) \\ &= 13-10 \qquad \text{(left subtraction before right subtraction)} \\ &= 3. \end{aligned}$$

For the second expression,

$$\begin{aligned} a-(b-c) &= (2)-\Big((-11)-(10)\Big) \\ &= 2-(-21) \qquad \text{(subtraction within grouping symbols first)} \\ &= 23. \end{aligned}$$

□

Example 212. Evaluate $3-(p-r\div t)$ if $p=\frac{3}{4}$, $r=-\frac{1}{8}$ and $t=\frac{5}{12}$.

Solution. Within grouping symbols, division comes first.

$$\begin{aligned} 3-(p-r\div t) &= 3-\left[\frac{3}{4}-\left(-\frac{1}{8}\right)\div\left(\frac{5}{12}\right)\right] \quad \text{(changing parentheses to brackets for clarity)} \\ &= 3-\left[\frac{3}{4}-\left(-\frac{1}{\cancel{8}_{2}}\right)\cdot\left(\frac{\cancel{12}^{3}}{5}\right)\right] \quad \text{(division as multiplication by the reciprocal)} \\ &= 3-\left[\frac{3}{4}-\left(-\frac{3}{10}\right)\right] \\ &= 3-\left[\frac{3}{4}+\frac{3}{10}\right] \quad \text{(subtraction within grouping symbols first)} \\ &= 3-\left[\frac{15}{20}+\frac{6}{20}\right] \quad \text{(using the LCD} = 20\text{)} \\ &= 3-\left[\frac{21}{20}\right] = \frac{60}{20}-\frac{21}{20} = \frac{39}{20} = 1\frac{19}{20}. \end{aligned}$$

□

Example 213. Evaluate $\dfrac{3x^3-5x^2+13x-4}{x^2+2}$ if $x=0$.

Solution. $\dfrac{3(0)^3-5(0)^2+13(0)-4}{0^2+2}=\dfrac{-4}{2}=-2.$ □

Example 214. Evaluate the expressions $\sqrt{a^2+b^2}$ and $a+b$ if $a=-0.6$, and $b=0.8$.

Solution. For the first expression,

$$\begin{aligned} \sqrt{a^2+b^2} &= \sqrt{(-0.6)^2+(0.8)^2} \\ &= \sqrt{.36+.64}=\sqrt{1}=1. \end{aligned}$$

For the second expression,

$$\begin{aligned} a + b &= -0.6 + 0.8 \\ &= 0.2. \end{aligned}$$

This example shows that, in general,

$$\sqrt{a^2 + b^2} \neq a + b.$$

□

6.1.1 Exercises

Evaluate the expressions for the given values of the variables. Reduce fractions to lowest terms and express improper fractions as mixed numbers.

1. $2a - b \qquad$ if $a = 11$ and $b = -5$
2. $2rt \qquad$ if $r = 5$ and $t = -6$
3. $-6x^2 \qquad$ if $x = -\dfrac{2}{3}$
4. $p^2 - q^2 \qquad$ if $p = -2$ and $q = 3$
5. $a - ab - b \qquad$ if $a = -0.6$ and $b = 0.8$
6. $-x^2 + 3y^2 - 8y + 5 \qquad$ if $x = -2$ and $y = 0$
7. $(d - e)(d^2 + ed + e^2) \qquad$ if $d = -1$ and $e = -4$
8. $\dfrac{2x + y}{x - 2y} \qquad$ if $x = 4$ and $y = -6$
9. $2pq - q^2 \qquad$ if $p = 1.2$ and $q = 2.3$
10. $\dfrac{3a - 2b}{5a + b} \qquad$ if $a = 1\dfrac{1}{2}$ and $b = \dfrac{1}{4}$
11. $\sqrt{x^2 + y^2} \qquad$ if $x = \dfrac{2}{3}$ and $y = -\dfrac{1}{2}$
12. $-b + \sqrt{b^2 - 4ac} \qquad$ if $a = -2$, $b = 7$ and $c = -3$

6.2 Using Formulae

A **formula** expresses one quantity in terms of others. For example, the formula for the area A of a rectangle of length l and width w is

$$A = lw.$$

If we know the length and width of a rectangle, we can calculate its area, using the formula. For example, the area of a rectangle of width 17 feet and length 50 feet is

$$A = lw = (17)(50) = 850 \text{ square feet.}$$

Example 215. The formula for the distance s (in feet) that an object falls in a time t (in seconds) is

$$s = 16t^2.$$

Use the formula to determine the distance an object falls in (a) 2 seconds; (b) 4 seconds.

