7.2. Adding and subtracting rational expressions Professor Luis Fernández

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Review: adding and subtracting fractions

Recall: To add or subtract two or more fractions, you have to

- Write all the fractions with the same denominator. To do this,
  - → Find a common multiple of all the denominators, preferably the Least Common Multiple (LCM).
  - → Multiply the numerator and denominator of each fraction by appropriate quantity so that all the fractions have the same denominator.
- Add or subtract the numerators; leave the same denominator.
- Simplify the fraction.

Example: Add $\dfrac{5}{6}+\dfrac{7}{8}$.

The LCM of 6 and 8 is 24. To get 24 as denominator of the first fraction we need to multiply numerator and denominator by 4, and to get 24 as denominator of the first fraction we need to multiply numerator and denominator by 3. Therefore,

$$\frac{5}{6}+\frac{7}{8}=\frac{4\cdot 5}{4\cdot 6}+\frac{3\cdot 7}{3\cdot 8}=\frac{20}{24}+\frac{21}{24}=\frac{41}{24}.$$

Remember that when the fractions already have the same denominator, you can skip the first step.

Practice exercises: Add or subtract.

1. $\dfrac{7}{18}+\dfrac{5}{18}$ **2.** $\dfrac{5}{16}+\dfrac{5}{6}$ **3.** $\dfrac{7}{4}-\dfrac{5}{8}$ **4.** $\dfrac{2}{5}-\dfrac{4}{3}$

Exactly the same steps, word by word, are used to add or subtract rational expressions. The only thing we need to learn is how to find the Least Common Multiple of two or more polynomials.

Finding the Least Common Multiple (LCM) of two polynomials

To find the LCM of two or more polynomials,

- If not already factored, factor the polynomials completely.
- The LCM is the LCM of the coefficients times the product of **all** the factors that you got, raised to the highest exponent in which they appear.

Example: Find the LCM of $4(x+2)^2(x-3)^4$ and $6x(x-3)^2(x-2)^3$.

The polynomials are already factored, so we can skip the first step. The factors are 4, 6, x, $(x+2)$ and $(x-3)$.

The LCM of the coefficients (4 and 6) is 12.

The highest exponent for x is 1 (in the second polynomial).

The highest exponent for $(x+2)$ is 3 (in the second polynomial).

The highest exponent for $(x-3)$ is 4 (in the first polynomial).

Therefore the LCM of $4(x+2)^2(x-3)^4$ and $6x(x-3)^2(x-2)^3$ is $12x(x-3)^4(x-2)^3$.

Example: Find the LCM of $5x^2-25x+30$ and $4x^2-24x+36$.

First we factor the polynomials: $5x^2-25x+30=5(x^2-5x+6)=5(x-3)(x-2)$ and $4x^2-24x+36=4(x^2-6x+9)=4(x-3)^2$. The factors are 4, 5, x, $(x-2)$ and $(x-3)$.

The LCM of the coefficients (4 and 5) is 20.

The highest exponent for $(x-3)$ is 2 (in the second polynomial).

The highest exponent for $(x-2)$ is 1 (in the first polynomial).

Therefore the LCM of $5x^2-25x+30$ and $4x^2-24x+36$ is $20(x-2)(x-3)^2$.

Practice exercises: Find the LCM of the following sets of polynomials.

5. $x-2$ and $2x+3$

6. $8(x-4)^2(x+3)$ and $6(x-4)^3(x+3)^2$

7. $x^2-7x+10$ and x^2-4

8. $2x^2+4x$ and $3x+9$

9. $3x(x+1)^2$, $6x^2(x+2)$ and $(x+3)(x+1)^2$

10. $x+5$ and $x-2$

Adding and subtracting rational expressions

Now that we know how to find the LCM of polynomials we can add and subtract fractions. To do this we proceed as with numbers:

- Write all the rational expressions with the same denominator. To do this,
  - → Find the Least Common Multiple (LCM) of all the denominators.
  - → Multiply the numerator and denominator of each rational expression by an appropriate quantity so that all the fractions have the same denominator.
- Add or subtract the numerators; leave the same denominator.
- Simplify the fraction.

Example: Add: $\dfrac{4}{x+3}+\dfrac{3}{x+2}$.

The denominators are $x+3$ and $x+2$. The LCM of $x+3$ and $x+2$ is $(x+3)(x+2)$. Thus the LCD is $(x+3)(x+2)$.

