## MTH 31 Spring 2025 Final Exam Review

**Concept Review:**

1. Explain the concepts of function, domain and range.
2. Explain the concepts of limit, one-sided limits.
3. State the squeeze theorem.
4. State the limit definition of the derivative of a function at a point.
5. Give examples to illustrate that not all continuous functions are differentiable.
6. State the product rule, quotient rule and chain rule.
7. State the Intermediate Value Theorem and mention some of its main applications.
8. State the Mean Value Theorem and some of its main applications.
9. State the definition of the critical number(s) of a function.
10. Explain how to find absolute and local maxima and minima of a function (Extreme Value Theorem, First derivative test and Second derivative test).
11. Explain how to sketch the graph of a function using the derivatives of the function.
12. State the definition of a Definite Integral as the limit of a Riemann sum.
13. State the Fundamental Theorem of Calculus and explain why it is fundamental.
14. Explain how to use Substitution Rule for definite and indefinite integrals.

**Review Problems:**

1. True or False? Give brief explanations.
   (a) If neither $\lim_{x\to a} f(x)$ nor $\lim_{x\to a} g(x)$ exists, then $\lim_{x\to a}[f(x)+g(x)]$ does not exist.
   (b) If $f$ is differentiable at $a$, then $\lim_{x\to a} f(x) = f(a)$.
   (c) If $f$ is continuous at $a$ then it is differentiable at $a$.
   (d) If $f$ is continuous on an open interval $(a,b)$, then $f$ attains an absolute minimum value at some number $c$ in $(a,b)$.
   (e) If $f''(x) < 0$ on $(a,b)$ then $f(x)$ is decreasing on $(a,b)$.
   (f) A critical number of a function $f$ is a number $c$ in the domain of $f$ such that $f'(c) = 0$.
   (g) If $f'(c) = 0$ and $f''(c) > 0$, then $f(x)$ has a local minimum at $c$.
   (h) If $f'(x)$ is increasing on $(a,b)$ then $f(x)$ is concave up on $(a,b)$.
   (i) If $f''(c) = 0$, then $f(x)$ has an inflection point at $c$.

(j) If $f(x)$ and $g(x)$ are increasing on an interval $I$, then $f(x)+g(x)$ is increasing on $I$.

(k) If $f'(x)=g'(x)$ for any $x$ then $f(x)=g(x)+C$ for some constant $C$.

(l) If $f(x)$ is continuous on on $(a,b)$ then $\int_a^b f(x)\,dx$ exists.

2. Find the following limits:

$$\text{(a)}\ \lim_{x\to 4}\frac{x^2-16}{x^2-2x-8} \quad \text{(b)}\ \lim_{x\to 0} x^4\sin\frac{6}{x} \quad \text{(c)}\ \lim_{x\to 4^+}\frac{8-2x}{|4-x|}$$

$$\text{(d)}\ \lim_{x\to 0}\frac{\tan^2(3x)}{3x^2} \quad \text{(e)}\ \lim_{x\to \pi^+}\frac{\cos x}{x-\pi} \quad \text{(f)}\ \lim_{x\to 0}\frac{\sqrt{x^2+25}-5}{x^2}$$

$$\text{(g)}\ \lim_{x\to \infty}\frac{4x^3-5x-3}{x^5+1} \quad \text{(h)}\ \lim_{x\to -\infty}\frac{x^7-5x-1}{x^4-5} \quad \text{(i)}\ \lim_{x\to \infty}\left(\sqrt{4x^2+x}-2x\right)$$

3. Let $f(x)=x^2-4x+1$. Find $f'(x)$ using the limit definition of the derivative.

4. Use linear approximation to estimate $\sqrt[5]{30}$.

5. Find the absolute maximum and absolute minimum values of the function $f(x)=\frac{x^4}{4}-\frac{x^3}{3}-x^2+2$ on the interval $[-1,1]$.

6. (a) State the Mean Value Theorem.

   (b) Verify that the function $f(x)=x^3+2x^2-x$ satisfies the hypotheses of the Mean Value Theorem on $[-1,2]$. Then find all the numbers $c$ in $(-1,2)$ which satisfy the conclusion of the Mean Value Theorem.