Solution. We substitute $t = 2$ and $t = 4$ into the formula. In 2 seconds the object falls

$$s = 16(2)^2 = 64 \text{ feet},$$

while, in 4 seconds it falls

$$s = 16(4)^2 = 256 \text{ feet}.$$

□

Example 216. The formula which converts temperature in degrees Celsius ($^\circ C$) to temperature in degrees Fahrenheit ($^\circ F$) is

$$F = \frac{9}{5}C + 32.$$

Find the temperature in $^\circ F$ if a thermometer reads $-10^\circ C$.

Solution. Substituting (-10) for C in the formula, we find

$$\begin{aligned} F &= \frac{9}{5}(-10) + 32 \\ &= -\frac{9}{\cancel{5}_1} \cdot \frac{\cancel{10}^2}{1} + 32 \\ &= -18 + 32 = 14. \end{aligned}$$

The temperature is 14° F. □

Recall that the *Pythagorean theorem* is a relation between the side-lengths of a right triangle. If c denotes the length of the hypotenuse (the long side opposite the right angle) and a and b the lengths of the two shorter sides (legs), the relation is

$$a^2 + b^2 = c^2.$$

From this relation, we can extract three formulae:

$$\begin{aligned} a &= \sqrt{c^2 - b^2}; \\ b &= \sqrt{c^2 - a^2}; \\ c &= \sqrt{a^2 + b^2} \end{aligned}$$

The first two are formulae for the length of one leg in terms of the length of the other leg and the length of the hypotenuse. The last is a formula for the length of the hypotenuse in terms of the lengths of the two legs.

Example 217. The hypotenuse of a right triangle is 169 feet, and one its legs is 119 feet. How long is the other leg?

Solution. In the formulae, the legs are denoted a and b. Suppose a is the unkown leg. Putting $c = 169$ and $b = 119$ in the first formula

$$a = \sqrt{c^2 - b^2} = \sqrt{169^2 - 119^2} = \sqrt{14400} = 120.$$

The other leg is 120 feet. □

Example 218. Find the length of the hypotenuse of a right triangle whose legs have lengths 5 and 12 feet.

Solution. We put $a = 5$ and $b = 12$ in the third formula.

$$c = \sqrt{a^2 + b^2} = \sqrt{(5)^2 + (12)^2} = \sqrt{25 + 144} = \sqrt{169} = 13 \text{ feet.}$$

□

Example 219. The *semi-perimeter* of a triangle with side lengths a, b and c is given by the formula

$$s = \frac{1}{2}(a + b + c).$$

Find the semi-perimeter of a triangle with side lengths $a = 6$ ft 8 in, $b = 9$ ft 4 in, and $c = 11$ ft 10 in.

Solution. We convert all measurements to inches, using the fact that 1 ft = 12 in. Thus

$$a = 80 \text{ in}, \qquad b = 112 \text{ in}, \qquad \text{and} \qquad c = 142 \text{ in}.$$

The semi-perimeter is

$$\begin{aligned} s = \frac{1}{2}(a + b + c) &= \frac{1}{2}(80 + 112 + 142) \\ &= 167 \text{ in} \\ &= 13 \text{ ft} \quad 11 \text{ in}. \end{aligned}$$

□

Example 220. The child's dosage for a medicine is given by the formula

$$C = \frac{t}{t + 12} \cdot A,$$

where C is the child's dosage, A is the adult dosage, and t is the child's age in years. Find the dosage for a four-year old child if the adult dosage is 48 mg (milligrams).