To write the first expression with the LCD as denominator, we have to multiply the numerator and denominator by $(x+2)$:

$$\frac{4}{x+3}=\frac{4(x+2)}{(x+3)(x+2)}.$$

To write the second expression with the LCD as denominator, we have to multiply the numerator and denominator by $(x+3)$:

$$\frac{3}{x+2}=\frac{3(x+3)}{(x+3)(x+2)}.$$

Therefore,

$$\frac{4}{x+3}+\frac{3}{x+2}=\frac{4(x+2)}{(x+3)(x+2)}+\frac{3(x+3)}{(x+3)(x+2)}=\frac{4x+8+3x+9}{(x+3)(x+2)}=\frac{7x+17}{(x+3)(x+2)}.$$

To see the whole procedure in general, let us do a much more difficult example (this is as hard as you will see in this course):

Example: Subtract: $\dfrac{x+1}{x^2+6x+5}-\dfrac{x+2}{x^2+3x-10}$.

First we factor the denominators: $x^2+6x+5=(x+1)(x+5)$ and $x^2+3x-10=(x+5)(x-2)$, so we can write:

$$\frac{x+1}{x^2+6x+5}-\frac{x+2}{x^2+3x-10}=\frac{x+1}{(x+1)(x+5)}-\frac{x+2}{(x+5)(x-2)}.$$

Let us find the LCM of the denominators.

The factors of the denominators are $(x+1)$, $(x+5)$ and $(x-2)$, all with highest exponent 1. Therefore the LCM of the denominators (and thus the LCD) is $(x+1)(x+5)(x-2)$.

Now we need to write the expressions with $(x+1)(x+5)(x-2)$ as denominator. To do this we need to multiply numerator and denominator of the first rational expression by $(x-2)$ and of the second expression by $(x+1)$, and we get

$$\frac{x+1}{(x+1)(x+5)}-\frac{x+2}{(x+5)(x-2)}=\frac{(x+1)(x-2)}{(x+1)(x+5)(x-2)}-\frac{(x+1)(x+2)}{(x+1)(x+5)(x-2)}.$$

$$=\frac{(x+1)(x-2)-(x+1)(x+2)}{(x+1)(x+5)(x-2)}.$$

Next we need to simplify the numerator. To do this, multiply everything out, **leaving each part in parenthesis**. Then subtract (do not forget to distribute the "−" in the second term!):

$$\frac{(x+1)(x-2)-(x+1)(x+2)}{(x+1)(x+5)(x-2)} = \frac{(x^2-x-2)-(x^2+3x+2)}{(x+1)(x+5)(x-2)} = \frac{x^2-x-2-x^2-3x-2}{(x+1)(x+5)(x-2)}$$
$$= \frac{-4x-4}{(x+1)(x+5)(x-2)}.$$

Finally, factor the numerator and simplify:

$$\frac{-4x-4}{(x+1)(x+5)(x-2)} = \frac{-4(x+1)}{(x+1)(x+5)(x-2)} = \frac{-4\cancel{(x+1)}}{\cancel{(x+1)}(x+5)(x-2)} = \frac{-4}{(x+5)(x-2)}.$$

Therefore,

$$\frac{x+1}{x^2+5x+6} - \frac{x+2}{x^2+3x-10} = \frac{-4}{(x+5)(x-2)}.$$

Notice that the process is a long one, but you know how to do each one of the steps. It only involves factoring and multiplying polynomials.

Practice exercises: Add or subtract, as indicated.

11. $\frac{5x}{x+1} + \frac{2x-2}{x+1}$

12. $\frac{5x}{x-3} - \frac{15}{x-3}$

13. $\frac{3x}{x^2+4x+3} - \frac{x-2}{x^2+4x+3}$

14. $\frac{5}{x+1} + \frac{2}{x-3}$

15. $\frac{1}{x} - \frac{3x-2}{x+3}$

16. $\frac{x-2}{x+3} + \frac{x+1}{x-2}$

17. $\frac{1}{x-1} - \frac{1}{x+1}$

18. $\frac{5}{x+4} + \frac{3x-12}{x^2-16}$

19. $\frac{2}{x+3} + 2$

20. $\frac{x}{x-1} + \frac{1}{x+1} - \frac{2}{x^2-1}$

21. $\frac{x+2}{x^2+4x-5} + \frac{x-12}{x^2-25}$

22. $\frac{2}{x-3} - \frac{4}{3-x}$