7. Sketch the graphs of the following functions. Find and label any intercepts and horizontal and vertical asymptotes; Determine the intervals of increase and decrease; Determine the intervals of concavity; Determine the points of inflection if any.

$$\text{(a)}\ f(x)=x^3-3x^2-1 \quad \text{(b)}\ f(x)=\frac{-2}{4-x^2} \quad \text{(c)}\ f(x)=\frac{x-2}{4-x^2}$$

8. Differentiate the following functions. You do not have to simplify your answers.

   (a) $f(x)=e^4$

   (b) $f(x)=(x^7-1)^6(x^{11}+2)$

   (c) $f(x)=x+\sqrt{1+x^2}$

   (d) $f(x)=\dfrac{x^2+3}{x^2-1}$

   (e) $f(x)=\sqrt{\cos\sqrt{x}}$

   (f) $f(x)=\sin^2(5x+1)$

   (g) $f(x)=\sec(2x)+\tan(2x)$

   (h) $f(x)=\int_0^{\tan x}(2+\sin t)dt$

9. Find an equation of the tangent line and an equation of the normal line to the following curve at the point $(1, \frac{\pi}{2})$:

$$\cos y + \sqrt{3 + x^2} = 2.$$

10. Evaluate the following integrals. In the case of a definite integral, your answer should be a real number. In the case of an indefinite integral, your answer should be the most general antiderivative as a function of the original variable. If the integral does not exist, explain why.

(a) $\displaystyle\int \frac{x^5 + \sqrt{x} - 2}{x^3}\, dx$

(b) $\displaystyle\int t^3 \sin(t^4 + 2)\, dt$

(c) $\displaystyle\int \sin^3 x \cos x\, dx$

(d) $\displaystyle\int x\sqrt{x - 5}\, dx$

(e) $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \csc^2(x)\, dx$

(f) $\displaystyle\int_0^1 \frac{1}{\sqrt{3x + 1}}\, dx$

(g) $\displaystyle\int_{\frac{\pi}{12}}^{\frac{\pi}{8}} \frac{1}{\sqrt{1 - 4x^2}}\, dx$

(h) $\displaystyle\int_{-\pi/4}^{\pi/4} \sin^7(x)\, dx$ (Hint: is the integrand a continuous odd function?)

(i) $\displaystyle\int_{-1}^{2} \frac{1}{x^2}\, dx$ (Hint: draw a graph of the function)

(j) $\displaystyle\int \left(x^2 + 8x + \frac{3}{x^2}\right) dx$

(k) $\displaystyle\int \frac{\cos\theta}{\sin^3\theta}\, d\theta$

(l) $\displaystyle\int_1^4 \frac{e^{1/u}}{u^2}\, du$

(m) $\displaystyle\int_1^e \frac{\ln x}{x}\, dx$

(n) $\displaystyle\int (1 + \ln x)\, x^x\, dx$

(o) $\displaystyle\int \frac{9}{t^2 + 1}\, dt$

(p) $\displaystyle\int \frac{t^2 - 2t}{t^3 - 3t^2 + 7}\, dt$

(q) $\int_0^4 \frac{xe^{\sqrt{x^2+9}}}{\sqrt{x^2+9}}\,dx$

(r) $\int \frac{\sec^2 x}{\tan x}\,dx$

11. Sketch the graph of $f(x) = |x-1|$, give a geometric interpretation for $\int_{-1}^{2} |x-1|\,dx$ and evaluate this definite integral.

12. (a) Use Intermediate Value Theorem to show that $x^3 - 3x^2 + 1 = 0$ has a solution in the interval $(0, 1)$.

    (b) Use Newton's method to find the third approximation of this solution (set $x_1 = 1$).

13. A paper cup has the shape of a cone with height $5$ inches and a radius of $2$ inches at the top. If water is poured into. the cup at a rate of $0.5$ inch$^3$/second, how fast is the water rising when the water is $3$ inches deep?

14. Find two positive numbers such their product is $192$ and sum of the first and three times the second is as small as possible.