Solution. Substituting $t = 4$ and $A = 48$,

$$C = \frac{4}{4 + 12} \cdot 48 = \frac{1}{4} \cdot \frac{48}{1} = 12.$$

The child's dosage is 12 mg. □

6.2.1 Exercises

1. Find the area of a rectangle whose length is 4.8 meters and whose width is 3.6 meters. Use the formula $A = lw$, where A is the area, l is the length, and w is the width.

2. Find the perimeter of the rectangle in the preceding exercise, using the formula $P = 2l + 2w$, where P is the perimeter and l, w are the length and width, respectively.

3. Find the length of the hypotenuse of a right triangle whose legs are 0.3 yards and 0.4 yards.

4. Straight roads between three towns form a right triangle. The longest road is 17 miles. The next longest road is 15 miles. How long is the third road?

5. How far does an object fall in 3 seconds? Use the formula $s = 16t^2$, where s is the distance fallen (in ft), and t is the time (in sec).

6. If a thermometer reads 22°C, find the temperature in °F. Use the formula $F = \frac{9}{5}C + 32$.

7. Heron's formula, $A = \sqrt{s(s-a)(s-b)(s-c)}$, gives the area ($A$) of a triangle (not necessarily a right triangle) with side lengths a, b and c, and semi-perimeter s. Use Heron's formula and the semi-perimeter formula $s = \frac{1}{2}(a + b + c)$ to find the area of a triangle with $a = b = c = 2$ ft.

8. The formula $A = P(1 + r)^t$ gives the amount A of money in a bank account t years after an initial amount P is deposited, when the annual interest rate is r. Find A after 2 years if the interest rate is 5% ($r = .05$), and the initial deposit was $P = \$500$.

9. Find the child's dosage for a 10-year old, if the adult dosage is 2.2 grams. Use the formula $C = \frac{t}{t+12} \cdot A$, where C is the child's dosage, t is the child's age, and A is the adult dosage.

6.3 Functions

Formulae show that one numerical quantity can depend on another. The proper dosage of a child's medicine depends on the child's age; the distance your car can drive (without stopping) depends on the amount of gas in your tank.

In mathematics, dependence of one quantity (say, y) on another (say, x) is expressed by saying that y is a **function** of x. We write

$$y = f(x),$$

and say "y equals f of x." In this notation, f does **not** denote a number, and the parentheses do **not** denote multiplication. f is short for "function." Another letter might be used, for example, as an abbreviation for a quantity (d for "dosage," say). The parentheses are there to receive numerical **input** in the form of a value of x. The function acts on the input in a definite way (often given by a formula) to produce an **output**, $f(x)$, which is a unique value of y.

A common type of function has the form

$$f(x) = Ax + B,$$

where A and B are any two fixed numbers. This is called a **linear** function; it acts on its input by first multiplying it by the number A, and then adding the number B.

Example 221. Let f be the linear function defined by

$$f(x) = -2x + 300.$$

Find $f(0)$, $f(10)$, $f(25)$ and $f(150)$.

Solution. The function takes an input x, multiplies it by -2, and then adds 300. Hence

$$\begin{aligned} f(0) &= -2(0) + 300 = 300 \\ f(10) &= -2(10) + 300 = 280 \\ f(25) &= -2(25) + 300 = 250 \\ f(150) &= -2(150) + 300 = 0. \end{aligned}$$

□

Example 222. New York City Taxi fares are calculated as follows: there is an initial charge of \$2.50. After that, there is a charge of \$0.50 per $\frac{1}{5}$ mile (and \$0.50 per minute in slow traffic). Assuming there is no traffic (!), write a linear function which expresses how the taxi fare depends on the distance travelled.

Solution. Let x denote the distance travelled (in miles), and let $F(x)$ be the function that determines the corresponding fare. You always pay the initial charge, so $F(x)$ is at least \$2.50, even if you go nowhere. Then you are charged \$0.50 for every $\frac{1}{5}$ mile. So for one mile you pay $5 \times \$0.50 = \$2.50$. If you go x miles, you pay x times \$2.50, or $2.50x$ dollars. Putting this all together,

$$F(x) = 2.50 + 2.50x.$$

□

There are many other types of function. A *quadratic* function has the form

$$q(x) = ax^2 + bx + c,$$

where a, b and c are three fixed numbers, and a is not 0 (otherwise, it's a linear function.) For example, the quadratic function

$$h(t) = -16t^2 + 100$$

expresses the height (h) above the ground of an object dropped from a 100-foot tower after t seconds. Here, the height h is a function of the time t.

Example 223. Using the height function $h(t)$ above, find the height of an object $1\frac{1}{2}$ seconds after it is dropped from a 100-foot tower.

Solution. We simply evaluate

$$\begin{aligned} h\left(1\frac{1}{2}\right) &= h\left(\frac{3}{2}\right) \\ &= -16\left(\frac{3}{2}\right)^2 + 100 \\ &= \frac{-16(9)}{4} + 100 \\ &= -36 + 100 \\ &= 64. \end{aligned}$$

After $1\frac{1}{2}$ seconds, the object is 64 feet above the ground. □

It is hard to resist asking: how long until the object hits the ground? In function terms, the question is: for what value of t is it true that $h(t) = 0$? See if you can figure it out.

6.3.1 Exercises

For the function $f(x) = 2x - 5$, find:

1. $f(-2)$
2. $f(0)$
3. $f(\frac{1}{2})$
4. $f(0.75)$

For the function $g(t) = -2t^2 + 4t + 1$, find:

5. $g(-3)$
6. $g(0)$
7. $g(1)$
8. $g(\dfrac{2}{3})$
9. $g(\dfrac{-2}{3})$
10. In 1960, New York City Taxis were much cheaper. The initial charge was \$0.25, after that, a charge of \$0.05 per $\frac{1}{5}$ mile, and no charge for sitting in traffic. Write a linear function which expresses how the taxi fare depended on the distance travelled in 1960. Compare the price of a 10 mile ride today, and a 10-mile ride in 1960.

6.4 Linear terms

A **linear term** is an expression of the form

$$Cx \quad \text{or} \quad Dy \quad \text{or} \quad Ez$$

where C (or D or E, etc.) is a nonzero number called the **coefficient**, and x (or y or z, etc.) is a variable. (You will note that linear terms show up in linear functions!) For example,

$$3x, \quad -5y, \quad z, \quad \frac{1}{2}w, \quad -t$$

are linear terms with coefficients 3, -5, 1, $\frac{1}{2}$, and -1, respectively. Note that a variable by itself (without a coefficient) is understood to have a coefficient of 1 (not 0). Also, a negative sign is sufficient to indicate the coefficient -1.

Linear terms with the *same variable* can be combined into a single term by addition and subtraction. For example

$$\begin{aligned}
5x + 4x &= (5+4)x &&= 9x \\
3x + 2x - x + 4x &= (3+2-1+4)x &&= 8x \\
t + 5t - 8t &= (1+5-8)t &&= -2t \\
-2y + \frac{2}{3}y &= \left(-2 + \frac{2}{3}\right)y &&= -\frac{4}{3}y.
\end{aligned}$$

We are simply using the distributive law here. This is possible because, in each expression, the value of the variable, though unspecified, is the same wherever it occurs. Note that it is **not** possible to combine linear terms with *different* variables. Thus, for example, the expressions

$$5x + 3y \quad \text{or} \quad z - 3t$$

cannot be simplified in any way. This is because x and y (or z and t) are independent variables – they need not take on the same value.

Linear terms with the same variable are called **like terms**.

Example 224. Simplify the following expression by combining like terms.

$$5x - 4y + 3y - 11x + 2t.$$

Solution. First survey the expression to determine the different types of like terms: there are x-terms, y-terms, and t terms. There is just one t term, so that term won't combine with any other term. There are two x-terms, and two y-terms. The following approach is convenient because it allows us to rearrange the expression so that like terms are next to eachother.

Treat subtractions as additions of *negative* terms.

(Recall that subtraction means 'adding the opposite.') Doing this, our expression becomes

$$5x + (-4y) + 3y + (-11x) + 2t.$$

which can be reorganized, by associativity of addition, to

$$\begin{aligned}5x + (-11x) + (-4y) + 3y + 2t &= [5 + (-11)]x + [-4 + 3]y + 2t \\ &= -6x + (-1y) + 2t \\ &= -6x - y + 2t.\end{aligned}$$

At the final step, we revert to subtraction. □

6.4.1 Exercises

Simplify the expressions by combining like terms.

1. $2x + 9x$
2. $-3c + 14c - 19c - d$
3. $3x - 2y + 11x - 9y$
4. $\frac{1}{2}x - \frac{3}{4}x$
5. $a - b + 2a - c + 2b$
6. $1.09x - 0.07x$
7. $1.2y + 3.03y - 50y$
8. $3x + \frac{1}{2}y - \frac{3}{2}x - z - \frac{2}{3}x + 2z$

6.5 Linear Equations in One Variable

An **equation** is a statement that two mathematical expressions are equal. If the statement has just one variable, and if

- the variable does not appear in the denominator of a fraction,
- the variable does not appear under a $\sqrt{\ }$ symbol,
- the variable is not raised to a power other than 1,

then we have a **linear equation in one variable**. Here are some examples of linear equations in one variable:

$$2x + 4 = 8 \qquad -3y = 12 \qquad z - 9 = -1 \qquad 2 - \frac{2}{3}t = 0.$$

Note that a linear equation contains only linear terms, constants and the equality symbol.

A **solution** to an equation in one variable is a number which, when substituted for the variable, makes a true statement.

Example 225. Show that -4 is a solution to the equation $-3y = 12$

Solution. When we substitute −4 for y in the equation, we get

$$\begin{aligned} -3(-4) &= 12 \\ 12 &= 12, \qquad \text{a } \textit{true} \text{ statement.} \end{aligned}$$

□

Example 226. Show that 10 is *not* a solution to the equation $z - 9 = -1$.

Solution. When we substitute 10 for z in the equation, we get

$$\begin{aligned} 10 - 9 &= -1 \\ -1 &= 1, \qquad \text{a } \textit{false} \text{ statement.} \end{aligned}$$

□

Example 227. Show that 3 is a solution to the equation $2 - \frac{2}{3}t = 0$, but 4 is not.

Solution. When we substitute 3 for t, we obtain

$$\begin{aligned} 2 - \frac{2}{\cancel{3}} \cdot \frac{\cancel{3}}{1} &= 0 \\ 2 - 2 &= 0 \\ 0 &= 0, \qquad \text{a } \textit{true} \text{ statement.} \end{aligned}$$

When we substitute 4 for t, we obtain

$$\begin{aligned} 2 - \frac{2}{3} \cdot \frac{4}{1} &= 0 \\ 2 - \frac{8}{3} &= 0, \\ -\frac{2}{3} &= 0, \qquad \text{a } \textit{false} \text{ statement.} \end{aligned}$$

□

Here is the formal definition of the subject of this section.

> A linear equation in one variable x is an equation that can be written in the form
>
> $$ax + b = c$$
>
> for some numbers a, b and c, with $a \neq 0$.

Such an equation always has a **unique** (i.e., exactly one) solution.

6.5.1 Finding solutions

We find solutions to equations using two common-sense principles:

- Adding equals to equals produces equals,
- Multiplying equals by equals produces equals.

For example, 2 and 2 are equals, and so are 3 and 3. Adding equals to equals,

$$\begin{aligned} 2 &= 2 \\ 3 &= 3 \\ 2+3 &= 2+3 \end{aligned}$$

produces equals: $5 = 5$.

Consider the linear equation

$$x - 3 = 6.$$

Although we cannot say that $x - 3$ and 6 are "equals" (without knowing the value of x) we can say that *if* they are equal *for some* x, then adding equals to both sides, or multiplying both sides by equals, will produce a new pair of equals *for the same* x. In particular, if $x - 3 = 6$ is true for some x, so is the equation obtained by adding 3 to both sides:

$$\begin{aligned} x - 3 &= 6 \\ \underline{\quad +3} &\ \underline{+3 \quad} \\ x &= 9. \end{aligned}$$

The solution to the last equation is obvious (and obviously unique): x must be 9. It is slightly less obvious that 9 is a solution of the original equation. Could there be some *other* solution to the original equation, say, $x = p$? If so, then $p - 3 = 6$, and adding 3 to both sides yields $p = 9$. So p is no different from the solution we already found. We conclude that 9 is the **unique** solution of $x - 3 = 6$.

Example 228. Find the solution of the equation $9 = -\frac{1}{5}z$ by multiplying both sides by equals. Check that you have indeed obtained the unique solution.

Solution. If we multiply both sides of the equation by -5, we get

$$\begin{aligned} -5(9) &= -5\left(-\frac{1}{5}\right)z \\ -45 &= z. \end{aligned}$$

To check that -45 is the solution, substitute -45 for z in the original equation, and verify that a true statement results.

$$\begin{aligned} 9 &= -\frac{1}{5}z \\ 9 &\stackrel{?}{=} -\frac{1}{5}(-45) \\ 9 &\stackrel{?}{=} \frac{\cancelto{9}{45}}{\cancelto{1}{5}} \\ 9 &= 9 \qquad \text{a true statement.} \end{aligned}$$

□

Equations which have the same solution are called **equivalent**. As the previous examples show, a strategy for finding the solution of an equation is to systematically transform it into simpler equivalent equations, until we arrive at an equation like $x = 8$, or $-45 = z$, whose solution is obvious.

> To find the solution (solve) a linear equation in one variable, we can do one or both of the following:
>
> - add the same number to both sides of the equation;
> - multiply both sides of the equation by the same nonzero number.

We can replace "add" by "subtract," and "multiply" by "divide," if convenient. (This is because subtracting a number is the same as adding the opposite number, and dividing by a nonzero number is the same as multiplying by the reciprocal number.)

Example 229. Solve the equation $2x + 4 = -10$ and check that the solution is correct.

Solution. First subtract 4 from both sides:

$$\begin{array}{rcl} 2x + 4 &=& -10 \\ \underline{\quad -4} & & \underline{-4 \quad} \\ 2x &=& -14. \end{array}$$

Then divide both sides by 2:

$$\begin{aligned} 2x &= -14 \\ \frac{\cancel{2} \cdot x}{\cancel{2}} &= \frac{\cancel{-14}^{\,-7}}{\cancel{2}^{\,1}} \\ x &= -7. \end{aligned}$$

The solution is -7. To check, we substitute -7 for x in the original equation, and see if a true statement results.

$$\begin{aligned} 2x + 4 &= -10 \\ 2(-7) + 4 &\stackrel{?}{=} -10 \\ -14 + 4 &\stackrel{?}{=} -10 \\ -10 &= -10 \qquad \text{a true statement} \end{aligned}$$

□

Example 230. Solve the equation $26 = 5 - \frac{7}{8}x$ and check the solution.

Solution. At some point it will be necessary to multiply both sides by the reciprocal of $-\frac{7}{8}$. But it is a mistake to do that immediately. If we first subtract 5 from both sides, to isolate the linear term, the computations are shorter.

$$\begin{array}{rl} 26 = & 5 - \frac{7}{8}x \\ -\ 5 & -\ 5 \\ \hline 21 = & -\frac{7}{8}x \end{array}$$

Now we multiply both sides by $\left(-\frac{8}{7}\right)$

$$-\frac{8}{7} \cdot 21 = \left(-\frac{8}{7}\right)\left(-\frac{7}{8}x\right)$$

$$-\frac{8}{\cancel{7}} \cdot \cancel{21}^{3} = \left(-\frac{\cancel{8}}{\cancel{7}}\right)\left(-\frac{\cancel{7}}{\cancel{8}}x\right)$$

$$-24 = x$$

The check is left to you. ☐

A linear equation given in the standard form

$$ax + b = c \qquad (a \neq 0)$$

is solved in two steps:

- *first,* subtract b from both sides;
- *second,* divide both sides by a.

Note that this is a reversal of the order of operations: in solving for x we "undo" the operations on x, from last to first.

Example 231. Solve the equation $-x + 5 = 12$.

Solution. Note that $-x$ means $-1x$. Following the two step procedure, we first subtract 5 from both sides, and then divide both sides by -1.

$$\begin{aligned} -x + 5 &= 12 \\ -x &= 7 \qquad \text{(subtracting 5 from both sides)} \\ x &= -7 \qquad \text{(dividing or multiplying both sides by } -1) \end{aligned}$$

Note that dividing by -1 is the same as multiplying by -1, since -1 is its own reciprocal! ☐

Example 232. Solve $3x + \frac{1}{2} = \frac{1}{2}$.

Solution. We first subtract $\frac{1}{2}$ from both sides.

$$\begin{aligned} 3x + \frac{1}{2} &= \frac{1}{2} \\ 3x &= \frac{1}{2} - \frac{1}{2} \\ 3x &= 0 \\ x &= 0. \end{aligned}$$

At the last step, we divided both sides by 3. Recall that $\frac{0}{3} = 0$. □

You may encounter a linear equation with more than one linear term, such as

$$2x - 4 = 3x + 1.$$

It is easy to write an equivalent equation in which both linear terms are on the same side of the equation: simply add or subtract a linear term from both sides of the equation. This is just another application, using linear terms instead of numbers, of the principle "adding (or subtracting) equals to equals produces equals." Two linear terms on the same side of an equation (if they are like terms) can be combined into a single term. At this point, we have an equivalent equation in the standard form $ax + b = c$.

In the example above, we can subtract $2x$ from both sides, combine like terms, and proceed as before.

$$\begin{aligned} 2x - 4 &= 3x + 1 & \\ -4 &= 3x - 2x + 1 & \text{(subtracting } 2x \text{ from both sides)} \\ -4 &= x + 1 & \text{(combining like terms)} \\ -5 &= x & \text{(subtracting 1 from both sides)} \end{aligned}$$

We could also have subtracted $3x$ from both sides at the beginning. The subsequent steps would have been different, but the solution $x = -5$ would be the same (try it!)

Example 233. Solve the equation $-5t + 6 = -8t$.

Solution. We have linear terms on both sides. If we add the linear term $8t$ to both sides, we obtain

$$3t + 6 = 0.$$

Subtracting 6 from both sides, and then dividing both sides by 3, we have

$$\begin{aligned} 3t + 6 &= 0 \\ 3t &= -6 \\ t &= \frac{-6}{3} \\ t &= -2. \end{aligned}$$

Let's check this. Substitute -2 for t in the original equation.

$$\begin{aligned} -5t + 6 &= -8t & \\ -5(-2) + 6 &\stackrel{?}{=} -8(-2) & \\ 16 &= 16 & \text{true!} \end{aligned}$$

□

6.5.2 Exercises

Determine if the given value of the variable is a solution of the equation.

1. $y + 1.6 = 14.4, \quad y = 2$
2. $2x + 3 = 19, \quad x = 8$

Solve the equations and check the solutions.

3. $x - 5 = 19$
4. $6 + y = -2$
5. $9x = 45$
6. $\frac{2}{5}t = 24$
7. $-y - 14 = 5$
8. $\frac{x}{8} = -4$
9. $19t - 5 = -5$
10. $4x - 5 = 7$
11. $2y - 1.6 = 13.2$
12. $2x - 4 = 3x - 6$
13. $1 - x = x - 1$
14. $\frac{2}{3} - \frac{x}{5} = \frac{-14}{15}$
15. Twelve times a number is 108. What is the number?
16. Five more than a number is $-9\frac{1}{2}$. What is the number?